Finally, it may be worth mentioning that our new polynomials $P_n(f; x), T_n(f; x)$ etc. can also be generalized to the cases of a complex variable and of several variables. Further investigation of these polynomials is being accomplished in a joint paper of the author with L. P. Hsu, which will appear elsewhere.

References


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On modular spaces

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In the present paper the authors investigate functionals $g(x)$ defined in a real linear space $X$, which are called modulars. An $F$-norm will be introduced in certain subspaces of the space $X$. In the second part of this paper some examples of modulars are considered.

1. First, the following definition of a modular and a pseudomodular will be given:

1.01. Given a linear space $X$, a functional $g(x)$ defined on $X$ with values $−∞ < g(x) ≤ ∞$ will be called a modular if the following conditions hold:

A.1. $g(x) = 0$ if and only if $x = 0$,
A.2. $g(−x) = g(x)$,
A.3. $g(ax + by) ≤ g(x) + g(y)$ for every $a, b ≥ 0$, $a + b = 1$.

If $g(x)$ satisfies the condition $g(0) = 0$ instead of A.1, then $g(x)$ will be called a pseudomodular.

1.02. We now give some simple properties of pseudomodulars.

Let us assume $g(x)$ to be a pseudomodular on $X$. Then

(a) $g(a) ≥ 0$,
(b) $g(ax)$ is a non-decreasing function of $a ≥ 0$ for each $x ∈ X$,
(c) $g(\sum_{i=1}^{n} a_i x_i) ≤ \sum_{i=1}^{n} g(a_i x_i)$ for $a_i ≥ 0$, $\sum_{i=1}^{n} a_i = 1$.

Moreover, if $X_s$ denotes the set of $x ∈ X$ such that $g(x) < ∞$, the set $X_s$ is convex and symmetric with respect to 0.

The properties (a) and (b) easily follow from A.3 and A.2; (c) is obtained by induction as follows:

$g(\sum_{i=1}^{n} a_i x_i) = g(\sum_{i=1}^{n-1} a_i x_i + a_n x_n) ≤ g(\sum_{i=1}^{n-1} a_i x_i) + g(a_n x_n) ≤ \sum_{i=1}^{n} g(x_i)$.

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It will be noted that for a modular \( q(x) \) the inequality \( q(x) < q(y) \) does not imply in general \( q(ax) \leq q(ay) \) for every real \( a \).

1.03. Denote by \( X_\omega \) a convex subset of \( X \), symmetric with respect to 0. Let \( X_\omega^* \) be the set of all \( x \in X \) such that \( kx \in X_\omega \) with a positive constant \( k \) depending on \( x \). In particular, \( X_\omega^* \) denotes the set of all \( x \in X \) such that, for a given \( k > 0 \), \( q(kx) \) is finite. It is easily seen that \( X_\omega^* \) are linear subspaces of \( X \).

1.04. We now introduce the concept of modular convergence and modular completeness. A sequence \( \{ x_n \} \subset X \) will be said to be:

(a) modular convergent or \( q \)-convergent to \( x \in X \) (in symbols: \( x_n \xrightarrow{q} x \)) if there exists a number \( k > 0 \) (depending on the sequence \( \{ x_n \} \)) such that \( q(k(x_n - x)) \rightarrow 0 \) as \( n \rightarrow \infty \);

(b) satisfying the modular Cauchy condition if \( q(x_n - x_m) \rightarrow 0 \) as \( n, m \rightarrow \infty \).

Further, a subset \( X_\omega \subset X_\omega^* \) will be said to be:

(a') modular complete or \( q \)-complete if the modular Cauchy condition implies modular convergence to an \( x \in X_\omega \);

(b') strongly modular complete or strongly \( q \)-complete if the modular Cauchy condition implies modular convergence to an \( x \in X_\omega \) with a constant \( k > 0 \) independent of the sequence \( \{ x_n \} \subset X_\omega^* \).

1.05. There result the following properties of modular convergence:

(a) if \( q(x) \) is a modular, then the modular limit is uniquely determined;
(b) if \( x_n \xrightarrow{q} y \), then \( x_n + \beta y \xrightarrow{q} x + \beta y \) for every real \( \alpha, \beta \).

1.1. In our further considerations the following conditions will be of importance:

B.1. \( a_n \rightarrow 0 \) implies \( q(ax_n) \rightarrow 0 \),
B.2. \( q(x_n) \rightarrow 0 \) implies \( q(ax_n) \rightarrow 0 \) for every real \( a \).

1.11. Denote by \( X_\omega \) the set of all \( x \in X_\omega^* \) such that B.1 holds. Then \( X_\omega \) is convex and the linear space \( X_\omega^* \) is closed with respect to \( q \)-convergence, as follows from the inequality \( q(ax) \leq q(2ax_n - x) + q(2ax_n) \). Hence, if \( X_\omega^* \) is \( q \)-complete or strongly \( q \)-complete, so is \( X_\omega \).

1.12. Let \( X_\omega^* \) be a finite-dimensional linear subspace of \( X_\omega^* \) and let \( q(x) \) be a modular. Denoting by \( \alpha_1, \alpha_2, \ldots, \alpha_m \) the basis of \( X_\omega^* \), the necessary and sufficient condition for a sequence \( x_n = \alpha_1 e_1 + \cdots + \alpha_m e_m \) to be \( q \)-convergent to \( x = \lambda_1 e_1 + \cdots + \lambda_m e_m \) is \( \lambda_n \rightarrow \lambda_n \) as \( n \rightarrow \infty \) for \( i = 1, 2, \ldots, m \).

