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Gaussian measures on locally compact Abelian topological groups

by

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I. The aim of the present note is a generalization of the concept of normal measures for locally compact Abelian topological groups which for vector groups and toroidal groups would coincide with the usual concept of normal measures. Moreover, we shall examine the connection between the topological structure of groups and the existence of generalized normal measures.

We shall first quote the notions of normal measures on Euclidean spaces and on finitely dimensional toroidal groups which will be needed in the sequel.

R^n will denote the n-dimensional Euclidean space with inner product $(x, y) = \sum_{j=1}^{n} x_j y_j$ and norm $\|x\| = \sqrt{(x, x)}$, where $x = \langle x_1, \ldots, x_n \rangle$, $y = \langle y_1, \ldots, y_n \rangle$. Let A be a symmetric square matrix of order n. If the quadratic form (Ax, x) $(x \in R^n)$ induced by A is positive for $x \neq 0$ in the ordinary sense of taking only positive values, then A will be called a *positive matrix*. Obviously, every positive matrix is invertible.

A measure μ defined on R^n is called *normal* if there are a vector $y \in R^n$ and a positive matrix A of order n such that

$$\mu(E) = \frac{1}{\sqrt{(2\pi)^n \det A}} \int_E \exp\{-\tfrac{1}{2}(A^{-1}(x-y), x-y)\} dx$$

for every Borel subset $E \subset R^n$.

It is well known that a measure μ on R^n is normal if and only if its characteristic function

$$\varphi_\mu(t) = \int_{R^n} e^{i(t,x)} \mu(dx) \qquad (t \in R^n)$$

is of the form

$$\varphi_\mu(t) = \exp\{i(t, y) - \tfrac{1}{2}(At, t)\}, \tag{1}$$

where $y \in R^n$ and A is a positive matrix of order n.

T will denote the circle $\{e^{iu}: 0 \leqslant u < 2\pi\}$ with ordinary topology. For every subset $E \subset T$ we set

$$\arg E = \{u: e^{iu} \in E,\ 0 \leqslant u < 2\pi\}.$$

A measure μ defined on T is called *normal* if there exists a pair of numbers $u_0, b\,(0 \leqslant u_0 < 2\pi,\ b > 0)$ such that

$$\mu(E) = \frac{1}{\sqrt{2\pi b}} \int_{\arg E} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{(u-u_0+2n\pi)^2}{2b}\right\} du$$

for every Borel subset $E \subset T$.

Introducing the system of Fourier coefficients

$$a_k(\mu) = \int_T x^k \mu(dx) \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2}$$

we have the following assertion:

A measure μ defined on T is normal if and only if there exist an element $y \in T$ and a positive number d such that

$$a_k(\mu) = y^k e^{-dk^2} \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{3}$$

Now let us consider the n-dimensional toroidal group T^n. Every measure μ defined on the σ-field of all Borel subsets of T^n can be characterized by its system of Fourier coefficients for all lattice points $m = \langle m_1, \ldots, m_n\rangle$,

$$a_{m_1,\ldots,m_n}(\mu) = \int_{T^n} x_1^{m_1} \cdot \ldots \cdot x_n^{m_n} \mu(dx),$$

where x_j denotes the j-coordinate of $x \in T^n$. In the sequel we shall denote by Δ^n the set of all lattice points $m = \langle m_1, \ldots, m_n\rangle$.

Let B be a symmetric matrix of order n. If the quadratic form (Bm, m) $(m \in \Delta^n)$ is positive for $m \neq \langle 0, 0, \ldots, 0\rangle$, then the matrix B will be called *positive with respect to* Δ^n. We remark that matrices positive with respect to Δ^n *can be not* invertible.

We say that a measure μ defined on T^n is *normal* if its Fourier coefficients are of the form

$$a_{m_1,\ldots,m_n}(\mu) = y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} e^{-(Bm, m)}, \tag{4}$$

where $\langle y_1, \ldots, y_n\rangle \in T^n$ and the matrix B is positive with respect to Δ^n. (For a detailed treatment of normal and more general measures on T^n see [1], p. 74).

We remark that a normal measure on T^n is absolutely continuous with respect to the Lebesgue measure on T^n if and only if the matrix B is invertible.

