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The limiting distributions of cumulative sums of independent two-valued random variables

by

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K. Urbanik [3] has raised the following problem:

Let $X_1, X_2, \ldots$ be a sequence of independent random variables such that each of them takes at most 2 values. Find the class $\mathcal{T}$ of all possible limiting distributions of the sequence

$$Y_n = \frac{1}{B_n}\sum_{k=1}^{n} X_k - A_n \qquad (n = 1, 2, \ldots) \tag{1}$$

where A_n, B_n are constants, $B_n \to \infty$ and the random variables $X_1/B_n, \ldots, X_n/B_n$ are uniformly asymptotically negligible, i. e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant n} P\left(\left|\frac{X_k}{B_n}\right| > \varepsilon\right) = 0.$$

In this paper we shall partially solve this problem. Let us denote by $\mathcal{R}$ the subclass of all limiting distributions of Y_n defined by (1) if it is additionally assumed that the variances of Y_n converge to that of the limiting distribution. Our aim is to characterise the subclass $\mathcal{R}$.

Let $\mathcal{K}$ denote the class of all infinitely divisible distributions for which the function $K(u)$ in Kolmogorov's formula (see [2], § 18) is of the form

$$K(u) = \begin{cases} 0 & \text{for} \quad u < A, \\ \mu - \dfrac{\mu}{A^2}u^2 & \text{for} \quad A \leqslant u < 0, \\ \dfrac{\nu}{B^2}u^2 + (1-\nu) & \text{for} \quad 0 < u \leqslant B, \\ 1 & \text{for} \quad u > B, \end{cases} \tag{2}$$

where $\nu, \mu \geqslant 0$, $\nu + \mu \leqslant 1$, $A \leqslant 0$, $B \geqslant 0$.

THEOREM. *The class $\mathcal{R}$ is equal to the class $\mathcal{K}$.*

The proof of this theorem gives also a method of getting each distribution of the class $\mathcal{K}$ as the limit of sequence (1).

Proof. We shall use here theorem 1, § 31 of [2]. Let

$$P(X_k = a_k) = p_k, \quad P(X_k = b_k) = q_k = 1 - p_k$$

where $a_k \geqslant b_k$ and let us write $z_k = a_k - b_k$. Let us put

$$B_n = \sqrt{\sum_{k=1}^{n} D^2(X_k)}, \qquad A_n = \frac{1}{B_n}\sum_{k=1}^{n} E(X_k) \tag{3}$$

and let us introduce the random variable

$$\xi_{nk} = \frac{X_k - E(X_k)}{\sqrt{\sum_{k=1}^{n} D^2(X_k)}}.$$

We have of course

$$Y_n = \sum_{k=1}^{n} \xi_{nk}$$

and

$$P\left(\xi_{nk} = \frac{z_k q_k}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right) = p_k, \quad P\left(\xi_{nk} = \frac{-z_k p_k}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right) = q_k.$$

Let us assume that the random variables ξ_{nk} $(k = 1, 2, \ldots, n)$ are uniformly asymptotically negligible and that there exists a limiting distribution for the sequence $Y_n = \sum_{k=1}^{n} \xi_{nk}$. We shall prove that this distribution belongs to the class $\mathcal{K}$.

In order to obtain the non-singular distributions we must assume that

$$\sum_{k=1}^{\infty} z_k^2 p_k q_k = \infty.$$

Since the random variables ξ_{nk} $(k = 1, 2, \ldots, n)$ are uniformly asymptotically negligible, then for every sequence $k(n) \to \infty$ $(k(n) \leqslant n)$

$$\lim_{n\to\infty} \min\left\{p_{k(n)}, q_{k(n)}, \max\left[\frac{z_{k(n)} q_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}, \frac{z_{k(n)} p_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right]\right\} = 0.$$

Indeed, let us suppose that there exist such a sequence $k(n) \to \infty$ $(k(n) \leqslant n)$, such a positive integer N and such $\varepsilon_0 > 0$ that for $n > N$

$$p_{k(n)} > \varepsilon_0, \quad q_{k(n)} > \varepsilon_0, \quad \max\left[\frac{z_{k(n)} q_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}, \frac{z_{k(n)} p_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right] > \varepsilon_0.$$

Therefore

$$P(|\xi_{nk(n)}| > \varepsilon_0) > \varepsilon_0 \qquad (n > N)$$

and this contradicts the assumption that ξ_{nk} $(k = 1, 2, \ldots, n)$ are uniformly asymptotically negligible.

