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Continuity of semi-norms on topological vector spaces

by

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1. Let E be a vector space over the real (or the complex) field and denote its zero element by $\emptyset$. A real function p on E is called a *semi-norm* if for all $x, y \in E$ and every real (or complex) number λ,

$$p(\lambda x) = |\lambda| p(x)$$

and

$$p(x+y) \leqslant p(x)+p(y).$$

It then follows that $p(\emptyset) = 0$, $0 \leqslant p(x) < +\infty$ and $|p(x)-p(y)| \leqslant p(x-y)$. Also p is a norm if and only if $p(x) = 0$ implies $x = \emptyset$. If E is a topological vector space (TVS) and the semi-norm p is continuous at one point of E, it is continuous everywhere. If E is a normed space, then p is continuous on E if and only if, for some constant M,

$$p(x) \leqslant M\|x\| \quad \text{for all} \quad x \in E.$$

A very useful theorem on semi-norms, due [1] independently to Gelfand ([7], [8]), Orlicz [11] and Bosanquet-Kestelman [3], states that every lower semi-continuous semi-norm on a Banach space is, in fact, continuous. More precisely, we have

THOREM A. *If*

(i) *E is a Banach space,*

(ii) *p is a semi-norm on E,*

(iii) *p is lower semi-continuous,*

then p is continuous.

Eberlein [6] extended Theorem A by replacing hypothesis (iii) by the much weaker hypothesis

(iii)′ *p has the Baire property.*

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[1] I am indebted to Professor W. Orlicz for the references [8] and [11].

Theorem A with (iii)′ in place of (iii) will be referred to in the sequel as Theorem B [2]. It may be recalled that a function f on one topological space E to another has the Baire property if there is a set H in E such that CH is of the first category and the partial function f/H is continuous on H. In this note we show that this theorem is true when E is any second category TVS and p is a semi-norm such that the set $\{x\colon\ p(x)\leqslant 1\}$ is a Baire set. Some simple consequences of this theorem are noted and applications are given to the convergence of a sequence of linear transformations on a TVS and the weak convergence of linear functionals on a TVS.

2. Before proceeding to prove the above result, we remark that a certain refinement of Theorem A by Bosanquet and Kestelman and of Theorem B by Eberlein is more apparent than real. In the formulation of Theorem A by Bosanquet and Kestelman it is assumed merely that E is a normed space and that the lower semi-continuous semi-norm p is defined on E but is finite on a set $D\subset E$ which is of the second category. These assumptions, however, imply that D is identical with E. For, in the first place, D is a vector subspace of E, and if, for each $n\geqslant 1$,

$$A_n=\{x\,\epsilon D\colon\ p(x)\leqslant n\}$$

then A is closed and

$$D=\bigcup_{n=1}^{\infty}A_n.$$

Thus D is a Borel set (in fact an F_σ) and, being of the second category, it is identical with E by a well-known theorem of Banach ([2], p. 36). Entirely similar remarks apply to the refinement of Theorem B by Eberlein: the sets A_n are then Baire sets and hence D is a Baire set and the same conclusion is obtained.

3. The main theorem is now proved.

THEOREM 1. *If*

(i) *E is a second category* TVS,

(ii) *p is a (finite) semi-norm on E,*

(iii) *the set $B=\{x\colon\ p(x)\leqslant 1\}$ is a Baire set,*

then p is continuous.

Proof. For $n\geqslant 1$, let $A_n=\{x\colon\ p(x)\leqslant n\}$. Then $A_n=nB$, and being the image of B under the homeomorphism $x\to nx$ of E into itself, A_n is a Baire set. Since p is finite, $E=\bigcup_{n=1}^{\infty}A_n$. Since E is of second category, A_m is of the second category, for some $m\geqslant 1$. It follows by the Banach-Kuratowski-Pettis theorem [3] that $(A_m-A_m)^\circ$ is non-empty and

$$\emptyset\,\epsilon\,(A_m-A_m)^\circ,$$

where S° is the interior of the set S. Since p is a semi-norm,

$$A_m-A_m\subset A_{2m},$$

so that

$$\emptyset\,\epsilon\,A_{2m}^\circ.$$

Hence there is an open set U in A_{2m} containing $\emptyset$ and such that

$$p(x)\leqslant 2m\quad\text{for all}\quad x\,\epsilon\,U.$$

This implies that p is continuous at $\emptyset$: For $\varepsilon>0$, take any $x\,\epsilon\,\frac{\varepsilon}{2m}U$, then $\frac{2m}{\varepsilon}x\,\epsilon\,U$ and

$$0\leqslant p(x)=p\left(\frac{\varepsilon}{2m}\cdot\frac{2m}{\varepsilon}x\right)\leqslant\frac{\varepsilon}{2m}\cdot 2m=\varepsilon.$$

Hence p is continuous on E.

Remark. It is interesting to enquire if Theorem 1 is true for groups in place of vector spaces, with the conditions on p suitably modified. Let us say that a real function p on (an additive) group G is a *pseudo-norm* if

$$p(\emptyset)=0,$$

$$p(-x)=p(x),$$

$$p(x+y)\leqslant p(x)+p(y),$$

for all $x,y\,\epsilon\,G$. It is then tempting to ask if the following proposition is true: „If

(1) G is a complete metric group,

(2) p is pseudo-norm on G,

(3) p has the Baire property (or the sets $\{x\colon\ p(x)\leqslant r\}$ are Baire sets for $r\geqslant 0$),

then p is continuous".

(2) Cf. Math. Review 8 (1947), p. 279 where Theorem B is misquoted.

(3) Banach [2], p. 21, Kuratowski [10], p. 82, and Pettis [12], p. 295. Cf. also Hille [9].

That the proposition is false may be seen by the following Example. Take $G = R$, the additive group of real numbers, and let

$$p(x) = \begin{cases} 0, & x \text{ rational}, \\ 1, & x \text{ irrational}. \end{cases}$$

All the conditions (1)-(3) are satisfied. But p is discontinuous at every point. What conditions (1)-(3) imply is that p is bounded on some non-empty open set in G. But this implies continuity only in the presence of scalar multipliers. Clearly the function p in the above example is not lower semi-continuous at every point of R. It is an open question whether *lower semi-continuous* pseudo-norms on complete metric groups are necessarily continuous.

4. Corollaries. Theorem 1 and the corollaries below are true, in particular, when E is a complete metrizable TVS. These spaces include the F-spaces of Banach (whose metric is invariant under translation) and the Fréchet spaces of Bourbaki [5] (which are locally convex).

COROLLARY 1. *On a second category* TVS, *every semi-norm which has the Baire property* (*or, in particular, is Borel measurable*) *is continuous.*

If p has the Baire property, then for every closed set F in the real line, $p^{-1}(F)$ is a Baire set. This follows from a theorem of Kuratowski ([10], p. 306). (The proof there given for functions on a metric space holds for functions on any topological space.)

COROLLARY 2. *On a second category* TVS, *every lower* (*or upper*) *semi-continuous semi-norm is continuous.*

For, if p is lower semi-continuous then $\{x\colon p(x) \leqslant 1\}$ is closed and therefore a Baire set. And if p is upper semi-continuous then the sets $\{x\colon p(x) < 1+1/n\}$ are open and

$$\{x\colon p(x) \leqslant 1\} = \bigcap_{n=1}^{\infty} \left\{x\colon p(x) < 1+\frac{1}{n}\right\}$$

so that $\{x\colon p(x) \leqslant 1\}$ is a G_δ and hence a Baire set.

COROLLARY 3. *Every second category locally convex space is tunnelled.*

For, by corollary 2, every lower semi-continuous semi-norm on the spaces is continuous, cf. Bourbaki ([5], p. 1-2).

Remark. We recall that in the Bourbaki terminology [4], a *Baire space* is a topological space in which every non-empty open set is of the second category. It is easily seen that a topological group is of the second category if and only if it is a Baire space. This follows from Banach's *first category theorem* ([1], [10], p. 51) and shows that corollary 3 is equivalent to a theorem of Bourbaki ([5], p. 1).

COROLLARY 4. *If E is a second category* TVS *and (p_n) is a sequence of continuous semi-norms on E such that $\lim_{n\to\infty} p_n(x)$ exists for every $x \in E$, then p, defined by*

$$p(x) = \lim_{n\to\infty} p_n(x),$$

is a continuous semi-norm on E.

Proof. That p is a semi-norm follows readily from the fact that the p_n are semi-norms. Since p is Borel measurable, the result follows from corollary 1.

THEOREM 2. *Let E be a second category* TVS *and F a Banach space and let u_n $(n \geqslant 1)$ be continuous linear transformations on E to F such that*

$$\sup_n \|u_n(x)\| < +\infty$$

for each $x \in E$. If $\lim_{n\to\infty} u_n(x)$ exists on a set S dense (*or fundamental*) *in E, then $\lim_{n\to\infty} u_n(x)$ exists for every $x \in E$.*

Proof. Put $p(x) = \sup_n \|u_n(x)\|$. Since the u_n are linear, p is a semi-norm and since the u_n are continuous, p is lower semi-continuous. By Theorem 1, p is continuous on E. Hence, to each $\varepsilon > 0$, there is an open neighbourhood U of $\emptyset$ such that

$$p(x) < \tfrac{1}{2}\varepsilon \quad \text{for all} \quad x \in U.$$

Choose any $y \in E$. Then $U+y$ is an open neighbourhood of y. Since S is dense in E, there is $z \in S$ such that $z \in U+y$, i. e., such that $z-y \in U$. Now

$$\begin{aligned} \|u_m(y)-u_n(y)\| &= \|u_m(y-z)+u_m(z)-u_n(z)+u_n(z-y)\| \\ &\leqslant \|u_m(y-z)\|+\|u_n(z-y)\|+\|u_m(z)-u_n(z)\| \\ &\leqslant 2p(z-y)|+\|u_m(z)-u_n(z)\| \\ &< \varepsilon+\|u_m(z)-u_n(z)\|. \end{aligned}$$

Since $z \in S$, this implies that

$$\lim_{m,n\to\infty} \|u_m(y)-u_n(y)\| = 0,$$

and the result follows from this, since F is a Banach space.

THEOREM 3. *Let E be a second category* TVS *and let f_n $(n \geqslant 1)$ be continuous linear functionals on E. The necessary and sufficient conditions for $\lim_{n\to\infty} f_n(x)$ to exist for every $x \in E$ are:*

(i) $\sup_n |f_n(x)| < +\infty$, *for each $x \in E$,*

(ii) $\lim_{n\to\infty} f_n(x)$ *exists on a fundamental set in E.*

Proof. The necessity of (i) and (ii) is trivial and their sufficiency follows from the previous theorem.

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Sur le problème de la division

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Le but de cet article est de démontrer un théorème qui confirme une hypothèse de L. Schwartz, selon laquelle la division d'une distribution par une fonction analytique réelle est toujours possible (cf. [10], p. 116 et [9], p. 181): *l'équation*

$$\Phi S = T$$

admet toujours une solution S, quelles que soient la distribution T et la fonction analytique réelle Φ.

Evidemment cette solution n'est pas unique (sauf le cas où $\Phi \neq 0$) et toutes les solutions de l'équation homogène ΦS sont portées par l'ensemble des zéros de Φ.

L. Schwartz a résolu ce problème (dans [9]) pour une fonction analytique de n variables complexes (considérés comme une fonction de $2n$ variables réelles).

Il résulte du théorème de L. Schwartz que l'on peut diviser par toute fonction analytique réelle de la forme

$$\varphi(x_1, y_1, \ldots, x_n, y_n) = |f(z_1, \ldots, z_n)|^2,$$

où f est une fonction holomorphe des variables $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy$.

Dans la première partie nous nous occuperons des distributions et des fonctions indéfiniment dérivables (sans faire intervenir la notion d'analycité) et nous démontrerons quelques théorèmes sur la division dans certains cas.

La deuxième partie est consacrée à la décomposition d'un ensemble analytique réel en sous-variétés. On obtient certaines propriétés de cette décomposition, qui résultent d'une inégalité de la forme

$$|f(\boldsymbol{x})| \geqslant \varepsilon \varrho(\boldsymbol{x})$$

(où f est analytique réelle, $\varrho(x)$ désigne la distance à l'ensemble des zéros); la démonstration de cette inégalité est assez difficile, bien que, dans le cas des variables complexes, elle soit banale. Finalement nous arrivons au théorème sur la division par une fonction analytique réelle.

L'idée de la démonstration a été signalée dans une note aux Comptes Rendus [6].