Supposing that the limit element \( x = 0 \), we have to prove that \( \lambda_n \rightarrow 0 \) as \( n \rightarrow \infty \). In the contrary case we could write

\[ \sum_{i=1}^{m} |\lambda_n| > d > 0, \]

and denoting

\[ \gamma_n = \frac{d \lambda_n}{\sum_{i=1}^{m} |\lambda_n|} \quad \text{for} \quad i = 1, 2, \ldots, m; n = 1, 2, \ldots, \]

we should obtain

\[ q\left( k \sum_{i=1}^{m} \gamma_n e_i \right) \leq q\left( k \sum_{i=1}^{m} \lambda_n e_i \right) - q(kx_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

On the other hand, it may be assumed that

\[ \gamma_n \rightarrow \gamma_i \quad \text{as} \quad n \rightarrow \infty \quad \text{for} \quad i = 1, 2, \ldots, m. \]

Then

\[ q\left( k \sum_{i=1}^{m} \gamma_i e_i \right) \leq \sum_{i=1}^{m} q(k(y_i - \gamma_i) e_i) + q\left( k \sum_{i=1}^{m} \gamma_i e_i \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

This implies \( \gamma_i = 0 \) and \( \lambda_n \rightarrow 0 \) as \( n \rightarrow \infty \) for \( i = 1, 2, \ldots, m \). The fact that if \( \lambda_n \rightarrow 0 \) as \( n \rightarrow \infty \) for \( i = 1, 2, \ldots, m \) then \( x_n \xrightarrow{q} 0 \) follows from B.1.

From 1.12 there results the following statements:

1.15. If \( X_\omega^* \) is a finite-dimensional linear space, \( X_\omega^* \subset X_\omega^* \), \( \phi(x) \) and \( \psi(x) \) being modulars on \( X \), then \( q \)-convergence of a sequence \( \{ x_n \} \subset X_\omega^* \) to \( x \in X_\omega^* \) is equivalent to \( \psi \)-convergence of \( x_n \) to \( x \).

1.2. The object of this paper is to introduce an \( F \)-norm in a linear space by means of a modular. Similar problems have been considered by H. Nakane and his school under the additional hypothesis of convexity (see e.g. [5]) or subadditivity [5] of the modular \( q(x) \). Moreover, the case of semi-ordered linear spaces and that of \( B \)-norms have been chiefly investigated. Our investigation aims at obtaining some results under weaker assumptions, fitted to the structure of the spaces under consideration. Neither convexity nor subadditivity of the modular will be assumed. It is to be noted that, in the case of \( l^\infty \) - and \( L^\infty \)-spaces for example (see [6] and [2]), the assumptions of continuity and monotony of \( M(x) \) suffice for \( q(x) \) to be a modular. In introducing the norm, a certain natural connection between the modular and the norm convergence will be required: norm convergence should imply modular convergence. Under the additional assumptions B.2, the two convergences are equivalent. A number of examples in the second part of our paper makes clear the application of the conditions introduced in the present part. The authors will return to this investigation in another paper.
1.21. Given a modular \( q(x) \), let us write
\[
|x| = \inf \{ e > 0 : q(x/e) \leq e \}.
\]
The functional \( |x| \) is an \( F \)-norm in \( X^*_p \), i.e.
1° \( |x| = 0 \) if and only if \( x = 0 \),
2° \( |x + y| \leq |x| + |y| \),
3° \( \| -x \| = |x| \),
4° \( a_n \to a \) and \( |x_n - x| \to 0 \) imply \( |a_n x_n - ax| \to 0 \).

The norm \( |x| \) is a non-decreasing function of \( a \geq 0 \) for every \( x \in X^*_p \). We have \( q(a) \leq |x| \) for \( |x| < 1 \); hence norm convergence implies modular convergence to the same limit. Moreover, if \( X^*_p \) is strongly \( e \)-complete, then it is complete in norm.

It will be noted that \( q(a/x) = a \) for an \( a > 0 \) implies \( |x| = a \) and that, if \( q(x) \) is a continuous function of \( a \geq 0 \) for every \( x \in X^*_p \), then \( q(a|x|) = |x| \) for every \( x \neq 0, x \in X^*_p \). Moreover, if \( q(x) \) is a pseudo-modular, then \( |x| \) is an \( F \)-pseudo-norm; if \( q(x) = 0 \) implies \( q(zx) = 0 \) and \( X^*_p \) is strongly \( e \)-complete, then it is complete in pseudo-norm.

Conditions 1° and 3° and the monotony of \( |x| \) as a function of \( a \geq 0 \) being trivial, we now prove the triangle inequality 2°. Given any \( x > 0 \) and \( y, z \in X^*_p \), we write \( a = |x| + |y|, \beta = |y| + |z| \). Then \( q(a/x) < a, \beta \leq \beta \), and
\[
q(\frac{a + y}{a + \beta}) = q\left(\frac{\frac{a}{a + \beta}}{\frac{a}{a + \beta} + \frac{y}{a + \beta}}\right) \leq q\left(\frac{a}{a + \beta}\right) + q\left(\frac{y}{a + \beta}\right) \leq a + \frac{1}{a + \beta}.
\]
Hence \( |x + y| \leq a + \beta \) and the triangle inequality follows. To prove 4°, let us first note that
(a') \( a_n \to 0 \) implies \( |x_n| \to 0 \),
(b') \( |x_n| \to 0 \) implies \( |a_n x_n| \to 0 \) for any real \( a_n \),
these conditions being easy to verify by the basic inequalities
\[
g\left(\frac{x}{\lambda |x|}\right) \geq |x| \quad \text{for} \quad \lambda < 1,
\]
\[
g\left(\frac{x}{\lambda |x|}\right) \leq |x| \quad \text{for} \quad \lambda > 1,
\]
where \( x \neq 0 \). Applying the triangle inequality \( |a_n x_n - x| \leq |a_n (x_n - x)| + + \) \( |x_n - x| \) and the monotony of \( |x| \) and of \( |y| \), and using (a') and (b'), we obtain 4°.

Now, let us assume \( X^*_p \) to be strongly \( e \)-complete and let \( |x_n| \to 0 \) as \( m, n \to \infty \). Then \( q(a_n x_n) \to 0 \) for any \( \varepsilon > 0 \). The strong \( e \)-completeness of \( X^*_p \) implies the existence of an \( \tilde{x}_p \) in \( X^*_p \) such that \( q(\tilde{x}_p) = q(a_n x_n) \to 0 \) as \( n \to \infty \). Obviously \( q(\tilde{x}_p) = q(a_n x_n) \to 0 \).

1.22. The following theorem establishes the uniqueness of the norm \( |x| \):

Let \( q(x) \) be a modular and let \( \|\cdot\| \) and \( \|\cdot\|' \) be two complete \( F \)-norms in \( X^*_p \). Such that norm convergence implies modular convergence of this sequence to zero. Let the \( \|\cdot\| \) and \( \|\cdot\|' \) are equivalent in the sense that \( |x_n - x| \to 0 \) if and only if \( |x_n - x|' \to 0 \).