Let us now denote by $\mathfrak{N}(n)$ the set of the first n positive integers:

$$\{1, 2, \ldots, n\} = \mathfrak{N}(n).$$

Let us denote by $\mathfrak{P}(n)$ and $\mathfrak{Q}(n)$ such subsets of $\mathfrak{N}(n)$ that

$$\min\left\{p_{k(n)}, q_{k(n)}, \max\left[\frac{z_{k(n)} q_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}, \frac{z_{k(n)} p_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right]\right\} = \begin{cases} p_{k(n)} & \text{for } k(n) \in \mathfrak{P}(n), \\ q_{k(n)} & \text{for } k(n) \in \mathfrak{Q}(n). \end{cases}$$

If for a k we have $p_k = q_k$, then let $k \in \mathfrak{P}(n)$ so that $\mathfrak{P}(n) \cap \mathfrak{Q}(n) = 0$. Let us further write

$$\mathfrak{M}(n) = \mathfrak{N}(n) \setminus (\mathfrak{P}(n) \cup \mathfrak{Q}(n)).$$

For $k(n) \in \mathfrak{M}(n)$ we have of course

$$\lim_{n\to\infty} \max\left[\frac{z_{k(n)} q_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}, \frac{z_{k(n)} p_{k(n)}}{\sqrt{\sum_{k=1}^{n} z_k^2 p_k q_k}}\right] = 0.$$

Let us assume that $\mathfrak{P}(n) = \mathfrak{N}(n)$. Let $k_1^{(n)}, k_2^{(n)}, \ldots, k_n^{(n)}$ denote a permutation of the numbers $1, 2, \ldots, n$ for which the following inequalities hold:

$$0 \leqslant z_{k_1^{(n)}} q_{k_1^{(n)}} \leqslant \ldots \leqslant z_{k_n^{(n)}} q_{k_n^{(n)}}.$$

Let us write $M_k = z_k^2 p_k q_k$. The distribution function $F_{nk}(x)$ of the random variable ξ_{nk} is of course given by the formula

$$F_{nk}(x)=\begin{cases}0 & \text{for} \quad x\leqslant -\sqrt{\dfrac{M_kp_k}{q_k\sum\limits_{k=1}^{n}M_k}},\\[2ex] q_k & \text{for} \quad -\sqrt{\dfrac{M_kp_k}{q_k\sum\limits_{k=1}^{n}M_k}}<x\leqslant\sqrt{\dfrac{M_kq_k}{p_k\sum\limits_{k=1}^{n}M_k}},\\[2ex] 1 & \text{for} \quad x>\sqrt{\dfrac{M_kq_k}{p_k\sum\limits_{k=1}^{n}M_k}},\end{cases}$$

whence

$$\int\limits_{-\infty}^{u}x^2dF_{nk}(x)=\begin{cases}0 & \text{for} \quad u\leqslant -\sqrt{\dfrac{M_kp_k}{q_k\sum\limits_{k=1}^{n}M_k}},\\[2ex] \dfrac{M_kp_k}{\sum\limits_{k=1}^{n}M_k} & \text{for} \quad -\sqrt{\dfrac{M_kp_k}{q_k\sum\limits_{k=1}^{n}M_k}}<u\leqslant\sqrt{\dfrac{M_kq_k}{p_k\sum\limits_{k=1}^{n}M_k}},\\[2ex] \dfrac{M_kp_k+M_kq_k}{\sum\limits_{k=1}^{n}M_k} & \text{for} \quad u>\sqrt{\dfrac{M_kq_k}{p_k\sum\limits_{k=1}^{n}M_k}}\end{cases}$$

and

$$K_n(u)=\sum_{k=1}^{n}\int\limits_{-\infty}^{u}x^2dF_{nk}(x)$$

$$=\begin{cases}\dfrac{\sum\limits_{k=1}^{n}M_kp_k}{\sum\limits_{k=1}^{n}M_k} & \text{for} \quad 0<u\leqslant\sqrt{\dfrac{M_{k_1^{(n)}}q_{k_1^{(n)}}}{p_{k_1^{(n)}}\sum\limits_{k=1}^{n}M_k}},\\ \cdots\cdots\cdots & \cdots\cdots\cdots\\ \dfrac{\sum\limits_{k=1}^{n}M_kp_k+\sum\limits_{j=1}^{l}M_{k_j^{(n)}}q_{k_j^{(n)}}}{\sum\limits_{k=1}^{n}M_k} & \text{for} \quad \sqrt{\dfrac{M_{k_l^{(n)}}q_{k_l^{(n)}}}{p_{k_l^{(n)}}\sum\limits_{k=1}^{n}M_k}}<u\leqslant\sqrt{\dfrac{M_{k_{l+1}^{(n)}}q_{k_{l+1}^{(n)}}}{p_{k_{l+1}^{(n)}}\sum\limits_{k=1}^{n}M_k}},\\ \cdots\cdots\cdots & \cdots\cdots\cdots\\ 1 & \text{for} \quad u>\sqrt{\dfrac{M_{k_n^{(n)}}q_{k_n^{(n)}}}{p_{k_n^{(n)}}\sum\limits_{k=1}^{n}M_k}}.\end{cases}$$