II. Let G be a locally compact Abelian topological group. In the present note regular completely additive measures μ defined on the class of all Borel subsets of G, with $\mu(G) = 1$, will be called briefly *measures*. A sequence of measures $\mu_1, \mu_2, \ldots$ is said to be *weakly convergent* to a measure μ if

$$\lim_{n\to\infty} \int_G f(x)\mu_n(dx) = \int_G f(x)\mu(dx)$$

for any complex-valued continuous function f defined on G which vanishes outside a compact set. We define the convolution of μ and ν, denoted by $\mu * \nu$, by the formula

$$\mu*\nu(E) = \int_G \mu(E\cdot x^{-1})\nu(dx),$$

where $E\cdot x^{-1} = \{yx^{-1}: y \in E\}$. A measure μ on G is called *symmetric* if for every Borel subset $E \subset G$ we have the equality $\mu(E) = \mu(E^{-1})$ where $E^{-1} = \{x^{-1}: x \in E\}$.

Let $\hat{G}$ be the character group of G. The complex-valued function

$$L_\mu(\chi) = \int_G \chi(x)\mu(dx) \qquad (\chi \in \hat{G})$$

is called the *characteristic function* of the measure μ. It is well known that every measure is uniquely determined by its characteristic function. Moreover, the weak convergence to μ of a sequence $\mu_1, \mu_2, \ldots$ is equivalent to the convergence to $L_\mu(\chi)$ of the sequence $L_{\mu_1}(\chi), L_{\mu_2}(\chi), \ldots$ for each $\chi \in \hat{G}$. The characteristic function of the convolution of two measures is given by the formula

$$L_{\mu*\nu}(\chi) = L_\mu(\chi)L_\nu(\chi).$$

Further, a measure is symmetric if and only if its characteristic function is real.

Let G_1, G_2 be two topological groups and let h be a homomorphism of G_1 onto G_2. For every measure μ on G_1 by μ_h we denote a measure in G_1 induced by μ as follows:

$$\mu_h(E) = \mu(h^{-1}(E))$$

for Borel subsets $E \subset G_2$. If χ is a character belonging to $\hat{G}_2$, then the superposition χh belongs to $\hat{G}_1$. Hence and from the definition of characteristic functions directly follows the equality

$$L_{\mu_h}(\chi) = L_\mu(\chi h) \qquad (\chi \in \hat{G}_2). \tag{5}$$

A measure μ on G will be called an *infinitely subdivisible* measure if there exists a family $\{\mu^t\}$ $(t > 0)$ of measures on G such that we have $\mu^1 = \mu$, $\mu^t * \mu^s = \mu^{t+s}$ for every pair t, s of positive numbers and, for $t \to 0$, μ^t is weakly convergent to the identity measure which is wholly concentrated at the unit element of G.

For example, normal measures on R^n or T^m are infinitely subdivisible. (For a characterization of all infinitely subdivisible measures on R^n and T^m see [1], Chapt. 3).

It is easy to verify that if μ is an infinitely subdivisible measure on G_1 and h is a homomorphism of G_1 onto G_2, then μ_h is also an infinitely subdivisible measure on G_2.

Our next object is a definition of generalized normal measures on locally compact Abelian groups, which will be called Gaussian measures.

An infinitely subdivisible measure μ on a locally compact Abelian group G is called *Gaussian* if for every non-trivial character χ $(\chi \in \hat{G})$ μ_χ is a normal measure on T.

At first sight it seems that the assumption of infinite subdivisibility in this definition can be omnitted. But, as shown by the following example, such modified definition of Gaussian measures does not coincide with the definition of normal measures on the two-dimensional toroidal group T^2. Put

$$g(u_1, u_2) = \sum_{n,m=-\infty}^{\infty} c_{mn} e^{-i(mu_1+nu_2)} \qquad (0 \leqslant u_1, u_2 < 2\pi),$$

where

$$c_{mn} = \begin{cases} 10^{-n^2} & \text{if } m = n, \\ 10^{-(n^2+m^2)} & \text{if } m \neq n. \end{cases} \tag{6}$$

