STUDIA MATHEMATICA, T. XIX. (1960)

On the locally bounded and m-convex topological algebras

by

W. ŻELAZKO (Warszawa)

The fundamental theorem of the theory of normed algebras is the Gelfand-Mazur theorem (see [8] and [11]). Gelfand's proof of that theorem is based upon an abstract theory of analytic functions. The original proof of Mazur, which has not been published so far, was based upon certain properties of harmonic functions. The existence of linear functionals is essential for that proof. There are also some other proofs of the theorem in question (see [19] and [9]), based upon some inequalities in which the homogeneity of the norm plays an important role. Some generalizations have also been given. Arens [2] proves an analogous theorem for the locally convex topological algebras in which the operation of taking inverse is continuous. Michael [14] proves a theorem for locally convex m-convex topological algebras. The problem of generalization of the Gelfand-Mazur theorem for the locally bounded topological algebras was suggested to me by A. Pełczyński, whose theorem, Theorem 1 of this paper, is the starting point for the theory developed below.

This paper consists of two parts. The first part is devoted to the theory of the locally bounded topological algebras. It will be seen that a greater part of Gelfand's theory [8] is also valid for the locally bounded complete metric commutative algebras. Hence in the theory of normed algebras the property of local boundedness is much more essential than the property of local convexity. The second part consists of some results concerning m-convex topological algebras. We point out some connections between locally bounded topological algebras and m-convex topological algebras. The Gelfand-Mazur theorem for m-convex topological algebras is obtained as a corollary to a representation theorem for those algebras. We also prove a theorem on the multiplicative linear functionals and give an example of a B_0-algebra which is not m-convex (cf. the definition given below).

Definitions and notation. A set R of elements $x, y, \ldots$ is called a *topological algebra* if

1° R is an algebra over the real or complex numbers,

2° R is a linear topological space,

3° the multiplication xy is continuous in each argument separately.

Since every topological algebra can be imbedded into an algebra with the unit element, we assume also that

4° there exists in R such an element e that $ex = xe = x$ for every $x \in R$.

The last assumption is not only a formality. We shall see later on (§ 1) that there exists a metric algebra with a submultiplicative metric function (see below) which cannot be topologically imbedded into a topological algebra with the unit element.

If q designates any class of topological linear spaces, then a topological algebra R is called a *q-algebra*, if R, considered as a linear space, is a member of the class q. As q we shall consider the following classes. The class of:

B-spaces, i. e. the complete normed spaces, or the Banach spaces,

B^*-spaces (1), i. e. the normed spaces not necessarily complete,

B_0-spaces, i. e. the locally convex metrisable complete topological linear spaces,

B_0^*-spaces, i. e. the locally convex metrisable topological linear spaces,

F-spaces, i. e. the metrisable complete topological linear spaces in which a topology may be introduced by a metric function $\varrho(x, y)$ satisfying $\varrho(x-z, y-z) = \varrho(x, y)$ (2) for arbitrary elements x, y, z.

F^*-spaces, i.e. the spaces defined as F-spaces, but without assuming completness.

It is known that every topological linear metrisable space is an F-space or an F^*-space. We shall also consider the class of locally convex topological algebras and the class of locally bounded topological algebras.

The metric functions ϱ and ϱ' defined on a linear metric space are called *equivalent* if they generate the same topology.

The metric function ϱ of a metric algebra R is called submultiplicative if

$$\varrho(xy, 0) \leqslant \varrho(x, 0)\varrho(y, 0) \quad \text{for} \quad x, y \in R.$$

(1) An asterisk means that completeness is not assumed.

(2) Setting $\|x\| = \varrho(x, 0)$ we get the so called *F-norm*, i. e. the non-negative function satisfying the following conditions:

(a) $\|x\| = 0$ if and only if $x = 0$,

(b) $\|x\| = \|-x\|$,

(c) $\|x+y\| \leqslant \|x\|+\|y\|$,

(d) if (a_n) is a sequence of scalars tending to 0, then for every element x $\lim\|a_n x\| = 0$,

(e) if $\lim\|x_n\| = 0$, then $\lim\|ax_n\| = 0$ for every scalar a (see [4], pp. 35, 37).

An F-algebra R is called a *p-normed algebra* if there exists in R an equivalent submultiplicative and p-homogeneous norm, i. e. if there exists on R a function $\|x\|$ such that

1° R is an F-space with the metric $\varrho(x, y) = \|x-y\|$, and ϱ is equivalent to the original metric in R,

2° $\|xy\| \leqslant \|x\|\|y\|$ for every $x, y \in R$,

3° $\|ax\| = |a|^p\|x\|$ for every $x \in R$ and every scalar a. It can be proved that $0 < p \leqslant 1$. If $p = 1$, then a p-normed algebra is a B-algebra.

A topological algebra R is called *m-convex* if:

1° the topology in R is introduced by means of a family ϱ_t, $t \in T$, of submultiplicative pseudometrics; the basis of neighbourhoods of an arbitrary element $x \in R$ is given by the family

$$V(x; \varepsilon; t_1, \ldots, t_k) = \{y \in R\colon \varrho_{t_i}(x, y) < \varepsilon,\ i = 1, 2, \ldots, k\},$$

where $\varepsilon > 0$, and $t_1, \ldots, t_k$ is an arbitrary finite subset of T;

2° the addition and scalar multiplication is continuous with respect to each pseudometric ϱ_t;

3° for each pseudometric ϱ_t, there is a pair of elements $x, y \in R$ such that $\varrho_t(x, y) \neq 0$.

The m-convex algebras are a generalization of locally convex m-convex topological algebras introduced by Michael [14].

The radical $\operatorname{rad} R$ of a topological algebra R is the set of such elements $x \in R$ that for every $y \in R$ there exists an inverse $(e+xy)^{-1} \in R$.

PART I. LOCALLY BOUNDED TOPOLOGICAL ALGEBRAS

§ 1. Some theorems on the existence of p-norms on locally bounded topological algebras

THEOREM 1 (3). *Let R be a complete metric algebra. The following statements are then equivalent:*

(a) *there exists in R an equivalent metric which is submultiplicative;*

(b) *R is a locally bounded algebra;*

(c) *R is a p-normed algebra.*

Proof. Let ϱ be a metric function defined in R such that $\varrho(xy, 0) \leqslant \varrho(x, 0)\varrho(y, 0)$. We shall show that the unit ball

$$K = \{x\colon \varrho(x, 0) < 1\}$$

is a bounded set. Indeed, let (a_n) be an arbitrary sequence of scalars tending to zero, and $x_n \in K$. We have to show that the sequence $(a_n x_n)$ also

(3) This theorem is due to A. Pełczyński.

tends to zero. This immediately follows from the inequality

$$\varrho(a_n x_n, 0) = \varrho(a_n e x_n, 0) \leqslant \varrho(a_n e, 0)\varrho(x_n, 0) \leqslant \varrho(a_n e, 0).$$

Consequently K is a bounded neighbourhood in R, and thus (a) implies (b).

Suppose now that R is locally bounded. By the theorem of Rolewicz [18] there exists an equivalent p-homogeneous norm $\|x\|_1$ in R. If we set

$$\|x\| = \sup_{y\neq 0} \frac{\|xy\|_1}{\|y\|_1}, \tag{1.1}$$

then by the same arguments as in Gelfand's proof (see [8], Satz 1) it may be shown that the norm $\|x\|$ satisfies 1°-3° of the definition of p-normed algebras, and that it is equivalent to the original norm $\|x\|_1$. Hence (b) implies (c).

That (c) implies (a) is evident, q. e. d.

Remark. The assumption that R is a complete algebra is essential. Later on, we shall give an example of a B^*-algebra for which there exists no equivalent submultiplicative norm. As it has been observed by C. Ryll-Nardzewski, the existence of a unit element in R is also essential. In fact, taking any F-space R which is not locally bounded and setting $xy = 0$ for every $x, y \in R$, we get a metric algebra which satisfies (a) but does not satisfy (b).