We shall now prove that one can obtain in the limit only the function $K(u)$ of the form

$$K(u)=\lim_{n\to\infty}K_n(u)=\begin{cases}0 & \text{for} \quad u\leqslant 0,\\ \dfrac{1}{B^2}u^2 & \text{for} \quad 0<u\leqslant B, \quad (B=\text{const}),\\ 1 & \text{for} \quad u>B.\end{cases} \tag{4}$$

Let us consider first the case when

$$\sqrt{\frac{M_{k_n^{(n)}}q_{k_n^{(n)}}}{p_{k_n^{(n)}}\sum\limits_{k=1}^{n}M_k}}\underset{n\to\infty}{\to}B,$$

i. e. when

$$p_{k_n^{(n)}}=\frac{M_{k_n^{(n)}}}{\sum\limits_{j=1}^{n}M_{k_j^{(n)}}}\cdot a_{k_n^{(n)}} \quad \text{where} \quad \lim_{n\to\infty}a_{k_n^{(n)}}=\frac{1}{B^2}.$$

We thus have

$$p_{k_l^{(n)}}=\frac{M_{k_l^{(n)}}}{\sum\limits_{j=1}^{l}M_{k_j^{(n)}}}\cdot a_{k_l^{(n)}} \quad (l=1,2,\ldots,n).$$

Let us take

$$u_{k_{l(n)}^{(n)}}=\sqrt{\frac{M_{k_{l(n)}^{(n)}}q_{k_{l(n)}^{(n)}}}{p_{k_{l(n)}^{(n)}}\sum\limits_{k=1}^{n}M_k}}$$

and let $\lim\limits_{n\to\infty}k_{l(n)}^{(n)}=\infty$. We have

$$K_n(u_{k_{l(n)}^{(n)}}+0)-\frac{1}{B^2}u^2_{k_{l(n)}^{(n)}}=\frac{\sum\limits_{k=1}^{n}M_kp_k+\sum\limits_{j=1}^{l(n)}M_{k_j^{(n)}}q_{k_j^{(n)}}-\dfrac{1}{B^2}\cdot\dfrac{M_{k_{l(n)}^{(n)}}q_{k_{l(n)}^{(n)}}}{p_{k_{l(n)}^{(n)}}}}{\sum\limits_{k=1}^{n}M_k}.$$

According to the theorem of Stolz ([1], p. 81) we have

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}M_kp_k}{\sum\limits_{k=1}^{n}M_k}=\lim_{n\to\infty}\frac{M_np_n}{M_n}=0.$$

Since

$$\lim_{n\to\infty}\frac{1}{B^2}\cdot\frac{q_{k_{l(n)}^{(n)}}}{a_{k_{l(n)}^{(n)}}}=1,$$

we have

$$\frac{1}{B^2}\cdot\frac{q_{k_{l(n)}^{(n)}}}{a_{k_{l(n)}^{(n)}}}=1+\beta_{k_{l(n)}^{(n)}} \quad\text{where}\quad \lim_{n\to\infty}\beta_{k_{l(n)}^{(n)}}=0.$$

In view of this we have

$$\frac{\sum_{j=1}^{l(n)} M_{k_j^{(n)}} q_{k_j^{(n)}} - \frac{1}{B^2}\cdot\frac{M_{k_{l(n)}^{(n)}} q_{k_{l(n)}^{(n)}}}{p_{k_{l(n)}^{(n)}}}}{\sum_{k=1}^{n} M_k}$$

$$= \frac{\sum_{j=1}^{l(n)} M_{k_j^{(n)}} q_{k_j^{(n)}} - \frac{1}{B^2}\cdot\frac{q_{k_{l(n)}^{(n)}}}{a_{k_{l(n)}^{(n)}}}\sum_{j=1}^{l(n)} M_{k_j^{(n)}}}{\sum_{k=1}^{n} M_k} = \frac{-\sum_{j=1}^{l(n)} M_{k_j^{(n)}} p_{k_j^{(n)}} - \beta_{k_{l(n)}^{(n)}}\sum_{j=1}^{l(n)} M_{k_j^{(n)}}}{\sum_{k=1}^{n} M_k}.$$