Obviously, $g(u_2, u_1)$ is a continuous real-valued function and

$$g(u_1, u_2) \geqslant 1 - \sum_{\langle n,m\rangle \neq \langle 0,0\rangle} c_{mn} \geqslant 1 - 4\sum_{n=1}^{\infty} 10^{-n^2} - 4\Big(\sum_{n=1}^{\infty} 10^{-n^2}\Big)^2$$

$$\geqslant 1 - 4\sum_{n=1}^{\infty} 10^{-n} - 4\Big(\sum_{n=1}^{\infty} 10^{-n}\Big)^2 = \frac{41}{81}.$$

Consequently, $g(u_1, u_2)$ can be regarded as a density function with respect to the Lebesgue measure on T^2. Denoting by μ the measure determined

by $g(u_1, u_2)$ and taking into account equalities (2) and (5) we infer that

$$a_k(\mu_\chi) = e^{-d(\chi)k^2} \qquad (\chi \in \hat{T}^2,\ k = 0, \pm 1, \pm 2, \ldots),$$

where

$$d(\chi) = \begin{cases} n^2 \log 10 & \text{if } \chi = \langle n, n\rangle, \\ (n^2+m^2)\log 10 & \text{if } \chi = \langle m, n\rangle,\ m \neq n. \end{cases}$$

Thus, for every non-trivial character $\chi \in \hat{T}^2$, μ_χ is a normal measure on T. Now we shall prove that μ is not normal on T^2. Let us suppose that it is normal. There is then, according to (4), a triplet of real numbers α, β, γ such that

$$c_{mn} = a_{m,n}(\mu) = e^{-(\alpha m^2 + \beta mn + \gamma n^2)} \qquad (m, n = 0, \pm 1, \pm 2, \ldots)$$

Hence and from (6) we get the equalities

$$\alpha m^2 + \beta mn + \gamma n^2 = \begin{cases} (m^2+n^2)\log 10 & \text{for } m \neq n, \\ n^2 \log 10 & \text{for } m = n, \end{cases}$$

and, consequently,

$$\alpha = \gamma = \log 10, \qquad \beta = 0, \qquad \alpha + \beta + \gamma = \log 10,$$

which are impossible. Thus μ is not normal on T^2.

From (3) and (5) directly follows the assertion

(*) *An infinitely subdivisible measure μ on G is Gaussian if and only if there is a function $c(\chi)$, with $|c(\chi)| = 1$, such that*

$$L_\mu(\chi^k) = c^k(\chi)|L_\mu(\chi)|^{k^2} \qquad (\chi \in \hat{G},\ k = 0, \pm 1, \pm 2, \ldots) \tag{7}$$

and

$$0 < |L_\mu(\chi)| < 1 \tag{8}$$

for all non-trivial characters $\chi \in \hat{G}$.

Hence, in particular, it follows that the convolution of Gaussian measures is the same one.

Now we shall prove that the notion of Gaussian measures coincides with the notion of normal measures on vector groups and on toroidal groups.

THEOREM 1. *A measure on R^n is Gaussian if and only if it is normal.*

Proof. The sufficiency of our condition is evident. It follows directly from (*), (1), (5) and from the infinite subdivisibility of normal measures. To prove the necessity let us suppose that μ is a Gaussian measure on R^n. Set

$$\varphi(t) = \int_{R^n} e^{i(t,x)}\mu(dx) \qquad (t \in R^n).$$

Since $\hat{R}^n = R^n$, i. e. every character of R^n is of the form $\chi(x) = e^{i(t,x}$ $(t \in R^n)$, we have, according to (7) and (8), the relations

(9) $$|\varphi(kt)| = |\varphi(t)|^{k^2} \qquad (t \in R^n,\ k = 0, \pm 1, \pm 2, \ldots),$$

(10) $$0 < |\varphi(t)| < 1 \quad \text{for } t \neq \langle 0, 0, \ldots, 0\rangle.$$

From the infinite subdivisibility of μ it follows that there are a matrix A of order n, a vector $y \in R^n$ and a measure λ (not necessarily normalized) defined on R^n such that

(11) $$\log\varphi(t) = i(t, y) - \tfrac{1}{2}(At, t) + \int_{R^n}\left\{e^{i(t,x)} - 1 - \frac{i(t, x)}{1+\|x\|^2}\right\}\lambda(dx),$$