THEOREM 2. *If R is a metric algebra with a submultiplicative metric ϱ, then it is isomorphic and homeomorphic with a dense subalgebra of a p-normed algebra.*

Proof. We may assume that R is an F^*-algebra with the norm $\|x\|$. It can easily be verified that R is locally bounded, so that we have only to show that the multiplication in R can be extended continuously to the multiplication in $\bar{R}$ where $\bar{R}$ is the completion of R in the norm $\|x\|$. It will be sufficient if we show that whenever (x_n) and (y_n) are Cauchy sequences in the norm $\|x\|$, then so is the sequence $(x_n y_n)$.

Let (k_n) and (l_n) be two increasing sequences of natural numbers. It follows that

$$\|x_{k_n} - x_{l_n}\| \to 0, \text{ or, which is equivalent, } \varrho(x_{k_n} - x_{l_n}, 0) \to 0, \tag{1.2}$$

as well as

$$\|y_{k_n} - y_{l_n}\| \to 0, \quad \text{or} \quad \varrho(y_{k_n} - y_{l_n}, 0) \to 0. \tag{1.3}$$

Observe now that there exists such an integer n_0 that $\varrho(x_n - x_{n_0}, 0) < M$, and $\varrho(y_n - y_{n_0}, 0) < M$, for a certain constant M; otherwise there would exist such sequences (k_n) and (l_n) of integers that (1.2) and (1.3) would not hold. It is to be shown that $\|x_{k_n} y_{k_n} - x_{l_n} y_{l_n}\| \to 0$. We have

$$\|x_{k_n} y_{k_n} - x_{l_n} y_{l_n}\| \leqslant \|x_{k_n}(y_{k_n} - y_{l_n})\| + \|(x_{k_n} - x_{l_n}) y_{l_n}\|,$$

and hence it is sufficient to prove that the first term of the right-hand member of this inequality tends to zero as $n \to \infty$. We have

$$\|x_{k_n}(y_{k_n} - y_{l_n})\| \leqslant \|(x_{k_n} - x_{n_0})(y_{k_n} - y_{l_n})\| + \|x_{n_0}(y_{k_n} - y_{l_n})\|, \tag{1.4}$$

and since the norm $\|x - y\|$ is equivalent to the metric $\varrho(x, y)$, it follows in virtue of (1.3) that

$$\varrho[(x_{k_n} - x_{n_0})(y_{k_n} - y_{l_n}), 0] \leqslant M\varrho(y_{k_n} - y_{l_n}, 0) \to 0.$$

Consequently the first term of the right-hand member of (1.4) tends to zero; likewise, it can be seen that so does the second term, and thus $(x_n y_n)$ is a Cauchy sequence in the metric $\|x - y\|$, q. e. d.

THEOREM 3. *If R is a locally bounded metric algebra, then it may be algebraically imbedded into a p-normed algebra R.*

Proof. Let $\|x\|_1$ be a p-homogeneous norm which gives the topology in R. The norm $\|x\|$ defined by (1.1) is a submultiplicative norm. The completion $\bar{R}$ of R in the norm $\|x\|$ is the desired p-normed algebra, q. e. d.

Remark. If R is not complete, then the norm $\|x\|$ is not necessarily equivalent to $\|x\|_1$. And here is an example of a B^*-algebra for which there exists no equivalent submultiplicative norm: Let R be the space of two-sided complex sequences (a_n), $n = \pm 1, \pm 2, \ldots$, such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, provided with the norm $\|(a_n)\| = \left(\sum_{n=-\infty}^{\infty} |a_n|^2\right)^{1/2}$, where the operation of multiplication is defined as convolution. If there existed an equivalent submultiplicative norm, then, by theorem 2, the completion of R, which is l_2, would be a B-algebra with the convolution as multiplication, which is known to be impossible (see e. g. [23]).

The continuity of multiplication in R follows from the well-known inequality

$$\|(c_n)\| \leqslant \|(a_n)\| \cdot \left(\sum_{n=-\infty}^{\infty} |b_n|\right), \quad \text{where} \quad c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k}.$$

§ 2. Examples of p-normed algebras

2.1. The algebra l_p, $0 < p \leqslant 1$, of all complex two-sided sequences $a = (a_n)$ for which

$$\|a\| = \sum_n |a_n|^p < \infty$$

with the multiplication defined as convolution.

2.2. The algebra W_p, $0 < p \leqslant 1$, of all holomorphic functions defined in the unit circle, $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, for which

$$\|\varphi\| = \sum_{n=0}^{\infty} |a_n|^p < \infty$$

with the ordinary multiplication.

2.3. The algebra $l_p^{\langle a_n \rangle}$, $0 < p \leqslant 1$, of all complex two-sided sequences $x = (x_n)$ such that

$$\|x\| = \sum_{n=-\infty}^{\infty} a_n |x_n|^p < \infty,$$

where (a_n) is a given sequence of positive reals such that $a_{n+k} \leqslant a_n a_k$, $a_0 = 1$. The multiplication is defined as convolution.

2.4. The algebra $W_p^{\langle a_n \rangle}$, $0 < p \leqslant 1$, of all holomorphic functions $\varphi z = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|\varphi\| = \sum_{n=0}^{\infty} a_n |a_n|^p < \infty,$$

where (a_n) is as in 2.3. The multiplication is the ordinary multiplication of holomorphic functions.

2.5. l_p-products of q-normed algebras. If R is a q-normed algebra, then the set R_p of all two-sided sequences $x = (x_n)$, where $x_n \in R$, such that

$$|||x||| = \sum_{n=-\infty}^{\infty} \|x_n\|^p < \infty$$

is a pq-normed algebra if multiplication is defined as convolution.

2.6. A Cartesian product of a finite number of p_i-normed algebras is also an algebra of this type.

2.7. The product of the complex plane and a space $L_p[0,1]$, where $0 < p \leqslant 1$, is a p-normed algebra if we put

$$\|(\alpha, f(x)\| = |\alpha|^p + \int_0^1 |f(x)|^p \, dx,$$

$$(\alpha, f(x)) \cdot (\beta, g(x)) = (\alpha\beta, \alpha g(x) + \beta f(x)).$$

§ 3. The spectral norm and the fundamental theorem on p-normed complex field

In this section we assume R to be a commutative p-normed algebra over the complex scalars.

LEMMA 1. *If $x \in R$, then either $\|x^n\| \to 0$, or for each $n = 1, 2, \ldots$ $\|x^n\| \geqslant 1$.*

Proof. We shall show that if there exists a natural number n_0 such that $\|x^{n_0}\| < 1$, then $\|x^n\| \to 0$. Let $\|x^{n_0}\| < 1$; it follows then that $\|x^{n_0 k}\| \to 0$ as $k \to \infty$. We put $M = \max_{1 \leqslant n \leqslant n_0} \|x^n\|$, and for a given $\varepsilon > 0$ we choose such a K that, for $k \geqslant K$, $\|x^{n_0 k}\| < \varepsilon/M$. Setting $N = n_0 K$ we find that, for $n \geqslant N$, $n = n_0 k + m$, where $k \geqslant K$ and $m < n_0$ and,

$$\|x^n\| = \|x^{n_0 k + m}\| \leqslant \|x^{n_0 k}\| \|x^m\| \leqslant \frac{\varepsilon}{M} \cdot M = \varepsilon, \qquad \text{q. e. d.}$$

Definition 1. The real function $\|x\|_s$ defined on a p-normed algebra R by

$$\|x\|_s = \left(\sup\{|\alpha| : \lim_n \|(\alpha x)^n\| = 0\}\right)^{-p} \text{ (4)}$$

will be called the *spectral norm* of R.

To justify this terminology we shall observe later on that

$$\|x\|_s = \lim_n \sqrt[n]{\|x^n\|},$$

and in the case where R is a B-algebra this definition coincides with the usual definition of spectral norm for B-algebras.