We have further

$$\frac{\sum_{j=1}^{l(n)} M_{k_j^{(n)}} p_{k_j^{(n)}}}{\sum_{k=1}^{n} M_k} \leqslant \frac{\sum_{k=1}^{n} M_k p_k}{\sum_{k=1}^{n} M_k} \underset{n\to\infty}{\longrightarrow} 0,$$

$$\frac{\beta_{k_{l(n)}^{(n)}}\sum_{j=1}^{l(n)} M_{k_j^{(n)}}}{\sum_{k=1}^{n} M_k} \leqslant \frac{\beta_{k_{l(n)}^{(n)}}\sum_{k=1}^{n} M_k}{\sum_{k=1}^{n} M_k} \underset{n\to\infty}{\longrightarrow} 0.$$

It follows hence that

$$\lim_{n\to\infty}\left[K_n\left(u_{k_{l(n)}^{(n)}}+0\right)-\frac{1}{B^2}\,u^2_{k_{l(n)}^{(n)}}\right]=0 \tag{5}$$

for all $k_{l(n)}^{(n)}$ such that $\lim_{n\to\infty} k_{l(n)}^{(n)}=\infty$. For the bounded $k_{l(n)}^{(n)}$'s equality (5) results immediately. We thus have (4).

Now let

$$\lim_{n\to\infty}\frac{M_{k_n^{(n)}}\,q_{k_n^{(n)}}}{p_{k_n^{(n)}}\sum_{k=1}^{n} M_k}=\infty.$$

We shall prove that in this case there exists no limiting distribution belonging to the class $\mathfrak{R}$. Let us assume for the sake of simplicity that $0\leqslant z_1 q_1\leqslant\ldots\leqslant z_n q_n$. Then

$$K_n(u)=\frac{\sum_{l=1}^{n} M_l p_l+\sum_{l=1}^{k} M_l q_l}{\sum_{l=1}^{n} M_l}\quad\text{for}\quad \sqrt{\frac{M_k q_k}{p_k\sum_{l=1}^{n} M_l}}<u\leqslant\sqrt{\frac{M_{k+1}q_{k+1}}{p_{k+1}\sum_{l=1}^{n} M_l}}.$$

We have assumed that

$$\lim_{n\to\infty}\frac{M_n q_n}{p_n\sum_{k=1}^{n} M_k}=\infty.$$

However, this relation can hold only when either

1° $\lim_{n\to\infty}\frac{M_n}{\sum_{k=1}^{n} M_k}=a>0$ or

2° $\lim_{n\to\infty}\frac{M_n}{\sum_{k=1}^{n} M_k}=0$ and $p_n=o\left(\frac{M_n}{\sum_{k=1}^{n} M_k}\right)$.

In case 1° one can get only a singular limiting distribution. For if the limiting distribution is non-singular the relation

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n}=1$$

holds. Hence taking into account our definition of B_n we shall have

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1} M_k}{\sum_{k=1}^{n} M_k}=1+\lim_{n\to\infty}\frac{M_{n+1}}{\sum_{k=1}^{n} M_k}=1.$$

The last relation implies

$$\lim_{n\to\infty}\frac{M_n}{\sum_{k=1}^{n} M_k}=0,$$

which contradicts 1°.

Let us consider case 2°. There exists such a sequence $k(n)$ that

$$\lim_{n\to\infty}\frac{M_{k(n)}}{p_{k(n)}\sum_{k=1}^{n}M_k}=a^2>0.$$

(If such a sequence does not exist, then there is no limiting distribution.) Let us take

$$u=u_{k(n)}=\sqrt{\frac{M_{k(n)}q_{k(n)}}{p_{k(n)}\sum_{k=1}^{n}M_k}}.$$

We have

$$\lim_{n\to\infty}K_n(u_{k(n)}+0)=\lim_{n\to\infty}\frac{\sum_{l=1}^{k(n)}M_lq_l}{\sum_{k=1}^{n}M_k}\leqslant\lim_{n\to\infty}\frac{\sum_{l=1}^{k(n)}M_l}{\sum_{k=1}^{n}M_k}$$

$$=\lim_{n\to\infty}\frac{p_{k(n)}\sum_{l=1}^{k(n)}M_l}{M_{k(n)}}\cdot\frac{M_{k(n)}}{p_{k(n)}\sum_{k=1}^{n}M_k}=0\cdot a^2=0.$$

We should then get only a limiting distribution with the function $K(u)$ of the form

$$K(u)=\begin{cases}0 & \text{for}\quad u\leqslant a,\\ 1 & \text{for}\quad u>a.\end{cases}$$

Now this is impossible since for the distributions of the class $\mathcal{L}$ the function $K(u)$ is continuous for $u\neq 0$.