We consider the operation \( U(x) = x \) from \( \langle X^*_p, \|\cdot\| \rangle \) to \( \langle X^*_p, \|\cdot\|' \rangle \).
Assuming \( |x_n - x| \to 0 \) and \( \|U(x_n) - y\|' \to 0 \), we have \( q(x_n - x) \to 0 \) and \( q(x_n - y) \to 0 \). The modular limit being unique, this implies \( x = y \). Then the Banach closed graph theorem implies \( U(x) = 0 \) to be linear. Theorem 1.22 follows immediately.

1.5. In the following theorem, condition B.2 is of importance:

1.51. Modular convergence is equivalent to norm convergence in a subset \( X_w \) of \( X^*_p \) if and only if condition B.2 holds for any sequence of elements of \( X_w \).

Indeed, let the modular convergence be equivalent to the norm convergence and let us assume \( q(x_n) \to 0 \). Then \( |x_n| \to 0 \); hence \( |x_n| \to 0 \). Conversely, let B.2 hold and let \( q(x_n) \to 0 \). Assuming \( |x_n| > 0 \), we obtain \( q(x_n) \to 0 \). On the other hand, the definition of the norm yields \( q(x_n) \to 0 \), in contradiction to the above convergence.

1.32. We now make some remarks concerning condition B.3:

(a) If B.2 holds for any sequence of elements of \( X_w \subseteq X^*_p \), and \( X_w \) is the closure of \( X_w \) with respect to the norm, then B.2 holds also for any sequence of elements of \( X_w \). Moreover, we obviously have \( X_w \subseteq X^*_p \).

(b) Assuming \( X_w \) to be linear, B.2 in \( X_w \) is equivalent to the following condition: if \( q(x) \to 0 \), then \( q(x_n) \to 0 \) for any \( x_n \in X_w \), being any fixed number larger than 1.

(c) \( X_w \) being a linear space, let us assume there the existence of positive numbers \( a \) and \( c \) such that the condition
\[
q(2a) \leq a q(x)
\]
holds for every \( x \in X_w \) satisfying the inequality \( q(x) < a \). Then B.2 holds for any sequence of elements of \( X_w \).

We first prove (a). Let \( x_n \in X_w \) and \( q(x_n) \to 0 \). We choose a sequence of elements \( y_n \in X_w \) such that \( |y_n - x_n| \to 0 \). Then \( q(y_n - x_n) \to 0 \) and the inequality \( q(2a) \leq q(y_n - x_n) + q(x_n) \to 0 \) and B.2 imply \( q(2a x_n) \to 0 \) for any real \( a \). Hence \( q(x_n) \to 0 \) and \( q(2a y_n) \to 0 \). If \( 2a y_n \to 0 \) and \( q(x_n) \to 0 \).

is not necessary for B.2. A counter example is provided by the case; $X$ — the space of reals, $\varphi(x)$ — a continuous monotone function not satisfying $(\Delta_2)$.

2. In the present section a number of examples of spaces with a modular will be considered. As special cases, the well-known examples of spaces $l^p$, $L^p$, and $V_M$ come under consideration.

2.1. Let $X_1, X_2, \ldots$ be linear spaces with modulars $\varphi_1(x_1), \varphi_2(x_2), \ldots$ respectively. The question arises of defining in a natural sense a modular $\varphi(x)$ in the Cartesian product $X = X_1 \times X_2 \times \ldots$ by means of the modulars $\varphi_1(x_1), \varphi_2(x_2), \ldots$. This question may be put in various ways. When $x = (x_1, x_2, \ldots)$, the following definitions may be introduced:

1. \[ \varphi^1(x) = \sum_{i=1}^{\infty} \varphi_i(x_i) \]
2. \[ \varphi^2(x) = \sup_i \varphi_i(x_i) \]
3. \[ \varphi^3(x) = \sup_n \sum_{i=1}^{n} \varphi_i(x_i) \]

It is easily seen that, if $\varphi_i(x_i)$ are modulars on $X_i$, respectively, then $\varphi^1(x)$, $\varphi^2(x)$, and $\varphi^3(x)$ are modulars on $X$.

2.11. The following inclusion holds:

\[ X_1 \times X_2 \times \cdots \subset X^3 \]

Indeed, if $x = (x_1, x_2, \ldots)$ belongs to the left side of this inclusion, e.g., $x \in X_j$, then there exists for every $\varepsilon > 0$ a number $N$ such that $\sum_{i=1}^{N} \varphi_i(x_i) < \varepsilon$; hence $\sum_{i=j}^{\infty} \varphi_i(x_i) < \varepsilon$ for any $0 \leq a \leq 1$. If $a_n \to 0$, then $\sum_{i=1}^{N} \varphi_i(x_i) < \varepsilon$ for sufficiently large $n$, and 2.11 follows. It will be noted that the inclusion 2.11 does not hold in general either for $\varphi^2(x)$ or for $\varphi^3(x)$.

2.12. In the sequel in 2.12–2.15 we shall always take $X_i = X^i \times \cdots \times X^i = X_i$ as the space of reals, $\varphi_i(x) = M_i(x) = \bar{M}_i(x)$ is an even continuous function, non-decreasing for $u \geq 0$, $M_i(0) = 0$, $M_i(u) \to 0$ for $u \to 0$, where $i = 1, 2, \ldots$. Then

(a) $X_i$ is strongly $\varphi$-complete,
(b) $X_i$ is strongly $\varphi^2$-complete,
(c) if $M_i(u)$ satisfy the condition

$$ (\Delta_4) \quad M_i(2u) \leq u \bar{M}_i(u) \quad \text{for} \quad 0 \leq \bar{M}_i(u) \leq a, $$

where $a > 0$ and $u$ are independent of $i$, then B.2 holds for any sequence of elements of $X_i$. 

The easy proofs will be omitted. Let us note that, when $M_i(u) = M(u)$ for $i = 1, 2, \ldots$, we obtain the spaces $l^p$, considered in many papers, with various additional assumptions (see e.g. [4], [5]). An $p$-norm of the form considered here was introduced for $l^p$ in the case of $M(u)$ satisfying the condition $(\Delta)$ for small $u$ in paper [3].