THEOREM 4. *The spectral norm $\|x\|_s$ has the following important properties:*

(S1) $\|x\|_s \geqslant 0$, *and* $\|x\|_s < 1$ *if and only if* $\|x^n\| \to 0$, *or if there exists an* n_0 *such that* $\|x^{n_0}\| < 1$,

(S2) $\|x+y\|_s \leqslant \|x\|_s + \|y\|_s$,

(S3) $\|xy\|_s \leqslant \|x\|_s \|y\|_s$,

(S4) $\|\alpha x\|_s = |\alpha|^p \|x\|_s$, *where α is an arbitrary complex number,*

(S5) $\|x^n\|_s = \|x\|_s^n$, *for* $n = 1, 2, \ldots$,

(S6) $\|x\|_s \leqslant \|x\|$,

(S7) $\|x\|_s$ *is continuous in the original norm* $\|x\|$,

(S8) *if* $x^{-1} \in R$, *then* $\|x\|_s \neq 0$.

Proof. The properties (S1), (S4), (S5), (S6) are obvious. We are going to prove the remaining ones.

(4) If the supremum equals ∞, we put $\|x\|_s = 0$.

Ad (S2). In virtue of (S4), $\|x\|_s$ is p-homogeneous and it is sufficient to show that if $\|x\|_s+\|y\|_s<1$, then $\|x+y\|_s<1$. Let $\|x\|_s=\alpha$, and $\|y\|_s=\beta$. We have $\alpha+\beta<1$. We then choose $\bar\alpha, \bar\beta$ such that $\bar\alpha>\alpha$, $\bar\beta>\beta$, and $\bar\alpha+\bar\beta<1$ and we put $x=\bar\alpha^{1/p}\bar x$, $y=\bar\beta^{1/p}\bar y$. The $\bar x, \bar y$ thus defined have their spectral norms less than 1:

$$\|\bar x\|_s<1, \qquad \|\bar y\|_s<1.$$

By virtue of (S1) we have only to find an N such that

$$\|(x+y)^N\|<1.$$

For this purpose we observe that

$$\|(x+y)^n\| = \|(\bar\alpha^{1/p}\bar x+\bar\beta^{1/p}\bar y)^n\| = \Big\|\sum_{k=0}^{n}\binom{n}{k}\bar x^{n-k}\bar y^k\bar\alpha^{(n-k)/p}\bar\beta^{k/p}\Big\|$$
$$\leqslant \sum_{k=0}^{n}\binom{n}{k}^p\bar\alpha^{n-k}\bar\beta^k\|\bar x^{n-k}\bar y^k\|.$$

Since it follows from (S1) that $\|\bar x^n\|\to 0$, and $\|\bar y^n\|\to 0$, we have

$$A=\max_n\|\bar x^n\|<\infty \quad\text{and}\quad B=\max_n\|\bar y^n\|<\infty,$$

and can therefore choose an m_0 and an n_0 such that

$$\|\bar x^n\|<\frac{1}{B} \quad\text{for}\quad n>m_0,$$

$$\|\bar y^n\|<\frac{1}{A} \quad\text{for}\quad n>n_0.$$

It can easily be verified that, for $N=m_0+n_0$,

$$\|\bar x^{N-k}\bar y^k\|<1 \quad\text{for}\quad k=0,1,2,\ldots,N.$$

Consequently, since, for $0<p\leqslant 1$, $\binom{N}{k}\geqslant\binom{N}{k}^p$, it follows that

$$\|(x+y)^N\|\leqslant\sum_{k=0}^{N}\binom{N}{k}^p\bar\alpha^{n-k}\bar\beta^k\leqslant(\bar\alpha+\bar\beta)^N<1.$$

Ad (S3). Because of the p-homogeneity of $\|x\|_s$, it is sufficient to show that

$$\text{if}\quad \|x\|_s<1 \quad\text{and}\quad \|y\|_s<1, \quad\text{then}\quad \|xy\|_s<1.$$

This, in turn, follows immediately from (S1) and the fact that $\|x^n\|\to 0$ and $\|y^n\|\to 0$ imply $\|xy\|^n\to 0$.

Ad (S7). This immediately follows from (S6) and (S2).

Ad (S8). If $\|x\|_s=0$ and x had an inverse in R, then

$$\|e\|_s\leqslant\|x\|_s\|x^{-1}\|_s=0$$

and in virtue of (S1) we should have $\|e^n\|\to 0$, which is a contradiction, q. e. d.

The theorem that follows states that the definition of the spectral norm $\|x\|_s$ coincides with the ordinary definition in the case where R is a B-algebra.

THEOREM 5. *For every $x\in R$ the limit $\lim\sqrt[n]{\|x^n\|}$ exists and it is equal to the spectral norm*

$$\|x\|_s=\lim\sqrt[n]{\|x^n\|}.$$

Proof. Let

$$\|x\|^*=\limsup\sqrt[n]{\|x^n\|}.$$

It is evident then that $\|x\|^*\leqslant\|x\|$ and $\|\alpha x\|^*=|\alpha|^p\|x\|^*$. We shall show that $\|x\|_s\geqslant\|x\|^*$. Indeed, if $\|x\|_s<\|x\|^*$, then there can be found an α such that $\|\alpha x\|_s<1<\|\alpha x\|^*$. Hence, by (S1), $\|(\alpha x)^n\|\to 0$, and for sufficiently large n, $\|(\alpha x)^n\|<1$, and consequently $\|\alpha x\|^*<1$, which is a contradiction.

We now define

$$\|x\|_*=\liminf\sqrt[n]{\|x^n\|}.$$

To complete the proof we have only to show that $\|x\|_s\leqslant\|x\|_*$; or, in virtue of the p-homogeneity of $\|x\|_s$ and $\|x\|_*$, it will be sufficient to show that if $\|x\|_*<1$, then $\|x\|_s<1$. But if $\|x\|_*<1$, then there exists an n_0, such that $\sqrt[n_0]{\|x^{n_0}\|}<1$. Hence, by (S1), $\|x\|_s<1$, q. e. d.

Now we pass to the proof of the theorem on p-normed fields.

LEMMA 2. *The set V of all elements invertible in R is an open set.*

LEMMA 3. *The operation of taking the inverse is continuous on V.*

The proofs are the same as in [8] (Hilfssätze 1 und 2, see also [17]).

LEMMA 4. *If R is a field and R_0 is a complete subalgebra of R, then R_0 is also a field.*

Proof. Let $x\in R_0$, $x\neq 0$. We shall show that $x^{-1}\in R_0$. Observe that if $\|x_n-x_0\|\to 0$, and $x_n^{-1}\in R_0$ for $n=1,2,\ldots$, then $x_0^{-1}\in R_0$. Indeed, by lemma 3, x_n^{-1} tends to x_0^{-1} in the norm $\|x\|$, whence by the completeness of R_0, $x_0^{-1}\in R_0$.

If $x=\alpha e$, where α is a scalar $\neq 0$, then obviously $x^{-1}\in R_0$. Assume now that x is not of the form $x=\alpha e$. Hence, for every complex number λ,

$(\lambda x-e)^{-1}$ exists and $(\lambda x-e)^{-1}\epsilon R$. By lemma 3, the function $\varphi(\lambda) = (\lambda x-e)^{-1}$ is a continuous mapping of the complex plane into R. If we put

$$\Lambda = \{\lambda: (\lambda x-e)^{-1}\epsilon R_0\},$$

then Λ is a closed subset of the complex plane. On the other hand, by lemma 2 and the continuity of $\varphi(\lambda)$ it is an open set. It is non-void for $0\epsilon\Lambda$, and hence it is the whole complex plane. We have now

$$\left(x-\frac{1}{n}e\right)^{-1}\epsilon R_0 \quad \text{for} \quad n = 1, 2, \ldots,$$

and since $x-\frac{1}{n}e \to x$, $x^{-1}\epsilon R_0$, which completes the proof, q. e. d.