If we assume that $\mathfrak{Q}(n)=\mathfrak{N}(n)$, then in a similar way we prove that in the limit one can get only a distribution with the function $K(u)$ of the form

$$K(u)=\begin{cases}0 & \text{for}\quad u<A,\\ 1-\dfrac{u^2}{A^2} & \text{for}\quad A\leqslant u<0,\\ 1 & \text{for}\quad u\geqslant 0.\end{cases}$$

In the case of $\mathfrak{M}(n)=\mathfrak{N}(n)$ we get

$$K(u)=\begin{cases}0 & \text{for}\quad u\leqslant 0,\\ 1 & \text{for}\quad u>0.\end{cases}$$

In the general case, when the number of elements in each of the sets $\mathfrak{P}(n)$, $\mathfrak{Q}(n)$ and $\mathfrak{M}(n)$ diverges to infinity as $n\to\infty$, the indices $k(n)$ which belong to $\mathfrak{M}(n)$ give at the point $u=0$ a jump $\lambda=1-\nu-\mu<1$ and the other indices give for $u<0$ and for $u>0$ the arc of a parabola. Consequently we get a limiting distribution with the function $K(u)$ given by formula (2). The idea of the proof in the general case is the same as in the particular case under consideration. We thus have proved that

$$\mathcal{R}\subset\mathcal{K}. \tag{6}$$

Now we shall prove that $\mathcal{K}\subset\mathcal{R}$. Let us take an arbitrary function $K(u)$ of the form (2). We shall show that one can so choose the two-valued random variables $X_1, X_2, \ldots$ that Kolmogorov's function of the limiting distribution of sequence (1) (with A_n, B_n of the form (3)) is equal to our function $K(u)$.

Let us suppose first that $\nu\neq 0$, $\mu\neq 0$. Let us write the set $\mathfrak{N}(n)$ (see p. 297) in the form $\mathfrak{N}(n)=\mathfrak{P}(n)\cup\mathfrak{Q}(n)\cup\mathfrak{M}(n)$.

Let us denote the elements of the set $\mathfrak{P}(n)$ by $k_1, k_2, \ldots, k_{\varphi(n)}$ and the elements of the set $\mathfrak{Q}(n)$ by $l_1, l_2, \ldots, l_{\psi(n)}$. Let us put

$$\varphi(n)\overset{\text{def}}{=}[n\nu].$$

Let us further define k_i as being equal to the smallest positive integer k for which $[k\nu]=i$. Let us denote the elements of the set $\mathfrak{N}(n)\setminus\mathfrak{P}(n)$ by $m_1<m_2<\ldots<m_{n-\varphi(n)}$. Let us put

$$\psi(n)\overset{\text{def}}{=}[n\mu].$$

If $\psi(l-1)=i-1$, $\psi(l)=i$, we put

$$l_i\overset{\text{def}}{=}m_{l-\varphi(l)}.$$

Let us observe that

$$\lim_{n\to\infty}\frac{\varphi(n)}{n}=\nu, \tag{7}$$

$$\lim_{n\to\infty}\frac{\psi(n)}{n}=\mu. \tag{8}$$

Let us now suppose that for an integer s

$$[(n-s-1)\nu]=\varphi(n-s)-1,$$

$$[(n-s)\nu]=[(n-s+1)\nu]=\ldots=[n\nu]=\varphi(n-s),$$

$$[(n+1)\nu]=\varphi(n-s)+1.$$

We thus have $k_{\varphi(n-s)} = n-s$, $k_{\varphi(n+1)} = n+1$, $k_{\varphi(n-i)} = n-s$ $(i = 0, 1, \ldots, s-1)$. It follows that $s\nu < 1$, i. e. $s < 1/\nu$.

Let us find the limit $\lim\limits_{n\to\infty} k_{\varphi(n)}/n$. For fixed n one can find such an integer s that $k_{\varphi(n-s)} = n-s$ (we do not assume now that $k_{\varphi(n)} = n$). We have

$$n-\frac{1}{\nu} < n-s = k_{\varphi(n-s)} \leqslant k_{\varphi(n)} \leqslant n,$$

whence

$$\lim_{n\to\infty}\frac{k_{\varphi(n)}}{n} = 1. \tag{9}$$

Let us now suppose that for an integer s

$$\begin{cases} [(n-s-1)\mu] = \psi(n-s)-1, \\ [(n-s)\mu] = [(n-s+1)\mu] = \ldots = [n\mu] = \psi(n-1), \\ [(n+1)\mu] = \psi(n-s)+1. \end{cases} \tag{10}$$