2.15. Let us take $X_1 = X^1 \times \cdots \times R_1$, $M_1(u) = M(u)$, where $M(u) \to 0$ as $u \to 0$. Then we denote by $X_1$ the set of all $x = [a_i]$ such that there exists a number $a$ with the following property:

$$ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M_i(x_i-a_i) = 0 \quad \text{for every real} \quad k. $$

Then $X_1$ is strongly $\varphi$-complete, $X_1$ is a linear space contained in $X_1$, complete with respect to the norm.

In order to prove the completeness of $X_i$, choose a sequence of elements $x_n = [a_i^n]$ of $X_1$ satisfying the modular Cauchy condition, i.e.

$$ \frac{1}{n} \sum_{i=1}^{n} M_i(x_i^n-x_i) \to 0 \quad \text{as} \quad p, q \to \infty $$

uniformly in $n$. Hence, the numerical sequence $[a_i^n]$ satisfies the Cauchy condition for each fixed $i$; hence $a_i^n \to a_i$ as $m \to \infty$ for each $i$ and there results the strong $\varphi$-completeness of $X_1$. We shall now prove the inclusion $X_1 \subset X_1$. Let $x = [a_i] \in X_1$ and let $0 \leq a \leq 1$. Then

$$ \frac{1}{n} \sum_{i=1}^{n} M_i(x_i^n-x_i) \leq \frac{1}{n} \sum_{i=1}^{n} M_i(x_i^n-a_i) + M_i(2a). $$

Since the first term on the right side of the above inequality tends to zero as $n \to \infty$ and the second tends to zero as $a \to 0$, the expression on the left side of this inequality is small for $u$ sufficiently large and $u$ sufficiently (but independently of $n$) small. Now, $\varphi^2(x) \to 0$ as $u \to 0$ results easily. Since $X_1$ is strongly $\varphi$-complete, it is complete in the norm. Then, to obtain the norm-completeness of $X_1$, it is sufficient to prove that $X_1$ is closed with respect to the norm. Assuming $[x_n] \subset X_1$ and $[x_n] \to 0$, where $x_n = [a_i^n]$, $x = [a_i^n]$, we have, for every $k > 0$, $q(x_k^n-x_k) \to 0$; thus

$$ \frac{1}{n} \sum_{i=1}^{n} M_i(x_k^n-x_k) \to 0 \quad \text{as} \quad m \to \infty \quad \text{uniformly in} \quad n, $$

$$ \frac{1}{n} \sum_{i=1}^{n} M_i(x_k^n-a_k^n) \to 0 \quad \text{as} \quad m \to \infty \quad \text{for} \quad m = 1, 2, \ldots, $$

$k$ being arbitrary.
Let us choose a number $\varepsilon > 0$; since
\[
M\left(\frac{a^n - a^q}{4}\right) \leq \frac{1}{n} \sum_{i=1}^{n} M(k(a^n - a_i^n)) + \frac{1}{n} \sum_{i=1}^{n} M(k(a^n - a_i^q)) + \frac{1}{n} \sum_{i=1}^{n} M(k(a^q - a_i^n)) + \frac{1}{n} \sum_{i=1}^{n} M(k(a^q - a_i^q)) \leq M\left(\frac{\varepsilon}{4}\right)
\]
for all sufficiently large $p, q$ and $n = n(p, q)$, the sequence $a^n > a$ as $m \to \infty$. Thus the inequality
\[
\frac{1}{n} \sum_{i=1}^{n} M(k(a_i^n - a_i^q)) \leq \frac{1}{n} \sum_{i=1}^{n} M(3k(a_i^n - a_i^q)) + \frac{1}{n} \sum_{i=1}^{n} M(3k(a_i^n - a_i^q)) + \frac{1}{n} \sum_{i=1}^{n} M(3k(a_i^n - a_i^q)) \leq M\left(\frac{\varepsilon}{4}\right)
\]
implies that $x = (a_i^n) \in X_n$, hence $X_n$ is closed with respect to the norm.

2.15. Assuming $M_1(u) = M_2(u) = \ldots = M(n)$, we have the equality
\[X_n = X_n^\infty \text{ if and only if for any } x \geq 0 \text{ numbers } A_i > 0 \text{ and } a_i \geq 0 \text{ exist such that } M(nw) < s M(nw) \text{ for every } 0 \leq a_i \leq a_i, n \geq A_i.\]
In order to prove the sufficiency, let us choose a number $\varepsilon > 0$ and take an $x \in X_n$. Then
\[
\frac{1}{n} \sum_{i=1}^{n} M(a_i^n) \leq M(a_{A_i}) + \varepsilon s^2(x) \text{ for } 0 \leq a_i \leq a_{A_i};
\]
hence $\Phi^n(x) \leq M(a_{A_i}) + \varepsilon s^2(x)$ and the sufficiency results. Now, we shall prove the necessity. Let us suppose that there exists an $\varepsilon > 0$ and two sequences of positive numbers: $a_k$, decreasing to zero, satisfying the inequality $M(a_{A_i} - a_k) > s M(a_{A_i})$ for $n = 1, 2, \ldots$ It may be assumed that $M(a_{A_i}) > \frac{1}{2}$. Now, we shall define two sequences of indices $n_1, n_2, \ldots; n_1, n_2, \ldots,$ and a sequence of numbers $a_1, a_2, \ldots$ by induction. Let $n_1$ be the least positive integer such that $M(a_{A_i}) < \frac{1}{2}$.