(12) $$\lambda(\langle 0, 0, \ldots, 0\rangle) = 0$$

and

(13) $$\int_{R^n}\frac{\|x\|^2}{1+\|x\|^2}\lambda(dx) < \infty$$

(cf. [1], p. 69). From (9) and (11) we get the equality

$$\int_{R^n}(\cos k(t, x) - 1)\lambda(dx) = k^2\int_{R^n}(\cos(t, x) - 1)\lambda(dx),$$

which implies, according to (13),

$$\int_{R^n}(\cos(t, x) - 1)\lambda(dx) = \lim_{k\to\infty}\int_{R^n}\frac{\cos k(t, x) - 1}{k^2}\lambda(dx) = 0.$$

Hence, in virtue of (12), we obtain the equality $\lambda(R^n) = 0$. Thus, according to (11),

$$\varphi(t) = \exp\{i(t, y) - \tfrac{1}{2}(At, t)\}.$$

Further, taking into account inequality (10), we infer that A is a positive matrix. Consequently, μ is a normal measure on R^n.

THEOREM 2. *A measure in T^m is Gaussian if and only if it is normal.*

Proof. As before the sufficiency of our condition is evident. Let μ be a Gaussian measure on T^n. Since $\hat{T}^n = \Delta^n$, i. e. every character of

T^n is of the form $\chi(\langle x_1, \ldots, x_n\rangle) = x_1^{m_1}\cdot\ldots\cdot x_n^{m_n}$, we have, according to (7) and (8), the following relations for Fourier coefficients

(14) $$|a_{km_1,\ldots,km_n}(\mu)| = |a_{m_1,\ldots,m_n}(\mu)|^{k^2} \qquad (\langle m_1, \ldots, m_n\rangle \in \Delta^n,\ k = 0, \pm 1, \pm 2, \ldots)$$

(15) $$0 < |a_{m_1,\ldots,m_n}(\mu)| < 1 \quad \text{for } \langle m_1, \ldots, m_n\rangle \neq \langle 0, 0, \ldots, 0\rangle.$$

Taking into account the infinite subdivisibility of μ we get the decomposition

(16) $$\mu = \mu_0 * \nu,$$

where μ_0 is a weak limit of symmetric normal measures on T^n and, writing $x = \langle x_1, \ldots, x_n\rangle$,

(17) $$a_{m_1,\ldots,m_n}(\nu) = \exp\left\{i\sum_{j=1}^{n} c_j m_j + \int_{T^n}\left(x_1^{m_1}\cdot\ldots\cdot x_n^{m_n} - 1 - i\sum_{j=1}^{n} m_j \operatorname{Im} x_j\right)\lambda(dx)\right\}$$

(18) $$\lambda(\langle 1, 1, \ldots, 1\rangle) = 0,$$

(19) $$\int_{T^n}\sum_{j=1}^{n}(\operatorname{Im} x_j)^2\lambda(dx) < \infty$$

(cf. [1], p. 74). From (14), (15) and (16) we get the equality

$$|a_{km_1,\ldots,km_n}(\nu)| = |a_{m_1,\ldots,m_n}(\nu)|^{k^2} \qquad (\langle m_1, \ldots, m_n\rangle \in \Delta^n,\ k = 0, \pm 1, \pm 2, \ldots)$$

which implies, in virtue of (17) and (19),

$$\log|a_{m_1,\ldots,m_n}(\nu)| = \int_{T^n}\left\{\cos\left(\sum_{j=1}^{n} m_j \arg x_j\right) - 1\right\}\lambda(dx)$$

$$= \lim_{k\to\infty}\int_{T^n}\frac{\cos k\left(\sum_{j=1}^{n} m_j \arg x_j\right) - 1}{k^2}\lambda(dx) = 0.$$

Hence and from (18) follows the equality $\lambda(T^m) = 0$. Consequently, in virtue of (16) and (17), there is an element $\langle y_1, \ldots, y_n\rangle \in T^n$ such that

(20) $$a_{m_1,\ldots,m_n}(\mu) = y_1^{m_1}\cdot\ldots\cdot y_n^{m_n} a_{m_1,\ldots,m_n}(\mu_0) \qquad (\langle m_1, \ldots, m_n\rangle \in \Delta^n).$$

Further, from (15) it follows that the measure μ_0, being a weak limit of symmetric normal measures, is the same one. Hence and from (20) we get the normality of T^m.