We thus have $l_{\psi(n-s)} = m_{n-s-\varphi(n-s)}$, $l_{\psi(n+1)} = m_{n+1-\varphi(n+1)}$, $l_{\psi(n-i)} = m_{n-s-\varphi(n-1)}$ $(i = 0, 1, \ldots, s-1)$. It follows that $s\mu < 1$, i. e.

$$s < \frac{1}{\mu}. \tag{11}$$

Let us now suppose that for an integer r

$$[(n-1)\nu] = [n\nu] < [(n+1)\nu] < \ldots < [(n+r)\nu] = [(n+r+1)\nu].$$

We thus have $\varphi(n-1) = \varphi(n) < \varphi(n+1) < \ldots < \varphi(n+r-1) < \varphi(n+r) = \varphi(n+r+1)$, $[(n+r+1)\nu] = [(n+r)\nu] = [n\nu]+r$, whence $r\nu > r-1$, i. e.

$$r < \frac{1}{1-\nu}. \tag{12}$$

Let us find the limit $\lim\limits_{n\to\infty} l_{\psi(n)}/n$. For fixed n one can find such an integer s that $l_{\psi(n-s)} = m_{n-s-\varphi(n-s)}$ and such an integer r that $m_{n-s-r-\varphi(n-s-r)} = n-s-r$. In view of (11) and (12) we have

$$n-\frac{1}{\mu}-\frac{1}{1-\nu} < n-s-r = m_{n-s-r-\varphi(n-s-r)} \leqslant m_{n-s-\varphi(n-s)}$$
$$= l_{\psi(n-s)} \leqslant l_{\psi(n)} \leqslant m_{n-\varphi(n)} \leqslant n,$$

whence

$$\lim_{n\to\infty}\frac{l_{\psi(n)}}{n} = 1. \tag{13}$$

We shall now prove that $\mu l_k - k$ $(k \leqslant \psi(n))$ is (with $n\to\infty$) bounded. One can so choose k', s and r that $k = \psi(k')$ and

$$l_k = l_{\psi(k')} \leqslant l_{\psi(k'+s)} = m_{k'+s-\varphi(k'+s)} \leqslant m_{k'+s+r-\varphi(k'+s+r)}$$
$$= k'+s+r < k'+\frac{1}{\mu}+\frac{1}{1-\nu}.$$

Similarly

$$l_k > k'-\frac{1}{\mu}-\frac{1}{1-\nu}.$$

Consequently we have

$$\mu k'-1-\frac{\mu}{1-\nu}-k < \mu l_k - k < \mu k'+1+\frac{\mu}{1-\nu}-k.$$

Since $k = \psi(k') = [k'\mu]$, we have

$$-1-\frac{\mu}{1-\nu} \leqslant \mu k' - [\mu k'] - 1 - \frac{\mu}{1-\nu} \tag{14}$$
$$< \mu l_k - k < \mu k' - [\mu k'] + 1 + \frac{\mu}{1-\nu} \leqslant 1+\frac{\mu}{1-\nu}.$$

Now we shall define the two-valued random variables $X_1, X_2, \ldots$ (i. e. we shall define p_k and z_k). There exist of course such integers i_0 and i_1 that $B^2k_i - 1 > 0$ for $i \geqslant i_0$ and $A^2l_i - 1 > 0$ for $i \geqslant i_1$. Let us put

$$p_{k_i} = \begin{cases} \dfrac{1}{B^2k_{i_0}} & \text{for } 1 \leqslant i < i_0, \\ \dfrac{1}{B^2k_i} & \text{for } i \geqslant i_0, \end{cases} \qquad z_{k_i} = \begin{cases} \dfrac{B^2k_{i_0}}{\sqrt{B^2k_{i_0}-1}} & \text{for } 1 \leqslant i < i_0, \\ \dfrac{B^2k_i}{\sqrt{B^2k_i-1}} & \text{for } i \geqslant i_0, \end{cases} \tag{15}$$

$$q_{l_i} = \begin{cases} \dfrac{1}{A^2l_{i_1}} & \text{for } 1 \leqslant i < i_1, \\ \dfrac{1}{A^2l_i} & \text{for } i \geqslant i_1, \end{cases} \qquad z_{l_i} = \begin{cases} \dfrac{A^2l_{i_1}}{\sqrt{A^2l_{i_1}-1}} & \text{for } 1 \leqslant i < i_1, \\ \dfrac{A^2l_i}{\sqrt{A^2l_i-1}} & \text{for } i \geqslant i_1, \end{cases} \tag{16}$$

$$p_k = \tfrac{1}{2}, \quad z_k = 2 \quad \text{for} \quad k \in \mathfrak{M}(n) = \mathfrak{N}(n) \setminus (\mathfrak{P}(n) \cup \mathfrak{Q}(n)). \tag{17}$$