Moreover, let $n_1$ be the largest positive integer satisfying the inequality $\frac{1}{n_1} M(a_{A_i}) > \frac{1}{2}$. We define the first $n_1$ terms of the sequence $a_1, a_2, \ldots$ as follows: $a_k = 0$ for $n \leq n_1$, $a_k = a_{n_1}$ for $n_1 < n \leq n_1$. Now, let us suppose that the sequences $n_1, n_2, \ldots$ are defined for $i < k$ and $j < n_{k-1}$ in such a way that
\[
\frac{1}{n_1} \sum_{i=1}^{n_1} M(a_i^n) < \frac{1}{4} \text{ for any } n \leq n_{k-1},
\]
\[
\frac{(n_{k-1} - n_{k-1}) M(a_{n_{k-1}})}{n_{k-1}} \geq \frac{3}{4} \varepsilon.
\]
Let $n_k$ be the least positive integer, larger than $n_{k-1}$ and satisfying the inequalities
\[
\frac{M(a_k^n)}{n_k} < \frac{1}{4} \quad \text{and} \quad \frac{1}{n_k} \sum_{i=1}^{n_k} M(a_i^n) < \frac{1}{4}.
\]
Moreover, let $n_k > n_k'$ be the largest positive integer satisfying the inequality
\[
\frac{(n_k - n_k') M(a_{n_k'})}{n_k} < \frac{5}{4}.
\]
Obviously we have, for any $n_k' \leq n \leq n_k$,
\[
(n - n_k') M(a_{n_k'}) < \frac{5}{4} \text{ and } \frac{3}{4} \leq \frac{(n_k - n_k') M(a_{n_k'})}{n_k} < \frac{5}{4}.
\]
Now, we put $a_n = 0$ for $n_{k-1} < n \leq n_k$, $a_n = a_{n_k}$ for $n_k' < n \leq n_k$.

It is easily seen that
\[
\frac{1}{n} \sum_{i=1}^{n} M(a_i^n) < \frac{3}{2} \text{ for any } n \leq n_k.
\]
Indeed, it is sufficient to prove the above inequality for $n_k' < n \leq n_k$; however, for such $n$,
\[
\frac{1}{n} \sum_{i=1}^{n} M(a_i^n) \leq \frac{1}{n} \sum_{i=1}^{n_{k-1}} M(a_i^n) + \frac{(n - n_{k-1}) M(a_{n_{k-1}})}{n} < \frac{1}{4} + \frac{5}{4} = \frac{3}{2}.
\]
Thus, ($+$) and ($+$) are satisfied if we put $n_k$ instead of $n_{k-1}$. We shall now prove that $x = (a_i^n) \in X_\varepsilon$ but does not belong to $X_\varepsilon$. Evidently, since $\Phi^n(x) \leq \frac{1}{2}$, it is sufficient to prove that $\Phi^n(a_{n_k}) > \frac{1}{2} \varepsilon$ for $k = 1, 2, \ldots$

However,
\[
\frac{\Phi^n(a_{n_k})}{n_k} \geq \frac{1}{n_k} \sum_{i=1}^{n_k} M(a_{n_k} a_{n_k}) = \frac{n_k - n_k'}{n_k} M(a_{n_k} a_{n_k})
\]
\[
\geq \frac{3}{4} \varepsilon \text{ for } k = 1, 2, \ldots
\]
Finally, $x$ cannot belong to $X_\varepsilon$, since it would belong to $X_\varepsilon$, $x$ belonging to $X_\varepsilon$. 2.2. The example of a modular which we shall give now is the generalized variation of a function. Let $M(a)$ be an even continuous function,
non-decreasing for \( u > 0 \), \( M(0) = 0 \), \( M(u) > 0 \) for \( u > 0 \). Given a real function \( x(t) \), defined in a closed finite interval \( \langle a, b \rangle \), the value

\[
V_M(x) = \sup \sum_{i=1}^M \left[ x(t_i) - x(t_{i-1}) \right],
\]

where \( M : a = t_0 < t_1 < \ldots < t_m = b \) is an arbitrary partition of the interval \( \langle a, b \rangle \), is called the \( M \)-th variation of \( x(t) \) in \( \langle a, b \rangle \) (see [3]). Denoting by \( X \) the class of all real functions in \( \langle a, b \rangle \) vanishing at \( a \), we define in \( X \)

\[
g(x) = V_M(x).
\]

Obviously \( g(x) \) is a modular and \( X^* \) is strongly \( g \)-complete. The authors defined a \( B \)-norm in the space \( X^* \) in the case when \( M(u) \) is a convex function in \( [0,\infty) \). Several results have been obtained for \( M(u) \) convex and satisfying the condition \( M(u) = o(u) \) as \( u \to 0 \). As an example of the modulars, an \( M \)-norm will be introduced in \( X^*_u \) for the opposite case, namely \( u = o(M(u)) \) as \( u \to 0 \). The following conditions will be of importance:

(\( \Lambda \)) \( M(u)/u \to \infty \) as \( u \to 0 \),

(\( \beta \)) there exists a constant \( \alpha \) such that \( M(u_1 + \ldots + u_\alpha) \leq \alpha M(u_1 + \ldots + u_\alpha) \) for \( u_1, \ldots, u_\alpha > 0 \).

Condition (\( \beta \)) is satisfied for example, if \( M(u) \) is concave or sub-additive for \( u > 0 \). The following lemma will be useful:

2.24. If \( M(u) \) satisfies (\( \Lambda \)), \( x \in X^*_u \) and \( x(t) \) is continuous in \( \langle a, b \rangle \), then \( x(t) \) is linear in \( \langle a, b \rangle \).

In order to prove 2.21, let us suppose \( x(t) \) to be continuous in \( \langle a, b \rangle \), \( x(a) = a, x(b) = \beta \), where \( c < a < \beta \), and let us take for each positive integer \( n \) a partition \( a = t_0 < t_1 < \ldots < t_m = \beta \) of \( \langle a, b \rangle \), such that \( x(t_i) = \alpha_i (d - \alpha_i) + c \); hence

\[
V_M(x) \geq \sum_{i=1}^m M[x(t_i) - x(t_{i-1})] = [d - \alpha_1] M(2^{-n} d - \alpha_1) \rightarrow \infty \text{ as } n \to \infty,
\]

and this contradicts the hypothesis \( x \in X^*_u \).

To formulate a further lemma, let us put for a function \( x(t) \) of bounded variation in the usual sense, \( x(t) \) and \( x(t) - x(t_0) \) are also of bounded variation in the usual sense; moreover, \( x(t) - x(t_0) \) is continuous in \( \langle a, b \rangle \). The following lemma holds:

2.22. Assuming \( M(u) \) to satisfy (\( \Lambda \)) and (\( \beta \)), we have \( x(t) \equiv x(t) \) in \( \langle a, b \rangle \) for any \( x \in X^*_u \).