LEMMA 5. *If R is a field, then for every complex $\alpha \neq 0$, for every $\varepsilon > 0$, and for every $x\epsilon R$, which is not of the form $x = \lambda e$, there exists a polynomial $W(x)$ with complex coefficients such that*

$$\|\alpha e-[e+xW(x)]^{-1}\| < \varepsilon. \tag{3.1}$$

Proof. Consider the subalgebra $R(x)$ generated by the element x. Namely $R(x)$ is the completion in the norm $\|x\|$ of the algebra of all polynomials in x with complex coefficients. By lemma 4, $R(x)$ is a field, whence $x^{-1}\epsilon R(x)$. Then there exists a sequence of polynomials $W_n(x)$ such that $W_n(x)$ tends to $(\alpha^{-1}-1)x^{-1}$ in the norm $\|x\|$. Hence, by lemma 3, the sequence $[e+xW_n(x)]^{-1}$ tends to αe in the norm $\|x\|$, and (3.1) holds for sufficiently large n. That $e+xW_n(x)$ has an inverse can be deduced from the fact that $x \neq \lambda e$ and that there are no divisors of zero in $R(x)$, q. e. d.

THEOREM 6. *If R is a field, then it is isomorphic and homeomorphic to the field of complex numbers.*

Proof. Let $x\epsilon R$. It is to be shown that $x = \alpha e$ for a complex α. We then assume that $x \neq \alpha e$ for every complex α, and we get a contradiction.

We set

$$f(\alpha) = \|(x^{-1}+\alpha e)^{-1}\|_s,$$

which is a positive and continuous function defined for every complex α. Moreover, $f(\alpha)$ can be written in the form

$$f(\alpha) = |\alpha|^{-p}\left\|\left(\frac{x^{-1}}{\alpha}+e\right)^{-1}\right\|_s$$

Hence, by lemma 3, and (S6) we have $\lim\limits_{|\alpha|=\infty} f(\alpha) = 0$. Consequently there exists an α_0 such that $f(\alpha_0) \geqslant f(\alpha)$ for every complex α. Setting

$$y = (x^{-1}+\alpha_0 e)^{-1}[f(\alpha_0)]^{-1/p}$$

we have

$$\|y\|_s = 1, \tag{3.2}$$

and

$$\|(y^{-1}+\alpha e)^{-1}\|_s \leqslant 1 \text{ for every complex } \alpha. \tag{3.3}$$

We are going to show that there exists such n_0 that $\|y^{n_0}\|_s < 1$, which, by (S5) and (3.2), would be a contradiction completing the proof.

Let $V_n(z)$ be an arbitrary polynomial of the form

$$V_n(z) = z^n+\alpha_1 z^{n-1}+\ldots+\alpha_n.$$

In virtue of the fundamental theorem of algebra, we have

$$V_n(y^{-1}) = (y^{-1}-\beta_1 e)\ldots(y^{-1}-\beta_n e);$$

thus it follows from (3.3) that

$$\|[V_n(y^{-1})]^{-1}\|_s \leqslant \|(y^{-1}-\beta_1 e)^{-1}\|_s\ldots\|(y^{-1}-\beta_n e)^{-1}\|_s \leqslant 1. \tag{3.4}$$

On the other hand,

$$[V_n(y^{-1})]^{-1} = y^n[e+yW_n(y)]^{-1}, \tag{3.5}$$

where

$$W_n(y) = \alpha_1+\alpha_2 y+\ldots+\alpha_n y^{n-1}. \tag{3.6}$$

For a given $\varepsilon > 0$ and an arbitrary α, we can now choose a polynomial V_{n_0} such that for $W_{n_0}(y)$ defined by (3.6) the relation (3.1) holds. Setting

$$z = \alpha e-[e+yW_{n_0}(y)]^{-1},$$

we have

$$\|z\|_s \leqslant \|z\| < \varepsilon,$$

and it follows from (3.4) and (3.1) that

$$\|y^{n_0}(\alpha e-z)\|_s \leqslant 1.$$

Consequently

$$\begin{aligned} 1 \geqslant \|\alpha y^{n_0}-zy^{n_0}\|_s &\geqslant \|\alpha y^{n_0}\|_s-\|zy^{n_0}\|_s \\ &\geqslant |\alpha|^p\|y^{n_0}\|_s-\|z\|_s\|y^{n_0}\|_s \geqslant (|\alpha|^p-\varepsilon)\|y^{n_0}\|_s. \end{aligned}$$

Hence

$$\|y^{n_0}\|_s \leqslant \frac{1}{|\alpha|^p-\varepsilon},$$

and by a suitable choice of α and ε, we can make $\|y^{n_0}\|_s < 1$, which is the announced contradiction.

Some generalizations of theorem 6 will be given in § 5.

§ 4. The Gelfand theory for the commutative p-normed algebras

In this section also it is assumed that R is a commutative complex p-normed algebra.

THEOREM 7. *Every ideal of the algebra R is contained in a maximal ideal. Every maximal ideal M of the algebra R is a closed set, and for each maximal ideal there exists a multiplicative linear functional f_M such that*

$$M = \{x \in R: f_M(x) = 0\}.$$

COROLLARY 7a. *There exists in R at least one non-zero multiplicative linear functional.*

COROLLARY 7b. *Every multiplicative linear functional defined on R is continuous.*

COROLLARY 7c. *If, for every multiplicative linear functional f, $f(x) \neq 0$, then x is an invertible element of R.*

THEOREM 8. *Let $\mathfrak{M}$ be the bicompact space of all maximal ideals of R* (5). *Then there exists an algebraical homomorphism h of R into the algebra $C(\mathfrak{M})$ of all continuous functions on the space $\mathfrak{M}$, given by the formula*

$$h(x) = \varphi_x(M) = f_M(x),$$

where $x \in R$, $\varphi_x \in C(\mathfrak{M})$, $M \in \mathfrak{M}$. Moreover, $\max_{M \in \mathfrak{M}} |\varphi_x(M)|^p \leqslant \|x\|$, and $\varphi_e(M) \equiv 1$. The kernel of the homomorphism h is the radical of R.

COROLLARY 8a. *If the algebra R is semisimple, then h is an algebraical isomorphism.*

The proofs are the same as in [8]; indeed, they are based upon the Gelfand-Mazur theorem and some lemmas of the same kind as in the previous section.

Now we pass to the properties of the radical of R.

THEOREM 9. *If $x \in R$, then*

$$\max_{M \in \mathfrak{M}} |f(x)|^p \leqslant \|x\|_s, \tag{4.1}$$

where $\mathfrak{M}$ is the set of all multiplicative linear functionals in R.

Proof. If there existed $f \in \mathfrak{M}$, and $x \in R$, such that $f(x) = 1$, and $\|x\|_s < 1$, then x^n would tend to zero as $n \to \infty$, and $f(x^n) = 1$, which contradicts corollary 7b, q. e. d.

THEOREM 10. *If $x \in R$, and $\|x\|_s = 0$, then $x \in \operatorname{rad} R$.*

Proof. Let $x \in R$ be such an element that $\|x\|_s = 0$. By (4.1), $f(x) = 0$ for every $f \in \mathfrak{M}$. On the other hand, for every $f \in \mathfrak{M}$ and every $z \in R$, $(e + xz) = 1$, whence, by corollary 7c, we have $(e + xz)^{-1} \in R$, i. e. $x \in \operatorname{rad} R$, q. e. d.

(5) The topology in $\mathfrak{M}$ is introduced in the same way as in [8].

Remark. If R is a B-algebra, then $\|x\|_s = 0$ is also a necessary condition for $x \in \operatorname{rad} R$. For the p-normed algebras this is an open problem. We shall pose the following

PROBLEM 1. *Is it true that, for a commutative p-normed algebra R,*

$$\operatorname{rad} R = \{x \in R: \|x\|_s = 0\}? \text{ (6)}$$

§ 5. Locally bounded division algebras

THEOREM 11. *Any locally bounded complex division algebra R is isomorphic and homeomorphic with the field of complex numbers.*

Proof. Let $x \in R$. It is to be shown that $x = ae$ for any complex a. Suppose that this does not hold. Hence every non-zero polynomial in x is invertible in R. By theorem 3, the locally bounded field of all rational functions of the element x may be imbedded in a p-normed commutative algebra R_1. By corollary 7a there exists on R_1 at least one non-zero multiplicative linear functional f. Consequently there exists a non-zero multiplicative linear functional on the algebra of all rational functions of x, which is impossible. We thus get a contradiction of the assumption that $x \neq ae$, q. e. d.