Remark. In a similar way we can prove the following statement:

(**) *A measure μ on the direct sum $R^n \times T^s$ is Gaussiau if and only if there are an element $\langle y_1, \ldots, y^{n+s} \rangle \in R^n \times T^n$ and a matrix A of order $n+s$ satisfying the inequality $(An, u) > 0$ for any $u \in R^n \times \Delta^s$ and $u \neq \langle 0, \ldots, 0, 1, \ldots, 1 \rangle$ such that*

$$\int\limits_{R^n \times T^s} e^{i(t_1 x_1 + \ldots + t_n x_n)} x_{n+1}^{t_{n+1}} \cdot \ldots \cdot x_{n+s}^{t_{n+s}} \mu(dx) = e^{i(t_1 y_1 + \ldots + t_n y_n)} y_{n+1}^{t_{n+1}} \cdot \ldots \cdot y_{n+s}^{t_{n+s}} e^{-(At,t)},$$

where $\langle t_1, \ldots, t_{n+s} \rangle = t \in R^n \times \Delta^s$ and $\langle x_1, \ldots, x_{n+s} \rangle = x \in R^n \times T^s$.

Hence we obtain by a simple reasoning the following property:

(***) *Gaussian measures on $R^n \times T^s$ are positive on open non-empty sets.*

III. Before proving the Theorem describing the structure of groups with Gaussian measures we shall prove two Lemmas.

As a direct consequence of formula (5) and condition (*) we obtain the following Lemma:

LEMMA 1. *Let G_0 be a closed subgroup of a locally compact Abelian group G and let π be the projection of G onto the quotient group G/G_0. If a measure μ is Gaussian on G, then μ_π is the same one on G/G_0.*

LEMMA 2. *Let $\{G_\xi\}_{\xi \in \Xi}$ be a family of locally compact Abelian groups which are compact except a finite number. If $\mu^{(\xi)}$ are Gaussian measures on G_ξ $(\xi \in \Xi)$, then the product measure $\mathcal{P}_{\xi \in \Xi} \mu^{(\xi)}$ is the same one on the product group $\mathcal{P}_{\xi \in \Xi} G_\xi$.*

Proof. It is well known that every character of the product group $G = \mathcal{P}_{\xi \in \Xi} G_\xi$ depends on finitely many ξ-coordinates only, i. e. for every $\chi \in \hat{G}$ there exist a finite set of indices $\xi_1, \ldots, \xi_r \in \Xi$ and characters $\chi_{\xi_1}, \ldots, \chi_{\xi_r}$ belonging to $\hat{G}_{\xi_1}, \ldots, \hat{G}_{\xi_r}$ respectively, such that

$$\chi(x) = \prod_{j=1}^{r} \chi_{\xi_j}(\pi_{\xi_j}(x)) \quad (x \in G), \tag{21}$$

where π_ξ denotes the projection of G onto G_ξ $(\xi \in \Xi)$ (see [3], p. 260). Let c_ξ be the function determined by formula (7) for $\mu^{(\xi)}$. Setting $\mu = \mathcal{P}_{\xi \in \Xi} \mu^{(\xi)}$, $c(\chi) = \prod_{j=1}^{r} c_{\xi_j}(\chi_{\xi_j})$ $(\chi \in \hat{G})$ and taking into account (5), (*) and (21) we infer that

$$L_\mu(\chi^k) = L_\mu\Big(\prod_{j=1}^{r} \chi_{\xi_j}^k \pi_{\xi_j}\Big) = \prod_{j=1}^{r} L_\mu^{(\xi_j)}(\chi_{\xi_j}^k)$$
$$= c^k(\chi) |L_\mu(\chi)|^{k^2} \quad (\chi \in \hat{G},\ k = 0, \pm 1, \pm 2, \ldots)$$

and

$$0 < |L_\mu(\chi)| < 1$$

for all non-trivial characters $\chi \in \hat{G}$. The infinite subdivisibility of μ is evident. Thus, according to (*), μ is a Gaussian measure on G.