The random variables

$$\xi_{nk} = \frac{X_k - E(X_k)}{\sqrt{\sum_{k=1}^{n} D^2(X_k)}} \qquad (k = 1, 2, \ldots, n)$$

are of course uniformly asymptotically negligible. Let us write as before $M_k = z_k^2 p_k q_k$. The function

$$K_n(u) = \sum_{k=1}^{n} \int_{-\infty}^{u} x^2 dF_{nk}(x)$$

can be written in the form

$$K_n(u) = K_n^{(\mathfrak{P})}(u) + K_n^{(\mathfrak{Q})}(u) + K_n^{(\mathfrak{M})}(u),$$

where

$$K_n^{(\mathfrak{P})}(u) = \begin{cases} \dfrac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j}}{\sum_{j=1}^{n} M_j} & \text{for } 0 < u \leqslant \sqrt{\dfrac{M_{k_1} q_{k_1}}{p_{k_1} \sum_{j=1}^{n} M_j}}, \\ \cdots\cdots\cdots & \\ \dfrac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j} + \sum_{j=1}^{l} M_{k_j} q_{k_j}}{\sum_{j=1}^{n} M_j} & \text{for } \sqrt{\dfrac{M_{k_l} q_{k_l}}{p_{k_l} \sum_{j=1}^{n} M_j}} < u \leqslant \sqrt{\dfrac{M_{k_{l+1}} q_{k_{l+1}}}{p_{k_{l+1}} \sum_{j=1}^{n} M^j}}, \\ \cdots\cdots\cdots & \\ \dfrac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j} + \sum_{j=1}^{\varphi(n)} M_{k_j} q_{k_j}}{\sum_{j=1}^{n} M_j} & \text{for } u > \sqrt{\dfrac{M_{k_{\varphi(n)}} q_{k_{\varphi(n)}}}{p_{k_{\varphi(n)}} \sum_{j=1}^{n} M_j}}, \end{cases}$$

$$K_n^{(\mathfrak{Q})}(u) = \begin{cases} 0 & \text{for } u \leqslant -\sqrt{\dfrac{M_{l_{\psi(n)}} p_{l_{\psi(n)}}}{q_{l_{\psi(n)}} \sum_{i=1}^{n} M_i}}, \\ \cdots\cdots\cdots & \\ \dfrac{\sum_{i=k}^{\psi(n)} M_{l_i} p_{l_i}}{\sum_{i=1}^{n} M_i} & \text{for } -\sqrt{\dfrac{M_{l_k} p_{l_k}}{q_{l_k} \sum_{i=1}^{n} M_i}} < u \leqslant -\sqrt{\dfrac{M_{l_{k-1}} p_{l_{k-1}}}{q_{l_{k-1}} \sum_{i=1}^{n} M_i}}, \\ \cdots\cdots\cdots & \\ \dfrac{\sum_{i=1}^{\psi(n)} M_{l_i} p_{l_i}}{\sum_{i=1}^{n} M_i} & \text{for } -\sqrt{\dfrac{M_{l_1} p_{l_1}}{q_{l_1} \sum_{i=1}^{n} M_i}} < u < 0, \end{cases}$$

$$K_n^{(\mathfrak{M})}(u) = \begin{cases} 0 & \text{for } u \leqslant \dfrac{\min\limits_{k \in \mathfrak{M}(n)} (-z_k p_k)}{\sqrt{\sum_{k=1}^{n} M_k}}, \\ \dfrac{\sum_{k \in \mathfrak{M}(n)} (M_k q_k + M_k p_k)}{\sum_{k=1}^{n} M_k} & \text{for } u > \dfrac{\max\limits_{k \in \mathfrak{M}(n)} z_k q_k}{\sqrt{\sum_{k=1}^{n} M_k}}. \end{cases}$$

In the sequel we shall use formulas (15)-(17) (let us observe that now $M_k = 1$ for $k = 1, 2, \ldots$). We shall prove that

$$K^{(\mathfrak{P})}(u) = \lim_{n\to\infty} K_n^{(\mathfrak{P})}(u) = \begin{cases} 0 & \text{for } u \leqslant 0, \\ \dfrac{\nu}{B^2} u^2 & \text{for } 0 < u \leqslant B, \\ \nu & \text{for } u > B. \end{cases} \tag{18}$$