THEOREM 12. (A generalization of the first theorem of Mazur [11]). *If R is a locally bounded topological algebra, then R is either the field of reals, or the field of complex numbers, or the division algebra of the real quaternions.*

Proof. Here we may apply without any changes the proof given by Arens ([2], theorem 2). Indeed, in that proof local convexity is essential only in the complex case. On its basis, the real case is treated in a purely algebraic way.

THEOREM 13. (A generalization of a theorem of Edwards [7]). *If R is any F^*-algebra with the submultiplicative norm $\|x\|$ such that*

$$\|x^{-1}\| = \|x\|^{-1} \tag{5.1}$$

for every x invertible in R, then R is either the field of reals, or the field of complex numbers, or the division algebra of the real quaternions.

Proof. By the same method as in [7] it may be shown that R is a division algebra. It remains only to apply theorems 2 and 12.

THEOREM 14. (A generalization of the second theorem of Mazur [11]; see also Aurora [3] and Ostrowski [16]). *If R is an F^*-algebra whose norm satisfies*

$$\|xy\| = \|x\|\,\|y\|$$

(6) The answer is positive. The proof will be published in the paper [24] (*added in proof*).

for every $x, y \in R$, then the conclusion of the previous theorem holds also for R.

This is the well-known (see [3]) corollary to theorem 13.

Later it will be shown that every m-convex topological division algebra satisfies the conclusion of Mazur's theorem.

On the other hand, there exist some classes of topological algebras for which the problem is open. For the sake of brevity we shall say that a *class of topological algebras possess the property* M if every division algebra of that class satisfies the conclusion of the Mazur theorem.

The following problems arise:

PROBLEM 2. *Does the class of B_0-algebras possess the property* M? (7)

The problem is solved by Arens in the case of separable algebras [2].

PROBLEM 3. *Does the class of B_0^*-algebras possess the property* M? (8)

Of course, the answer is unknown in the case of non-m-convex B_0-algebras or B_0^*-algebras.

PROBLEM 4. *Does the class of locally convex topological algebras possess the property* M?

The algebra R of all rational functions of the real variable is an F^*-field if we put

$$\|w\| = \int \frac{|w(t)|}{1+|w(t)|}\,dt \quad \text{for} \quad w \in R.$$

Hence the answer to an analogous question for F^*-algebras is negative. The following problem is open:

PROBLEM 5. *Does the class of F-algebras possess the property* M?

§ 6. Linear multiplicative functionals in concrete p-normed algebras. The Wiener theorem for the class l_p, $0 < p \leqslant 1$

THEOREM 15. *Every multiplicative linear functional f defined on $W_p^{\langle a_n\rangle}$* (see example 2.4) *is of the form*

$$f_q(x) = \sum_{n=0}^{\infty} x_n q^n, \tag{6.1}$$

where $x = \sum_{n=0}^{\infty} x_n z^n \in W_p^{\langle a_n\rangle}$, and $|q| \leqslant r$, $r = \lim_n \sqrt[n]{a_n}$.

(7) The answer is positive. The proof will appear in [25] *(added in proof)*.

(8) The answer is negative; a counterexample, constructed by Williamson was published in [20]. This is also, of course, a negative answer to the problem 4 *(added in proof)*.

Proof. It may easily be verified that if $|q| \leqslant r$, then the series $\sum_{n=0}^{\infty} x_n q^n$ is convergent for every $x \in W_p^{\langle a_n\rangle}$, and that for every q, $|q| > r$, there exists an $x \in W_p$ such that the series is divergent. Hence only the multiplicative linear functionals defined on the subalgebra of all polynomials in z and continuous in $W_p^{\langle a_n\rangle}$, are functionals (6.1), and only they may be extended to $W_p^{\langle a_n\rangle}$, q. e. d.

COROLLARY 15a. *Every multiplicative linear functional defined on W_p is of the form*

$$f(\varphi) = \varphi(z_0), \quad \textit{where} \quad |z_0| \leqslant 1.$$

THEOREM 16. *Every multiplicative linear functional f defined on $l_p^{\langle a_n\rangle}$ is of the form*

$$f_q(x) = \sum_{n=-\infty}^{\infty} x_n q^n, \tag{6.2}$$

where $x = (x_n) \in l_p^{\langle a_n\rangle}$, $1/r \leqslant |q| \leqslant r'$, $r = \lim_n \sqrt[n]{a_{-n}}$, $r' = \lim_n \sqrt[n]{a_n}$.

Proof. Let f be any multiplicative linear functional defined on $l_p^{\langle a_n\rangle}$. The functional f restricted to the subalgebra $\{x \in l_p^{\langle a_n\rangle} : x_n = 0 \text{ for } n < 0\}$ must be of the form (6.1), where $|q| \leqslant r'$. But $(\delta_{-1}^n) = (\delta_1^n)^{-1}$ (δ_j^i is the Kronecker symbol; it is 1 for $i = j$, and 0 otherwise), and $f[(\delta_1^n)] = q$. Hence $f[(\delta_{-1}^n)] = q^{-1}$. The desired conclusion follows from the fact that the series $\sum_n x_{-n} q^{-n}$ converges for every $x \in l_p^{\langle a_n\rangle}$ only if $|q^{-1}| \leqslant r$, q. e. d.

COROLLARY 16a. *In the algebra l_p every multiplicative linear functional f is of the form*

$$f_t(x) = \sum_{n=-\infty}^{\infty} x_n e^{int}, \quad 0 \leqslant t < 2\pi. \tag{6.3}$$

THEOREM 17. *Let $\varphi(z) = \sum a_n z^n$ be a holomorphic function for $|z| < r$, and let $\sum a_n |a_n|^p < \infty$, $0 < p \leqslant 1$, $a_0 = 1$, $a_n > 0$, $a_{n+k} \leqslant a_n \cdot a_k$, $r = \lim \sqrt[n]{a_n}$. If $\varphi(z) \neq 0$ for $|z| \leqslant r$, then $1/\varphi(z) = \sum b_n z^n$, where $\sum a_n |b_n|^p < \infty$.*

Proof. This theorem follows from theorem 15, and corollary 7c.

THEOREM 18. (A generalization of Wiener's theorem). *Let $(a_n)_{-\infty}^{\infty}$ be a sequence of positive numbers such that $a_0 = 1$, $a_{n+k} \leqslant a_n a_k$, and $\lim \sqrt[n]{a_n} = \lim \sqrt[n]{a_{-n}} = 1$. Let $x(t)$ be a complex function of a real variable, $0 \leqslant t < 2\pi$, equal to its Fourier expansion*

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}.$$

Then, if $\sum a_n |x_n|^p < \infty$ *for* $0 < p \leqslant 1$, *and* $x(t) \neq 0$ *for* $0 \leqslant t < 2\pi$, *then the function* $1/x(t)$ *may be written in the form*

$$1/x(t) = \sum_{n=-\infty}^{\infty} y_n e^{int}, \quad 0 \leqslant t < 2\pi,$$

where $\sum a_n |y_n|^p < \infty$.

This theorem immediately follows from theorem 16, and corollary 7c.

As further applications of the previous theory, we give

THEOREM 18. *Let X be a locally bounded F-space. If there exists a multiplication on X which makes X a semisimple commutative topological algebra, then there exists in X a total family of linear functionals.*

Proof. This family is the family of all multiplicative linear functionals.

THEOREM 19. *If a p-normed commutative algebra R contains a subspace isomorphic with $L_p[0,1]$, then that subspace must be a subset of* rad R.

Proof. The conclusion follows from the fact that there are no continuous linear functionals in $L[0,1]$.