THEOREM 3. *There exists a Gaussian measure on a locally compact Abelian group G if and only if G is connected.*

Proof. Necessity. Let μ be a Gaussian measure on G. If the group G is not connected, then there is a subgroup G_0 of G such that the quotient group G/G_0 is discrete and contains at least two elements (see [3], p. 137 and Theorem 16 p. 139). By Lemma 1 the measure μ induces a Gaussian measure ν on G/G_0. Every regular measure on a discrete space is purely atomic. Consequently, ν is purely atomic and, moreover, for every non-trivial character $\chi \in \hat{G}/G_0$, μ_χ is the same one. But this contradicts the absolute continuity of μ_χ with respect to the Lebesgue measure on T. Thus the group G is connected.

Sufficiency. Let G be a connected locally compact Abelian group. Then G decomposes into the direct sum $G = R^n \times G_0$, where G_0 is a compact connected Abelian group (see [4], § 29, [3] § 39). By Lemma 2, to prove our assertion it is sufficient to show that there exists a Gaussian measure on the group G_0.

In the sequel S will denote the solenoid, i. e. the character group of the discrete additive group of all rational numbers. The group $\hat{G}_0$ has no elements of finite order (see [3], p. 262). Therefore $\hat{G}_0$ is a subgroup of a direct sum of the discrete additive groups of rational numbers (see [2], p. 191). Hence it follows that there are a product group H of solenoids and a subgroup H_0 of H such that $G_0 = H/H_0$. Consequently, by Lemmas 1 and 2, to prove that there exists a Gaussian measure on G_0 it is sufficient to show this for the solenoid only.

Set $f_t(r) = e^{-tr^2}$ for any rational r and $t > 0$. The functions f_t $(t > 0)$ are continuous and positively definite on the discrete group of rational numbers. By Bochner's Theorem there are measures μ^t on S such that

$$\int\limits_S \chi(x) \mu^t(dx) = e^{-tr^2} \quad (t > 0),$$

where χ denotes the character of S determined by r (see [4], § 30). Obviously, for $t \to 0$, μ^t is weakly convergent to the identity measure and $\mu^t * \mu^s = \mu^{t+s}$ for every pair t, s of positive numbers. Moreover,

$$L_{\mu^t}(\chi^k) = L_{\mu^t}(\chi)^{k^2} \quad (k = 0, \pm 1, \pm 2, \ldots)$$

and

$$0 < L_{\mu^t}(\chi) < 1$$

for any non-trivial character $\chi \in \hat{S}$. Consequently, according to (*) μ^1 is a Gaussian measure on S. The sufficiency of our condition is thus proved.

THEOREM 4. *An infinitely subdivisible measure μ on G is Gaussian if and only if there exists an element $x_0 \in G$ such that*

$$L_\mu(\chi^k) = \chi^k(x_0)|L_\mu(\chi)|^{k^2} \quad (\chi \in \hat{G},\ k = 0, \pm 1, \pm 2, \ldots)$$

and

$$0 < |L_\mu(\chi)| < 1$$

for every non-trivial character χ.

Proof. The sufficiency of our condition is obvious. Now let us suppose that μ is a Gaussian measure on a group G. According to (*), to prove our Theorem it is sufficient to show that there exists an element $x_0 \in G$ for which $c(\chi) = \chi(x_0)\ (\chi \in \hat{G})$. By Theorem 3 the group G is connected. Thus it is a projective limit of groups $R^n \times T^s$ (see [4], § 29). For every pair χ_1, χ_2 of characters belonging to $\hat{G}$ there is then a subgroup H of G such that $G/H = R^n \times T^s$ and, further, there are characters $\tilde{\chi}_1, \tilde{\chi}_2 \in \hat{G}/H$ such that

$$\chi_1 = \tilde{\chi}_1 \pi, \quad \chi_2 = \tilde{\chi}_2 \pi,$$

where π denotes the projection of G onto G/H (see [3], p. 260).