Since

$$\frac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j}}{\sum_{j=1}^{n} M_j} = \frac{\sum_{j=1}^{\varphi(n)} p_{k_j}}{n} = \frac{\sum_{j=i_0}^{\varphi(n)} \dfrac{1}{B^2 k_j} + (i_0 - 1) \dfrac{1}{B^2 k_{i_0}}}{n} \leqslant \frac{\sum_{j=1}^{n} \dfrac{1}{i}}{B^2 n}$$

$$= \frac{1}{B^2 n} \{\log n + C + \alpha_n\},$$

where $C = \text{const}$, $\lim\limits_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j}}{\sum_{j=1}^{n} M_j} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\varphi(n)} p_{k_j} = 0, \tag{19}$$

whence $K^{(\mathfrak{P})}(u) = 0$ for $u \leqslant 0$.

In view of (9) we have

$$\lim_{n\to\infty} \frac{M_{k_{\varphi(n)}} q_{k_{\varphi(n)}}}{p_{k_{\varphi(n)}} \sum_{j=1}^{n} M_j} = \lim_{n\to\infty} \frac{B^2 k_{\varphi(n)}}{n} \left(1 - \frac{1}{B^2 k_{\varphi(n)}}\right) = B^2$$

and in view of (7)

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{\varphi(n)} M_{k_j} p_{k_j} + \sum_{j=1}^{\varphi(n)} M_{k_j} q_{k_j}}{\sum_{j=1}^{n} M_j} = \lim_{n\to\infty} \frac{\varphi(n)}{n} = \nu.$$

Thus we have $K^{(\mathfrak{P})}(u) = \nu$ for $u > B$.

Let us now take

$$u_{k_{l(n)}} = \sqrt{\frac{M_{k_{l(n)}} q_{k_{l(n)}}}{p_{k_{l(n)}} \sum_{j=1}^{n} M_j}} = \sqrt{\frac{B^2 k_{l(n)} - 1}{n}}$$

and let $\lim_{n\to\infty} k_{l(n)} = 0$. We have

$$K_n^{(\mathfrak{P})}(u_{k_{l(n)}}+0) - \frac{\nu}{B^2} u_{k_{l(n)}}^2 = \frac{\sum_{j=1}^{\varphi(n)} p_{k_j} + \sum_{j=1}^{l(n)} (1-p_{k_j}) - \nu k_{l(n)}\left(1-\frac{1}{B^2 k_{l(n)}}\right)}{n}.$$

According to (19) and to the definition of k_i we have

$$\lim_{n\to\infty}\left\{K_n^{(\mathfrak{P})}(u_{k_{l(n)}}+0) - \frac{\nu}{B^2} u_{k_{l(n)}}^2\right\} = \lim_{n\to\infty}\frac{1}{n}\{l(n) - \nu k_{l(n)}\} = 0.$$

Formula (18) is thus proved.

In a similar way we obtain in view of (8), (13), (14)

$$(20)\qquad K^{(\mathfrak{Q})}(u) = \lim_{n\to\infty} K_n^{(\mathfrak{Q})}(u) = \begin{cases} 0 & \text{for } u < A, \\ \mu - \dfrac{\mu}{A^2}u^2 & \text{for } A \leqslant u < 0, \\ \mu & \text{for } u \geqslant 0. \end{cases}$$

Eventually, since

$$\lim_{n\to\infty}\frac{\sum_{k\in\mathfrak{M}(n)} M_k}{\sum_{k=1}^{n} M_k} = \lim_{n\to\infty}\frac{n-\varphi(n)-\psi(n)}{n} = 1-\nu-\mu,$$

we obtain

$$(21)\qquad K^{(\mathfrak{M})}(u) = \lim_{n\to\infty} K_n^{(\mathfrak{M})}(u) = \begin{cases} 0 & \text{for } u \leqslant 0, \\ 1-\nu-\mu & \text{for } u > 0. \end{cases}$$

In view of (18), (20), (21) $\lim_{n\to\infty} K_n(u)$ is equal to our function $K(u)$. We have assumed that $\nu \neq 0$, $\mu \neq 0$. In order to obtain $\nu = 0$ (respectively $\mu = 0$) one can put $\mathfrak{P}(n) = 0$ (respectively $\mathfrak{Q}(n) = 0$). In the limit we can then obtain every function of the form (2), i. e.

$$(22)\qquad \mathcal{K} \subset \mathcal{R}.$$

From (6) and (22) it follows that $\mathcal{R} = \mathcal{K}$. Our theorem is thus proved.

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