PART II. m-CONVEX TOPOLOGICAL ALGEBRAS

§ 7. Examples

(7.1) The algebra l^{q+}, $0 \leqslant q < 1$. Let l^{q+} be the set of all two-sided complex sequences $x = (x_n)$ such that

$$\|x\|_p = \sum_n |x_n|^p \quad \text{for every } p \text{ satisfying } q < p \leqslant 1.$$

l^{q+} is an m-convex algebra with the family of metrics $\varrho_p = \|x-y\|_p$, $q < p \leqslant 1$, and with the multiplication defined as convolution.

(7.2) Every subdirect product of any family of subalgebras dense in p-normed algebras is an m-convex algebra.

It will be shown below that every m-convex algebra is such a subdirect product.

§ 8. The representation theorem for m-convex topological algebras

THEOREM 20. *Let R be an m-convex topological algebra with the family ϱ_t, $t \in T$, of pseudometrics. If we put*

$$I_t = \{x \in R\colon \varrho_t(x, 0) = 0\},$$

then I is a closed two-sided ideal in R.

Proof. It is to be shown that:

(a) if $x, y \in I_t$, then $x+y \in I_t$;
(b) if $x \in I_t$, $y \in R$, then $xy \in I_t$, and $yx \in I_t$;
(c) $I_t \neq R$;
(d) I_t is a closed subset of R.

(a) From the continuity of addition it follows that for given $\varepsilon > 0$ there exists a δ such that if $\varrho_t(x, 0) < \delta$, $\varrho_t(y, 0) < \delta$, then $\varrho_t(x+y, 0) < \varepsilon$. It follows, that, for every $\varepsilon > 0$, $\varrho_t(x+y, 0) < \varepsilon$, whence $x+y \in R$.

(b) If $x \in I_t$, then for every $y \in \Omega$ $\varrho_t(xy, 0) \leqslant \varrho_t(x, 0)\varrho_t(y, 0) = 0$, whence $xy \in I_t$. Similarly $yx \in I_t$.

(c) If $e \in I_t$, then, for every $x \in R$, $\varrho_t(x, 0) \leqslant \varrho_t(e, 0)\varrho_t(x, 0) = 0$, and $\varrho_t(x, y) \leqslant \varrho_t(x, 0) + \varrho_t(y, 0) = 0$ for every $x, y \in R$, which contradicts the definition of an m-convex algebra.

(d) I is closed in R because the function $\varrho_t(x, 0)$ is continuous as a function of x, q. e. d.

Definition 2. Let R be an m-convex topological algebra. We define $R_t = R/I_t$. R_t is a metric algebra with the distance function ϱ_t. From theorem 2, and 21 follows

COROLLARY 20a. *The algebra R_t is locally bounded, and there exists an equivalent p_t-homogenous norm $\|x\|_t$.*

COROLLARY 20b. *If R is a m-convex topological algebra, then the topology in R may be introduced by the family of submultiplicative p_t-homogeneous pseudonorms $\|x\|_t$, $t \in T$.*

In the sequel we assume that the topologies in m-convex algebras are given by such families.

We have thus obtained

THEOREM 21. *The topological algebra R is m-convex if and only if it is isomorphic and homeomorphic with the subdirect product of topological algebras dense in p-normed algebras.*

THEOREM 22. *The family of m-convex algebras possesses the property M.*

Proof. Let R be an m-convex division algebra. Let $t_0 \in T$. The pseudonorm $\|x\|_{t_0}$ is a norm in R; otherwise there would exist a non-zero ideal. The conclusion follows from observing that R provided with the norm $\|x\|_{t_0}$ satisfies the assumptions of theorem 12, q. e. d.

§ 9. Multiplicative linear functionals in m-convex topological algebras

As was shown before, the m-convex topological algebras are subdirect products of some algebras R_t. The following theorem holds for the subdirect product of any family of topological algebras R_t, $t \in T$, i. e. for such subalgebras of Cartesian product $\prod_{t \in T} R_t$ that, for every $t \in T$ and every

$x_t \in R_t$, there exists such an $x \in R$ that $x_{|t} = x_t$ (by $A_{|t}$ we denote the projection of the set $A \subset R$ on the axis R_t; thus $x_{|t}$ is the projection of element x on R_t).

Definition 3. If f is a multiplicative linear functional defined on the subdirect product R of algebras R_t, $t \in T$, then it is called an *axial functional* if there exist such a $t_0 \in T$ and such a multiplicative linear functional f_{t_0} defined on R_{t_0} that

$$f(x) = f_{t_0}(x_{|t_0}) \quad \text{for every} \quad x \in R. \tag{9.1}$$

It is known that if R is the Cartesian product of the algebras R_t, and if the power of the set T is less than the first aleph of measure different from zero, then every multiplicative linear functional defined on R is of the form (9.1), (see [5]). For the subdirect products we prove

THEOREM 23. *If R is the subdirect product of the family R_t, $t \in T$, of topological algebras, then every continuous multiplicative linear functional defined on R is of the form* (9.1).

Proof. Put $M_f = \{x \in R: f(x) = 0\}$, where f is a multiplicative linear functional defined on R. Let $t \in T$; then, as may easily be verified, either $M_{f|t} = R_t$, or there exists such a multiplicative linear functional f_t defined on R_t that (9.1) holds. It is to be shown that if $M_{f|t} = R_t$ for every $t \in T$, then the functional f cannot be continuous. It is enough to show that in this case, for every neighbourhood V of e of R, there exists such a $z \in V$ that $f(z) = 0$. Let V be a neighbourhood of the form

$$V = \{x \in R: x_{|t_i} \in V_{t_i},\ i = 1, 2, \ldots, k\}, \tag{9.2}$$

where V_{t_i} is any neighbourhood of the unit element of R_{t_i}, $i = 1, 2, \ldots, k$. By our assumption, there exist such elements $z_1, \ldots, z_k$ that $z_{i|t_i} = e_{|t_i}$ and $f(z_i) = 0$ for $i = 1, 2, \ldots, k$. We put $y = (e-z_1)(e-z_2)\ldots(e-z_k)$, and $z = e-y$, and then observe that $f(z) = 0$, and $z_{|t_i} = e_{|t_i}$. The conclusion follows from the fact that the neighbourhoods (9.2) form the basis of neighbourhoods in R, q. e. d.

COROLLARY 23a. *If A is an arbitrary set, and R any algebra of complex functions defined on A containing all constants, then every multiplicative linear functional continuous in the topology of pointwise convergence is of the form*

$$f(x) = x(t_0), \quad \textit{where} \quad x \in R,\ t_0 \in A.$$

COROLLARY 23b. *Let A be a locally compact topological space, and R any algebra of continuous complex functions defined on A containing the constants. The topology in R is the topology of uniform convergence on every compact set. Then, for every continuous multiplicative linear functional f, and for every family G of open subsets covering A, there exists such a $G_0 \in G$ that if an arbitrary function $x(t)$ is zero in G_0, then $f(x) = 0$.*

Remark. If the multiplicative linear functional is continuous in R, then the corresponding functional f_{t_0} of (9.2) is not necessarily continuous in R_t. Here is an example of a subdirect product $R \subset R_1 \times R_2$, and a continuous functional f such that $f(x) = f_1(x_{|1}) = f_2(x_{|2})$ but no functional f_i is continuous in R_i, $i = 1, 2$. Let R be the algebra of all functions holomorphic for $|z| < 1$ and continuous for $|z| \leqslant 1$. The topology of uniform convergence may be given in R by the norms

$$\|x\|_1 = \sup_{0 \leqslant t \leqslant \pi} |x(e^{it})|, \qquad \|x\|_2 = \sup_{\pi \leqslant t \leqslant 2\pi} |x(e^{it})|.$$

Then R_i is R equipped with the norm $\|x\|_i$. The multiplicative linear functional $f(x) = x(0)$ is evidently continuous in R but is continuous neither in the norm $\|x\|_1$ nor in the norm $\|x\|_2$.