Let ν be a Gaussian measure induced by μ on G/H. According to (**) there is an element $y_0 \in G/H$ such that

$$L_\mu(\chi_1^{k_1}\chi_2^{k_2}) = L_\nu(\tilde{\chi}_1^{k_1}\tilde{\chi}_2^{k_2}) = \tilde{\chi}_1^{k_1}(y_0)\tilde{\chi}_2^{k_2}(y_0)|L_\nu(\tilde{\chi}_1^{k_1}\tilde{\chi}_2^{k_2})|$$
$$= \tilde{\chi}_1^{k_1}(y_0)\tilde{\chi}_2^{k_2}(y_0)|L_\mu(\chi_1^{k_1}\chi_2^{k_2})| \quad (k_1, k_2 = 0, \pm 1, \pm 2, \ldots)$$

Hence we get the equality

$$c(\chi_1^{k_1}\chi_2^{k_2}) = c^{k_1}(\chi_1)c^{k_2}(\chi_2) \quad (k_1, k_2 = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the function $c(\chi)$ is a character of the group $\hat{G}$. There is then an element $x_0 \in G$ such that $c(\chi) = \chi(x_0)$ for every $\chi \in \hat{G}$ (see [3], p. 278). The Theorem is thus proved.

THEOREM 5. *Gaussian measures are positive on open non-empty sets.*

Proof. Let μ be a Gaussian measure on a group G. Contrary to our statement let us suppose that there is an open non-empty subset $E \subset G$ for which the equality

$$\mu(E) = 0 \tag{22}$$

is true. Further, let E_0 be an open non-empty subset of E and let V be a neighbourhood of the unit element of the group G such that

$$E_0 V \subset E. \tag{23}$$

By Theorem 3, G is connected. Consequently, the group G is a projective limit of groups $R^n \times T^s$. There is then a subgroup H of G contained in the neighbourhood V such that $G/H = R^n \times T^s$. Denoting by π the projection of G onto G/H we have, in virtue of (22) and (23), the relation

$$\mu_\pi(\pi(E_0)) = \mu(E_0 V) = 0.$$

Consequently, the Gaussian measure μ_π vanishes on the open non-empty set $\pi(E_0)$, which contradicts the statement (***). The Theorem is thus proved.

IV. Let G be a solenoidal group, i. e. a product group $\mathcal{P}_{\xi \in \Xi} S_\xi$ of solenoids S_ξ $(\xi \in \Xi)$. For every $\xi \in \Xi$ there is a continuous function a_ξ defined on S_ξ such that every character χ_ξ of the group S_ξ is of the form

$$\chi_\xi(x) = e^{ia_\xi(x)r} \quad (x \in S_\xi),$$

where r is a rational number. Hence it follows that every character χ of the group G is of the form

$$\chi(x) = \exp\Big\{i\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j})\Big\} \quad (x = \{x_\xi\} \in G), \tag{24}$$

where $r_1, \ldots, r_s$ are rationals and $\xi_1, \ldots, \xi_s \in \Xi$.

It is well known that the mapping $F_n\colon x \to \sqrt[n]{x}$ of a solenoidal group G onto itself is one-to-one and continuous.

Now we shall prove a theorem which can be regarded as a Central Limit Theorem for solenoidal groups.

THEOREM 6. *Let G be a solenoidal group and let ν be a measure on G positive on all open non-empty sets. There is then an element $y \in G$ such that the sequence of measures $\nu_1, \nu_2 * \nu_2, \nu_3 * \nu_3 * \nu_3, \ldots$, where*

$$\nu_n(E) = \nu\big(F^{-1}_{[\sqrt{n}]}(E)y\big) \quad (n = 1, 2, \ldots)\ ^{(1)}, \tag{25}$$

converges weakly to a symmetric Gaussian measure on G.

Proof. By the continuity of the function a_ξ there is an element $y_\xi \in S_\xi$ such that

$$\int_G a_\xi(x_\xi)\nu(dx) = a_\xi(y_\xi). \tag{26}$$

Let us introduce the notation $y = \{y_\xi\}$,

$$\lambda_n = \underbrace{\nu_n * \nu_n * \ldots * \nu_n}_{n \text{ times}} \quad (n = 1, 2, \ldots), \tag{27}$$

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(1) $[z]$ denotes the integral part of z.

where ν_n are defined by formula (25). Further, let χ be a character of G given by expression (24). From (25) and (27) follows the equality

$$(28)\quad L_{\lambda_n}(\chi^k) = \{L_{\nu_n}(\chi^k)\}^n = \Big\{\int_G \exp\Big(ik[\sqrt{n}]^{-1}\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j}y_{\xi_j}^{-1})\Big)\nu(dx)\Big\}^n.$$

Taking into account equality (26), we infer that

$$\int_G \exp\Big(ik[\sqrt{n}]^{-1}\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j}y_{\xi_j}^{-1})\Big)\nu(dx) = 1-\frac{k^2}{2n}\int_G\Big\{\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j}y_{\xi_j}^{-1})\Big\}^2\nu(dx)+o\Big(\frac{1}{n}\Big),$$

whence, according to (28), we get the convergence

$$(29)\qquad \lim_{n\to\infty} L_{\lambda_n}(\chi^k) = e^{-k^2 d(\chi)}\qquad (\chi\in\hat{G}),$$

where

$$d(\chi) = \tfrac{1}{2}\int_G\Big\{\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j}y_{\xi_j}^{-1})\Big\}^2\nu(dx).$$

If $d(\chi)=0$, then $\sum_{j=1}^{s} r_j a_{\xi_j}(x_{\xi_j})$ is constant for almost all $x\in G$ with respect to the measure ν. Since ν is positive on open non-empty sets, the last assertion holds for all $x\in G$. Hence, in virtue of (24), it follows that the character χ is trivial. Consequently, $d(\chi)$ is positive for all non-trivial characters $\chi\in\hat{G}$. In other words, the limit in (29) is the characteristic function of a Gaussian measure on G. Thus the sequence $\lambda_1, \lambda_2, \ldots$ is weakly convergent to a Gaussian measure on G, which implies the assertion of the Theorem.

References

[1] S. Bochner, *Harmonic analysis and the theory of probability*, Berkeley-Los Angeles 1955.

[2] А. Г. Курош, *Теория групп*, Москва 1953.

[3] Л. С. Понтрягин, *Непрерывные группы*, Москва 1954.

[4] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris 1940.

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Bernsteinsche Potenzreihen

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Die der Funktion $f(x)$ zugeordneten Bernsteinschen Polynome

$$B_n(x) = B_n^f(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\Big(\frac{k}{n}\Big)\qquad (n=1,2,\ldots)$$

stehen in einer bekannten formalen Beziehung zu der durch

$$b_{nk} = b_{nk}(x) = \binom{n}{k}x^k(1-x)^{n-k}\qquad (n=1,2,\ldots;\ k=0,\ldots,n)$$

gegebenen Bernoullischen (oder binomischen) Verteilung der Wahrscheinlichkeitsrechnung (vgl. [2], S. 4). Liegt eine unbegrenzte Folge unabhängiger Bernoulli-Versuche mit der Erfolgswahrscheinlichkeit x und der Mißerfolgswahrscheinlichkeit $1-x$ vor, so ist b_{nk} die Wahrscheinlichkeit, in den n ersten Versuchen k Erfolge zu erzielen.

Mit der Bernoullischen Verteilung ist verwandt (vgl. [1] S. 155) die Pascalsche Verteilung (negative Binomialverteilung). Wieder liege eine unbegrenzte Folge unabhängiger Bernoulli-Versuche vor. Aus einem nachher einleuchtenden Grund sei diesmal die Erfolgswahrscheinlichkeit mit $1-x$, die Mißerfolgswahrscheinlichkeit mit x bezeichnet. Dann ist die Wahrscheinlichkeit, den n-ten Erfolg beim k-ten Wurf zu erzielen, gegeben durch

$$p_{nk} = p_{nk}(x) = (1-x)^n\binom{k-1}{n-1}x^{k-n}\qquad (n=1,2,\ldots;\ k=n,n+1,\ldots).$$

Es erhebt sich die Frage, ob wir — in Analogie zu den b_{nk} und $B_n(x)$ — mit Hilfe der p_{nk} den $B_n(x)$ verwandte und ähnliche Eigenschaften besitzende Ausdrücke bilden können. Dies ist der Fall. Es wird sich dabei nicht um Polynome, sondern um Potenzreihen handeln, die wir als Bernsteinsche Potenzreihen bezeichnen.