If we choose an arbitrary partition \( a = t_0 < t_1 < \ldots < t_m = b \) of the interval \( \langle a, b \rangle \), we obtain

\[
\sum_{i=1}^m M[x(t_i) - x(t_{i-1})] \leq \sum_{i=1}^m \left[ \sum_{t_{i-1} < t_i < t_0} [x(t_0 + 0) - x(t_0)] + [x(t_i) - x(t_{i-1})] + [x(t_{i-1}) - x(t)] \right]
\]

\[
\leq \sum_{i=1}^m \left[ \sum_{t_{i-1} < t_i < t_0} M[x(t_0 + 0) - x(t_0)] + \sum_{t_{i-1} < t_i < t_0} M[x(t) - x(t_{i-0})] + \sum_{t_{i-1} < t_i < t_0} M[x(t_{i-1}) - x(t)] \right]
\]

\[
= \sum_{i=1}^m \sum_{t_{i-1} < t_i < t_0} M[x(t)] \leq 3M(x),
\]

where we write by convention \( x(t_0) = x(t_0) \); hence \( V_M(x) < 3M(x) \) and \( 1/2 \alpha \) of \( e \). Thus, Lemma 2.21 implies, by the continuity of \( x(t) - x(t_0) \), the equality \( x(t) - x(t_0) \text{ constant in } \langle a, b \rangle \). This yields \( x(t) = x(t_0) \) for every \( t \in \langle a, b \rangle \).

2.23. If \( M(u) \) satisfies the conditions (\( \Lambda \)) and (\( \beta \)), then \( X^*_u = X^*_u \).

Assuming \( x \in X^*_u \) and defining \( \widehat{x}(t) = x(t_0) \) for \( a < t < b \), \( \widehat{x}(a) = x(a) - 0 \) and \( \gamma(t) = x(t) - \widehat{x}(t) \), we easily obtain \( V_M(\widehat{x}) \leq V_M(x) \).

Therefore \( \gamma = 1/2 \gamma \) and \( x = 1/2 \gamma \).

The following inequalities hold:

\[
2 \sum_{i=1}^m M[x(t_i)] - M[x(t_0)] \leq V_M(x) \leq 2 \sum_{i=1}^m M[x(t_i)] + c
\]

Since \( V_M(x) < \infty \), the series \( \sum_{i=1}^m M[x(t_i)] \) is convergent. Let us fix the arrangement of the sequence \( t_0, t_1, \ldots \) and let us choose a number \( s > 0 \); then there exists an integer \( N \) such that

\[
\sum_{i=0}^{n-1} M[x(t_i)] \leq \sum_{i=0}^{n-1} M[x(t_i)] < 2/k \]

for any \( 0 < \alpha \leq 1 \). Moreover, \( M[x(t_i)] < 2/k \alpha \) for sufficiently small \( \alpha \); hence

\[
\sum_{i=0}^{n-1} M[x(t_i)] \leq 2/k \alpha
\]

and the second of the inequalities (+) implies \( V_M(x) < \epsilon \) for sufficiently
small $a$. The relation $\hat{x}^+X_e^*$ may be deduced from the inequalities

$$(+) \quad \frac{1}{n} \sum_{n \geq n} M [\delta(t_0 + 0) - \delta(t)] \leq V_M (\delta)$$

in the same way as the relation $\sigma^+X_e^*$ from the inequalities $(\ast)$. We still have to prove $(+)$.

Lemma 2.23 implies $\hat{x}(t) = \hat{x}(t_0) = \hat{x}(a + 0) + \sum_{t < t_0} [\hat{x}(t_0 + 0) - \hat{x}(t)]$ for $a < t < b$,

$$\hat{x}(a) = 0,$$

whence, given an arbitrary partition $a = \tau_1 < \tau_2 < \ldots < \tau_m = b$ of the interval $(a, b)$, we have

$$\sum_{t \in I_{\tau_k}} M [\hat{x}(\tau_k) - \hat{x}(\tau_{k-1})] = M [\hat{x}(a + 0) + \sum_{t \in I_{\tau_k}} [\hat{x}(t_0 + 0) - \hat{x}(t)]] +$$

$$+ \sum_{t \in I_{\tau_k}} \sum_{t \in I_{\tau_k + 1}} [\hat{x}(t_0 + 0) - \hat{x}(t)]$$

$$\leq M [\hat{x}(a + 0) + \sum_{t < t_0} [\hat{x}(t_0 + 0) - \hat{x}(t)]]$$

2.24. Let $M(u)$ satisfy the following condition: there exists a $u_0 > 0$ such that $\sup_{a \to 0} M(u_0)/M(u) \to 0$ as $a \to 0$. Then $X_e^* = X_e^*$.

It will be noted that all convex functions $M(u)$ satisfy 2.24.

2.25. If $M(u)$ satisfies the condition

$$M(2u) \leq K M(u) \text{ for small } u,$$

then $\psi(x)$ also satisfies the condition $(\Delta_2)$ (see 1.32(e)); hence $(2.2)$ holds for any sequence of elements of $X_e^*$.

2.3. In many known examples the space $X$ consists of $M$-integrable functions. These examples will be generalized as follows. Given a set $E$ and a $\sigma$-additive and $\sigma$-finite measure $\mu$ defined on a $\sigma$-algebra $\mathcal{F}$ of subsets of the set $E$, we take a real function $M(u, v)$ defined in $\mathbb{R} \times E$, $\mathbb{R}'$ being the space of reals, satisfying the following conditions:

(a) $M(u, v) > 0$; $M(u, v) = 0$ if and only if $v = 0$,

(b) $M(u, v)$ is an even, continuous and non-decreasing (for $v > 0)$ function of $v$, for every $u \in E$,

(c) $M(u, v)$ is measurable as a function of $u$ for every $v \in \mathbb{R}'$.

We take as $X_1$, the set of all real-valued $\mu$-measurable functions $x(t)$ in $E$, finite almost everywhere on $E$. It is easily seen that $M[t, x(t)]$ is measurable on $E$ for any $x \in X_1$; denoting by $X$ the quotient space $X = X_1/X_\alpha$, $X_\alpha$ being the set of all $x(t) = 0$ almost everywhere, we may define a modular $\varphi(x)$ on $X$ by

$$\varphi(x) = \int_E M(t, x(t)) d\mu,$$

the uniqueness of this definition being granted by (a).

2.31. The modular $\varphi(x)$ satisfies the condition $X_e^* = X_e^*$; moreover, $X_e^*$ is strongly $\rho$-complete.

The equality $X_e^* = X_e^*$ being implied by the Lebesgue bounded-convergence theorem, we have only to prove the strong $\rho$-completeness.

First, let us assume $\mu(E) = \infty$. We apply the following lemma:

(*) If $f(t) > 0$ is measurable in a set of finite measure $E$, then for any $\varepsilon > 0$ there exists a $\nu > 0$ such that $\int f(t) d\mu < \eta$ implies $\mu(A) < \varepsilon$ for any $A \in E$, $\eta$ being independent of $A$.

Choose a number $\varepsilon > 0$. Writing $f(t) = M(t, e)$ we apply the above lemma. We find an $\eta > 0$ such that for every $A \in E$, $\int M(t, e) d\mu < \eta$ implies $\mu(A) < \varepsilon$. If a sequence $[x_n]$ satisfies the modular Cauchy condition, then we may find a number $N$ such that

$$\int_E M(t, x_n(t) - x_0(t)) d\mu < \eta$$

Writing $A_{m,n} = \{t \in E: |x_n(t) - x_0(t)| > \varepsilon\}$, we obtain

$$\int_{A_{m,n}} M(t, e) d\mu < \eta,$$

hence $\mu(A_{m,n}) < \varepsilon$ for $m, n > N$. Thus the sequence $x_n(t)$ is convergent in measure to a function $x(t)$ in $E$. Taking an arbitrary subsequence $x_{n_k}$, we may extract from $x_{n_k}$ a sequence $x_{n_k}(t) \to x(t)$ almost everywhere.

Hence $M[t, x_{n_k}(t) - x_{n_k}(t)] 	o M[t, x(t) - x(t)]$ almost everywhere, $n$ being fixed, and Fatou's lemma yields

$$\varphi(x_n - x) \leq \lim_{k \to \infty} \varphi(x_n - x_{n_k}) < \varepsilon$$

for sufficiently large $n$.

Now, let $\mu(E) = \infty$, $E = \bigcup_1^\infty E_k$, where $\mu E_k < \infty$ for $k = 1, 2, \ldots$ and the sets $E_k$ are ascending. Applying the above results we obtain a func-
tion \( x(t) \) such that \( x_n(t) \) converges in measure to \( x(t) \) in each \( E \) and that
\[
\int_{E} M\{t, x_n(t) - x(t)\}dmu \to 0 \text{ as } n \to \infty
\]
for each \( b \). Let us write for a fixed \( n \),
\[
y_n(t) = M\{t, x_n(t) - x(t)\}x_{E_n}(t),
\]
where \( x_{E_n}(t) \) denotes the characteristic function of the set \( E_n \),
\[
y(t) = M\{t, x_n(t) - x(t)\}.
\]
Applying once more the inequality (\( + \)) with \( E_n \) instead of \( E \) we obtain
\[
\int_{E} y_n(t)dmu = \int_{E} \left[ M\{t, x_n(t) - x(t)\}d\mu \right] \leq \lim_{k \to \infty} \int_{E} y_n(t)dmu \leq \varepsilon
\]
for sufficiently large \( k \) (independently of \( b \)). However, \( y_n(t) \) is convergent to \( y(t) \) everywhere; hence Fatou's lemma yields
\[
\int_{E} \left[ y_n(t)dmu \right] \leq \liminf_{k \to \infty} \int_{E} y_n(t)dmu \leq \varepsilon
\]
for sufficiently large \( k \).

2.32. (a) If there exists a constant \( \kappa > 0 \) such that \( E_1 \) denoting the set of all \( u \in E \) which satisfy the inequality \( M(u, 2v) \leq \kappa \mu(u, v) \) for all \( v \),
\[
\mu(B-E) = 0,
\]
then B.2 holds for any sequence of elements of \( X_{\pi}^* \).

(b) Let \( \mu \mu < \infty \), and assume that \( M(u, v) \) is integrable in \( E \) for each \( u \in E \) and satisfies the following condition: there exists a \( \varepsilon > 0 \) such that \( M(u, v) \) satisfies the inequality \( M(u, 2v) \leq \kappa \mu(u, v) \) for any \( u \in E \) and \( \varepsilon \mu \leq \kappa \mu \). Then B.2 holds for any sequence of elements of \( X_{\pi}^* \).

The assumption (a) easily implies the condition \( (A) \) for \( \varepsilon \); hence, (a) follows from 1.32 (c). To prove (b), let us write \( E_1 = \{ u \in E : M(u, 2v) \leq \kappa \mu(u, v) \} \), \( E_{n+1} = \{ t \in E : |x_{n}(t)| \leq |v_{n}(t)| \} \),
\[

3.33. Finally, it will be noted that example 2.12 may be obtained from the present example if we put \( E = \) the set of all positive integers,
\[
\mu(A) = \text{the number of elements of the set } A \subset E, \quad M(u, v) = M_{u, v}
\]
for \( u = v \). Another special case of the function \( M(u, v) \) may be obtained by putting
\[
M(u, v) = M_{u, v}\mu(v),
\]
where \( M(v) \) is an even, continuous and non-decreasing (for \( v > 0 \) function,
\[
M(0) = 0, \quad M(v) > 0 \text{ for } v > 0, \quad p(u) \text{ is a measurable positive function on } E.
\]

2.44. Given an even, continuous function \( M(u) \) non-decreasing for \( u > 0 \), \( M(0) = 0 \), \( M(u) > 0 \) for \( u > 0 \). We define, in the class \( X \) of all real functions \( x(t) \) measurable in \( (0, \infty) \), a pseudomodular \( \varrho(x) \) as follows:
\[
\varrho(x) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} M[x(t)]dt.
\]

2.441. The class \( X_{\pi}^* \) with the pseudomodular \( \varrho(x) \) defined above is strongly \( \varrho \)-complete. Moreover, \( X_{\pi}^* = X_{x}^* \) if and only if for every \( \varepsilon > 0 \) there exist numbers \( A_\varepsilon > 0 \) and \( A_\varepsilon > 0 \) such that \( M(u, v) < \varepsilon M(u) \) for any \( 0 \leq u \leq A_\varepsilon \).

First, we prove the strong \( \varrho \)-completeness of \( X_{\pi}^* \) (see also [1]). Let the sequence \( \{x_n \in X_{\pi}^* \} \) satisfy the modular Cauchy condition. Given an arbitrary sequence of numbers \( s_n \) decreasing to zero, let us choose an increasing sequence of indices \( n_1, n_2, \ldots \) satisfying the inequalities \( \varrho(x_n - x_{n+1}) < s_n \) for \( n > n_0 \). We now define a sequence \( T_1, T_2, \ldots \) by induction. Put \( T_1 = 0 \); if \( T_{k-1} < T_k \), \( T_{k+1}, \ldots \) are defined, we choose \( T_k \) with the following properties:

1. If \( T_k \leq T_{k+1} \),
\[
\frac{1}{T_k} \int_{0}^{T_k} M(x_n(t) - x_n(0))dt \leq s_k \quad \text{for } k = 1, 2, \ldots, i-1,
\]
\[
\frac{1}{T_k} \int_{0}^{T_k} M(x_n(t) - x_{n+1}(t))dt \leq s_i;
\]

2. \( T_{k+1} \leq 2T_k - 1 \).

Now, we define the function \( x(t) \) by the equalities \( x(t) = x_n(t) \) for \( T_{i-1} \leq t \leq T_i \). We shall prove that \( \varrho(x(t) - x_n(t)) \to 0 \) as \( n \to \infty \). Take an arbitrary index \( k \), \( n \geq k \) and a positive number \( T_k \), where \( T_{k+1} \leq T_k < T_{k+1} \). Then
\[
\frac{1}{T_k} \int_{0}^{T_k} M(x_n(t) - x(t))dt \leq \frac{1}{T_k} \int_{0}^{T_k} M(x_n(t) - x_{n+1}(t))dt + \frac{1}{T_k} \sum_{i=1}^{k} \int_{T_{i-1}}^{T_i} M(x_n(t) - x_{n+1}(t))dt + \frac{1}{T_k} \int_{0}^{T_k} M(x_n(t) - x_{n+1}(t))dt.
\]
Since
\[
\frac{1}{T} \sum_{n}^{\infty} \int_{\xi_{n}^{-1}}^{\xi_{n}} M\left[\nu_{m}(t) - \nu_{n}(t)\right] dt \leq \frac{1}{T_{m}} \sum_{n=1}^{m} T_{n} \int_{t_{n}}^{t_{n+1}} M\left[\nu_{m}(t) - \nu_{n}(t)\right] dt
\]
and
\[
\frac{1}{T} \int_{\xi_{m}}^{\xi_{m}} M\left[\nu_{m}(t) - \nu_{m+1}(t)\right] dt \leq \frac{1}{T} \int_{\xi_{m}}^{\xi_{m}} M\left[\nu_{m}(t) - \nu_{n}(t)\right] dt + \int_{\xi_{m}}^{\xi_{m+1}} M\left[\nu_{m+1}(t) - \nu_{m}(t)\right] dt < 2\varepsilon_{m}
\]
we have \(g\left(\frac{1}{2}(n_{m} - n)\right) \leq 3\varepsilon_{m} \to 0\) as \(k \to \infty\). Hence,
\[
g\left(\frac{1}{2}(n_{m} - n)\right) \leq g\left(\frac{1}{2}(n_{m} - n)\right) + g\left(\frac{1}{2}(n_{m} - a)\right) \leq 4\varepsilon_{m} \quad \text{for} \quad n \geq a_{k}
\]
In order to prove the sufficiency in the second part of 2.41, let us take a sequence \(a_{m} \to 0\). Writing \(E_{1} = \{t \geq 0: \|x(t)\| \leq A_{1}\}\) and \(E_{2} = (0, \infty) - E_{1}\), we obtain
\[
\frac{1}{T} \int_{E_{1}}^{E_{1}} M\left[\nu_{m}(t)\right] dt = \frac{1}{T} \int_{E_{1}}^{E_{1}} M\left[\nu_{m}(t)\right] dt + \frac{1}{T} \int_{E_{2}}^{E_{2}} M\left[\nu_{m}(t)\right] dt
\]
for \(0 \leq a_{m} \leq a_{n}\). In proving the necessity, we apply the method used in the proof of 2.15, with an arbitrary sequence \(v_{m}\) such that \(M(v_{m}) > \frac{1}{2}\) for \(m = 1, 2, \ldots\) instead of the sequence \(u_{m}\). Defining the sequences \(a_{1}, a_{2}\) and \(a_{3}\) as in 2.15 but with \(v_{m}\) instead of \(u_{m}\), we obtain
\[
\frac{1}{n} \sum_{k=1}^{n} M(a_{m}) < \frac{3}{2}
\]
for \(k = 1, 2, \ldots; \; n = 1, 2, \ldots\). Then we choose \(v_{m} = u_{m}, u_{m}\) being defined as in the proof of 2.15, where \(v_{k}\) is the sequence \(1, 1, 2, 1, 2, 3, \ldots, ..., 1, 2, 3, \ldots, u_{m}, \ldots\) and put \(x(t) = a_{m} \in (t_{1}, t_{2})\); thus, we obtain \(g(x) < \frac{1}{2}\) and \(g(a_{m}) \geq \frac{1}{2} + \varepsilon\), the sequence \(a_{m}\) and the number \(\varepsilon > 0\) being chosen as in 2.15.

\textbf{2.42.} The class \(X^{*}\) is an \(F\)-space with respect to the pseudonorm induced by the pseudomodular \(g(x)\), assuming \(g(x) = 0\) implies
\(g(2x) = 0\) (1). Denote by \(X_{1}\) the class of all \(x \in X_{1}^{*}\) such that the pseudonorm \(\|x\| = 0\); then the quotient space \(X_{1}^{*}/X_{0}\) is an \(F\)-space complete with respect to the norm generated by the above pseudonorm.

\textbf{2.45.} Let us note that the conditions sufficient for any sequence of elements of \(X_{1}^{*}\) to satisfy B.2 may be formulated similarly to 2.32.

\textbf{References}


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\(g(x) = 0\) (1) The theorem remains true without last hypothesis. Namely, the norm-completeness of the \(X_{1}^{*}\) may be proved directly by a slight modification of the argument of the proof on p. 63-64.

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