§ 10. An example of a non m-convex B_0-algebra

The m-convexity of locally bounded algebra is a natural consequence of its completeness. For the B_0-algebras it need not be so. The first example of a non-m-convex B_0-algebra was constructed by Arens [1]. Here we shall construct another example. It will be studied in this section. But first we prove the following

THEOREM 24. *If R is a B_0-algebra, then the topology in R may be introduced by means of a denumerable family of homogeneous pseudonorms $\|x\|_i$, $i = 1, 2, \ldots$, such that*

$$\|xy\|_i \leqslant \|x\|_{i+1} \|y\|_{i+1} \quad \textit{for} \quad x, y \in R. \tag{10.1}$$

Proof. Let the topology in R be introduced by the family $\|x\|_i$, $i = 1, 2, \ldots$ It may be assumed that $\|x\|_i \leqslant \|x\|_{i+1}$ for $i = 1, 2, \ldots$ Then if A is any continuous linear transformation defined on R with values from R, then (see [6]) for every natural i there exist such a natural j and a positive C that

$$\|Ax\|_i \leqslant C\|x\|_j \quad \text{for every} \quad x \in R. \tag{10.2}$$

It is evident that the set

$$A^{(i)}_{n,j} = \{x \in R: \|xy\|_i \leqslant n\|y\|_j \quad \text{for every } y \in R\}$$

is a closed subset of R. Hence, by (10.2), we have

$$R = \bigcup_{n,j} A^{(i)}_{n,j} \quad \text{for} \quad i = 1, 2, \ldots \tag{10.3}$$

Consequently, by means of the category method, it follows, that for every natural i there exist such a $j(i)$, an $n(i)$ and such a point $x_i \in R$ and a neighbourhood V of x_i that $V \subset A^{(i)}_{n(i),j(i)}$. This implies that such integer $k(i) \geqslant j(i)$ can be chosen that the set $\{x \in R: \|x - x_i\|_{k(i)} \leqslant \delta\}$

is a subset of V for a certain $\delta > 0$. Hence we have

$$\|xy\|_i \leqslant \|(x+x_i)y\|_i + \|x_i y\|_i \leqslant 2n(i)\|y\|_{j(i)} \leqslant 2n(i)\|y\|_{k(i)} \tag{10.4}$$

or every $y \in R$ and every x such that $\|x\|_{k(i)} \leqslant \delta$. Consequently

$$\|xy\|_i = \frac{\|y\|_{k(i)}}{\delta}\left\|x\frac{\delta y}{\|y\|_{k(i)}}\right\|_i \leqslant \frac{2n(i)}{\delta}\|x\|_{k(i)}\|y\|_{k(i)} \tag{10.5}$$

for every x and every y such that $\|y\|_{k(i)} > 0$, but if $\|y\|_{k(i)} = 0$, the inequality between the first and the last members of (10.5) is also true. Hence if we put

$$\|x\|_1' = \|x\|_1, \qquad \|x\|_{i+1}' = \left[\frac{2n(i)}{\delta}\right]^{1/2}\|x\|_{k(i)+1},$$

we get an equivalent family of pseudonorms for which (10.1) hold, q. e. d.

COROLLARY 24a. *In every B_0-algebra the multiplication is continuous in both variables simultaneously* (9).

Now we define the algebra l^{1+} as the set of all complex sequences $x = (x_n)$, $0 \leqslant n < \infty$, for which

$$\|x\|_k = \sqrt[1+1/p_k]{\sum_{n=0}^{\infty}|x_n|^{1+1/p_k}} < \infty, \tag{10.6}$$

where $p_1 = 1$, $p_{k+1} = 2p_k+1$, $k = 1, 2, \ldots$

THEOREM 25. *l^{1+} is a B_0-algebra if the multiplication is defined as convolution.*

Proof. l^{1+} is evidently a B_0-space. It is to be shown that the multiplication is defined and continuous on l^{1+}. This is a corollary of the following theorem of Young [21], p. 71. If $x = (x_n) \in l_p$, $y = (y_n) \in l_q$, $p, q \geqslant 1$, then their convolution $x*y$ is a member of l_r, where $1/p+1/q-1 = 1/r$, and $\|x*y\|_r \leqslant \|x\|_p\|y\|_q$, where $\|z\|_s = \left[\sum_n |z_n|^s\right]^{1/s}$. Thus if we put $p = q = 1+1/p_{k+1}$, we get $r = 1+1/p_k$ and the inequalities (10.1) hold for the norms (10.6), q. e. d.

Remark. The algebra l^{1+} may also be interpreted as the algebra of holomorphic functions in the unit circle $|\lambda| < 1$, $\varphi(\lambda) = \sum_{n=0}^{\infty}\alpha_n\lambda^n$, such that the sequence (α_n) satisfies (10.6) for $k = 1, 2, \ldots$ In the sequel we shall use this interpretation.

(9) Using the category method we can show that in every F-algebra the multiplication is continuous in the two variables simultaneously; that is a result obtained by Arens in [2], theorem 5.

THEOREM 26. *The algebra l^{1+} is not m-convex.*

Proof. Suppose, conversely, that it is m-convex. Then there would exist a sequence $\|x\|_n'$ of submultiplicative pseudonorms equivalent to the norms (10.6). It may be assumed that $\|x\|_n' \leqslant \|x\|_{n+1}'$, otherwise we should take the system of submultiplicative pseudonorms $\|x\|_n'' = \max(\|x\|_1', \|x\|_2', \ldots, \|x\|_n')$ equivalent to the family $\|x\|_n'$. By the well-known theorem (see [12], 1.511) it may be assumed that there exist such positive constants C_k, D_k that

$$\|x\|_k \leqslant C_k\|x\|_k' \leqslant D_k\|x\|_{k+1}, \qquad k = 1, 2, \ldots \tag{10.7}$$

Consider now the multiplicative linear functionals of the form

$$f_q(x) = \sum_{n=0}^{\infty} x_n q^n, \tag{10.8}$$

where $x \in l^{1+}$, $|q| < 1$. These functionals are continuous with respect to every $\|x\|_k$, whence, by (10.7), they are also continuous with respect to every $\|x\|_k'$. We choose any integer k_0. $\|x\|_{k_0}$ is a norm in l^{1+}, i. e. $\|x\|_{k_0} = 0$ if and only if $x = 0$; this is a consequence of (10.7) and of the fact that the pseudonorms (10.6) are norms themselves. Consider the algebra R defined as the completion of l^{1+} in the norm $\|x\|_{k_0}$. The functionals (10.8) are continuous with respect to $\|x\|_{k_0}$, whence they may be extended onto R, where those extensions are also multiplicative and linear. Hence we must have $|f_q(x)| \leqslant \|x\|_{k_0}$ for every $x \in l^{1+}$, and every q, $|q| < 1$, but taking $x = (1/n)$ we have

$$\sup_q |f_q(x)| = \sup_q \left|\sum_n q^n/n\right| = \infty.$$

Hence we get a contradiction of the assumption that l^{1+} is m-convex, q. e. d.

THEOREM 27. *Every non-zero multiplicative linear functional defined on l^{1+} is of the form* (10.8).

Proof. Following the previous remark, we shall treat l^{1+} as the algebra of holomorphic functions of variable λ, $|\lambda| < 1$. We put $z = z(\lambda) \equiv \lambda$. Let f be any non-zero multiplicative linear functional defined on l^{1+}. Let $f(z) = \lambda_0$. We shall divide the proof in two steps:

1° We shall show that $|\lambda_0| < 1$.

2° We shall show that if $x(\lambda) = \sum_{n=0}^{\infty} x_n\lambda^n$ is a holomorphic function in the unit circle, $\sum_{n=0}^{\infty}|x_n|^p < \infty$ for $p > 1$, and $x(\lambda_0) = 0$ for $|\lambda_0| < 1$, then $x(\lambda) = (\lambda-\lambda_0)y(\lambda)$, where $y(\lambda) = \sum_{n=0}^{\infty} y_n\lambda^n$, and $\sum_{n=0}^{\infty}|y_n|^p < \infty$.

From 1° and 2° our conclusion may easily be deduced. Indeed, if $f(z) = \lambda_0$, and x is any element of l^{1+}, then by 1° and 2° the element $x - x(\lambda_0)e$ may be written in the form

$$x - x(\lambda_0)e = (z - \lambda_0 e)y,$$

where $y \in l^{1+}$. Hence

$$f[x - x(\lambda_0)e] = f(z - \lambda_0 e)f(y) = 0,$$

and consequently $f(x) = x(\lambda_0)$. It remains only to prove 1° and 2°.

Ad 1°. We cannot have here $|\lambda_0| > 1$ because the element $z - \lambda_0 e$ is invertible in l^{1+}. By the observation that the rotations $\varphi(\lambda) \to \varphi(e^{it}\lambda)$, $t \in (0, 2\pi]$, are isometric automorphisms of l^{1+}, it is enough to show that $\lambda_0 \neq 1$. To prove this, consider the function

$$v(\lambda) = \frac{1}{\lambda}\ln(1-\lambda).$$

It may easily be verified that $v \in l^{1+}$. On the other hand,

$$v^{-1} = \sum_{n=0}^{\infty} p_n \lambda^n, \quad \text{where} \quad p_n = o\left(\frac{1}{n[\ln n]^2}\right)$$

(see [10], p. 93, example 8.4). We then have not only $v^{-1} \in l^{1+}$, but also $v^{-1} \in l_1$. If we had $f(z) = 1$, then the functional f restricted to $l_1 \subset l^{1+}$ would be of the form $f(u) = u(1)$ for every $u \in l_1$ (see e. g. theorem 15). Hence $f(v^{-1}) = v^{-1}(1) = 0$, which contradicts the fact that v^{-1} is invertible in l^{1+}. Consequently $|\lambda_0| < 1$.

Ad 2°. If x, x_n, y, y_n are such as in 2°, then $x_n = y_{n-1} - \lambda_0 y_n$ for $n = 1, 2, 3, \ldots$ Hence

$$\infty > \Big(\sum_1^{\infty} |x_n|^p\Big)^{1/p} \geqslant \Big(\sum_1^{N} |y_{n-1} - \lambda_0 y_n|^p\Big)^{1/p}$$

$$\geqslant \Big|\Big(\sum_1^{N} |y_{n-1}|^p\Big)^{1/p} - |\lambda_0|\Big(\sum_1^{N} |y_n|^p\Big)^{1/p}\Big|$$

for $N = 1, 2, \ldots$ We then have

$$|S_{n-1} - |\lambda_0| S_n| \leqslant \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p} + |y_0| < \infty, \tag{10.9}$$

where $S_n = \big(\sum_{k=1}^{\infty} |y_k|^p\big)^{1/p}$, $n = 1, 2, 3, \ldots$

It is to be shown that $\lim S_n < \infty$. Suppose then that $\lim S_n = \infty$. Hence, by (10.9), $\lim S_n/S_{n+1} = |\lambda_0| < 1$. Consequently, also $\lim |y_n| = \infty$;

otherwise there would exist a subsequence (y_{n_k}) and a constant M such that $|y_{n_k}| < M$ and we should have $1 = S_{n_k}/S_{n_k} \leqslant (S_{n_{k-1}} + M)/S_{n_k} \to |\lambda_0| < 1$. Since $|x_{n+1}| = |y_n - \lambda_0 y_{n+1}|$ tends to zero, it follows that $|y_n/y_{n+1} - \lambda_0|$ also tends to zero and there exists $\lim y_n/y_{n+1} = |\lambda_0| < 1$, and the radius of convergence of $\sum y_n \lambda^n$ is equal to $|\lambda_0| < 1$, which contradicts the fact that $y(\lambda)$ is a holomorphic function for $|\lambda| < 1$, q. e. d.

Remark. For the element $z = z(\lambda) \equiv \lambda$ we have $f(z - e) \neq 0$ for every multiplicative linear functional f. On the other hand, the element $z - e$ is not invertible in l^{1+}. This situation is impossible for m-convex algebras, which may be regarded as another proof of non-m-convexity of l^{1+}. The example given by Arens [1] also possesses this property (it is a B_0-algebra with unit element in which there exists no non-zero multiplicative linear functional). Hence we may pose the following

PROBLEM 6. *Is it true that B_0-algebra R is m-convex if and only if for every non-invertible element $x \in R$ there exists such a multiplicative linear functional f that $f(x) = 0$?*

References

[1] R. Arens, *The space L^ω and convex topological rings*, Bull. Amer. Math. Soc. 52 (1946), p. 931-935.

[2] — *Linear topological division algebras*, ibidem 53 (1947), p. 623-630.

[3] S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math. 7 (1957), p. 1279-1304.

[4] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[5] A. Białynicki-Birula, W. Żelazko, *On the multiplicative linear functionals on the Cartesian product of algebras*, Bull. Acad. Polon. Sc. Cl. III. 5, nr 6 (1957) p. 589-593.

[6] J. Dieudonné, *La dualité dans les espaces linéaires topologiques*, Annales de l'École Normale Supérieure 59 (1942), p. 107-139.

[7] R. E. Edwards, *Multiplicative norms on Banach algebras*, Proc. Cambr. Phil. Soc. 47 (1951), p 473-474

[8] I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sb.) N. S. 9 (1941), p. 3-24.

[9] S. Kametani, *An elementary proof of the fundamental theorem of normed fields*, J. Math. Soc. Japan 4 (1952), p. 96-99.

[10] J. E. Littlewood, *Lectures on the Theory of Functions*, London 1944.

[11] S. Mazur, *Sur les anneaux linéaires*, C. R. Acad. Paris 207 (1938), p. 1025-1027.

[12] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (I)*, Studia Mathematica 10 (1948), p. 184-208.

[13] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (II)*, ibidem 13 (1953), p. 137-179.

[14] E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[15] M. A. Наймарк, *Нормированные кольца*, Москва 1956.

[16] A. Ostrowski, *Über einige Lösungen der Funktionalgleichung* $\Phi(xy) = \Phi(x)\Phi(y)$, Acta Math. 4 (1918), p. 271-284

[17] M. Pavel, *On quasi normed spaces*, Bull. Acad. Polon. Sc. Cl III, 5 (1957), p. 479-484.

[18] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sc. Cl III 5 (1957), p. 479-484.

[19] L. Tornheim, *Normed fields over the real and complex fields*, Michigan Math. J. 1 (1952), p. 61-68.

[20] J. H. Williamson, *On topologizing of the field* $C(t)$, Proc. Amer. Math. Soc. 5 (1954), p. 729-734.

[21] A. Zygmund, *Trigonometrical Series*, Warszawa 1935.

[22] W. Żelazko, *On a certain class of topological division algebras*, Bull. Acad. Polon. Sc., Sér. Sc. Math. Astr. Phys. 7 (1959), p. 201-203.

[23] — *On the algebras* L_p *of locally compact groups*, Coll. Math. (in print).

[24] — *On the radicals of p-normed algebras*, Studia Math. (in print).

[25] — *A theorem on the* B_0 *division algebras*, Bull. Acad. Pol. Sci. (in print).

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 22. 1. 1960

STUDIA MATHEMATICA publient des travaux de recherches (en langues des congrès internationaux) concernant l'Analyse fonctionnelle, les méthodes abstraites d'Analyse et le calcul de probabilité. Chaque volume contient au moins 300 pages.

Les manuscrits dactylographiés sont à adresser à

M. Hugo Steinhaus

Wrocław 12 (Pologne), ul. Orłowskiego 15,

ou à

M. Marceli Stark

Warszawa 10 (Pologne), ul. Śniadeckich 8.

Les auteurs sont priés d'indiquer dans tout renvoi bibliographique le nom de l'auteur et le titre du travail cité, l'édition, le volume et l'année de sa publication, ainsi que les pages initiale et finale.

Adresse de l'échange:

Warszawa 10 (Pologne), ul. Śniadeckich 8.

STUDIA MATHEMATICA sont à obtenir par l'intermédiaire de

ARS POLONA

Warszawa (Pologne), Krakowskie Przedmieście 7.

Le prix de ce fascicule est 2$.

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA