Nous avons $K_p(u_n^*) = 0$ pour $n \leqslant p$ et $\|u_n^*(t) - u_n(t)\| = 2/n$. Nous obtenons, par analogie avec le calcul du travail [13],

$$\|\hat{K}_p u_n^* - \hat{K}_p u_n\|_{H^\mu} \geqslant \frac{\ln n}{n} = \frac{1}{2}\ln n \, \|u_n^* - u_n\|_{H^\mu} \qquad (\text{pour } n \leqslant p).$$

On peut choisir un nombre p tel que $2/\ln p < \lambda_0$. Donc, il en résulte que pour $p > \exp 2/\lambda_0$ et pour toutes les fonctions K_p l'inégalité suivante est satisfaite: $1/q < \lambda_0$.

Le domaine d'existence des solutions de l'équation (13) est alors essentiellement plus grand que le domaine d'unicité.

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Holomorphic vector-valued functions and Hartogs' theorems

by
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1. The aim of this note is to show how certain parts of the existing theory of vector-valued holomorphic functions may be used to obtain close analogues of theorems of Hartogs (see e. g. [1], pp. 137-142) about functions of several complex variables. Hartogs' theorems themselves are not obtained in as much as we find it necessary to impose a priori conditions of local boundedness. However, repayment for this initial expense comes in the form of increased generality and the weakening of other hypotheses involved.

Holomorphic functions with values in a Banach space are discussed in [3], pp. 92 et seq. A briefer, more general, and in some respects more convenient account is given in § 2 of [2].

2. General results. In this section $\mathcal{E}$ will denote a Fréchet space, $\mathcal{E}'$ its topological dual, and $\langle\ ,\ \rangle$ the bilinear form expressing the duality between $\mathcal{E}$ and $\mathcal{E}'$.

PROPOSITION 1. *Let M be a locally compact space, μ a positive measure on M, φ a function mapping M into $\mathcal{E}$. Assume that the following conditions are satisfied:*

(1) *φ is olmost separably-valued;*

(2) *there exists a subset Λ of $\mathcal{E}'$ generating a vector subspace $[\Lambda]$ which is sequentially weakly dense in $\mathcal{E}'$ and such that, for each $L \in \Lambda$, the function*

$$t \to \langle \varphi(t), L \rangle$$

is μ-measurable;

(3) *$\int^* p(\varphi(t))\, d\mu(t) < +\infty$ for each continuous seminorm p on $\mathcal{E}$.*

Then the weak integral $\int \varphi(t)\, d\mu(t)$, a priori an element of the algebraic dual $\mathcal{E}'^*$ of $\mathcal{E}'$, in fact belongs to $\mathcal{E}$.

Proof. Thanks to (1) we may assume that $\mathcal{E}$ itself is separable. (2) shows at once that $t \to \langle \varphi(t), L \rangle$ is measurable for each L in E', and (3) shows that in addition this same function is μ-integrable. So the weak integral certainly exists as an element of $\mathcal{E}'^*$, say u. In order to show that u lies in $\mathcal{E}$, it suffices to show that the linear form $L \to \langle u, L\rangle$ on $\mathcal{E}'$ is such that its restriction to each equicontinuous subset Q of $\mathcal{E}'$ is weakly continuous. However, $\mathcal{E}$ being separable, the weak topology induces on Q a metrisable topology. Moreover, there is a continuous seminorm p on $\mathcal{E}$ such that $|\langle x, L\rangle| \leqslant p(x)$ for x in $\mathcal{E}$ and L in Q. If then a sequence (L_n) extracted from the set Q converges weakly to $L \in Q$, we have $|\langle \varphi(t), L_n\rangle| \leqslant p(\varphi(t))$ for all t and all n, and $\lim\limits_{n\to\infty}\langle \varphi(t), L_n\rangle = \langle\varphi(t), L\rangle$ for all t. Thus, by Lebesgue's theorem,

$$\lim_n \langle u, L_n\rangle = \lim_{n\to\infty}\int \langle\varphi(t), L_n\rangle d\mu(t) = \int\langle\varphi(t), L\rangle d\mu(t) = \langle u, L\rangle,$$

and the desired continuity is established. Thus Proposition 1 is proved.

PROPOSITION 2. *Let $\mathcal{E}$ be a separable Fréchet space, and let φ be a mapping of the polydisk*

$$P = \{t = (t_i)_{1\leqslant i\leqslant m} \in C^m\colon \operatorname*{Sup}_{1\leqslant i\leqslant m}|t_i| < 1\}$$

into $\mathcal{E}$ which satisfies the following conditions:

(a) *there exists a subset Λ of $\mathcal{E}'$ such that $[\Lambda]$ is sequentially weakly dense in $\mathcal{E}'$ and such that, if $t^0 = t_i^0 \in P$, numbers r_i can be found satisfying $0 < r_i < 1 - |t_i^0|$ for which*

$$(2.1)\qquad (\theta_1, \ldots, \theta_m) \to \langle\varphi(t_1^0 + r_1 e^{i\theta_1}, \ldots, t_m^0 + r_m e^{i\theta_m}), L\rangle$$

is measurable for each $L \in \Lambda$, whilst

$$(2.2)\qquad \int_0^{2\pi}\cdots\int_0^{2\pi *} p\big(\varphi(t_1^0 + r_1 e^{i\theta_1}, \ldots, t_m^0 + r_m e^{i\theta_m})\big)\, d\theta_1\ldots d\theta_m < +\infty$$

for each continuous seminorm p on $\mathcal{E}$;

(b) *there exists a total subset Θ of the space $\mathcal{E}'$ such that, for each $T \in \Theta$, $t \to \langle\varphi(t), T\rangle$ is holomorphic on P.*

The conclusion is that φ is holomorphic on P.

Proof. Let c_i be the positively oriented circumference with centre t_i^0 and radius r_i, and let $P' \subset P$ be the open polydisk with distinguished boundary $c_1 \times \ldots \times c_m$. By (a) and Proposition 1, if t lies in P' the weak integral

$$\psi(t) = (2\pi i)^{-m}\int_{c_1}\cdots\int_{c_m}\varphi(s)\, ds_1\ldots ds_m/(s_1 - t_1)\ldots(s_m - t_m)$$

lies in $\mathcal{E}$; and it is clear that ψ is holomorphic on P'. It $T \in \Theta$, then

$$\langle\psi(t), T\rangle = (2\pi i)^{-m}\int_{c_1}\cdots\int_{c_m}\langle\varphi(s), T\rangle ds_1\ldots ds_m/(s_1 - t_1)\ldots(s_m - t_m),$$

and this is equal to $\langle\varphi(t), T\rangle$ because of (b). Since Θ is total, it follows that φ and ψ agree on P'. Thus φ is holomorphic on some neighbourhood of each point of P, and the proof is complete.

Remark. Condition (a) will certainly be satisfied if φ is locally bounded and (2.1) is measurable for each $L \in \Lambda$.

3. A theorem of the Hartogs' type. Throughout this section X and Y will denote complex manifolds. If D is a relatively compact open set in Y, $\mathcal{E}(D)$ will be the space of functions continuous on $\overline{D}$ and holomorphic on D. This is a Banach space when equipped with the norm.

$$\|g\| = \operatorname*{Sup}_{y\in D}|g(y)|;$$

the supremum here could equally well be taken over $\overline{D}$ or over D^*, the frontier of D relative to Y. $\mathcal{E}(D)$ is isomorphic with a vector subspace of the space $C(D^*)$ of continuous functions on D^*, likewise equipped with the sup norm. D^* being metrisable and compact, $C(D^*)$ is separable; hence the same is true of $\mathcal{E}(D)$. (For our purposes it is somewhat more convenient to take $\mathcal{E}(D)$ thus defined, rather than the Fréchet space of functions holomorphic on D with the compact-open topology.)

We shall wish to apply Proposition 2 to spaces of the type $\mathcal{E}(D)$ with D a relatively compact domain. So it is convenient to observe here that each point y of $\overline{D}$ defines an element ε_y of $\mathcal{E}(D)'$ defined by $\langle g, \varepsilon_y\rangle = g(y)$ for g in $\mathcal{E}(D)$. Further, if S is a dense subset of the set D^*, then $\Lambda = \{\varepsilon_y\colon y \in S\}$ generates a sequentially weakly dense vector subspace $[\Lambda]$ of $\mathcal{E}(D)'$. Indeed, in view of the isomorphic imbedding of $\mathcal{E}(D)$ into $C(D^*)$, any element of $\mathcal{E}(D)'$ is of the type

$$g \to \int g(y)\, d\mu(y),$$

where μ is a Radon measure on D^*; and any such μ is the vague limit of a sequence of measures of the form

$$\sum_n c_n \varepsilon_{y_n}\qquad (y_n \in S).$$

Again, if E is any somewhere-dense (i. e. not nowhere-dense) subset of D, then $\theta = \{\varepsilon_y\colon y \in E\}$ is a total subset of $\mathcal{E}(D)'$. If Y is of (complex) dimension one, the same is true whenever $E \subset D$ admits a limiting point in D.

Other choices of θ are possible. For example, one might take Θ to consist of the linear forms

$$g \to \int g(y)\, d\sigma(y),$$

where σ ranges over any total set of measures on D^*.

With these remarks in mind, we proceed to prove a general theorem of the Hartogs type. In what follows, measurability of a function h on X will be understood in this sense: if h is expressed locally as a function $H(z_1, \ldots, z_m)$ of local coordinates, then

$$H(z_1+r_1e^{i\theta_1}, \ldots, z_m+r_me^{i\theta_m})$$

is a measurable function of $(\theta_1, \ldots, \theta_m)$.

THEOREM 1. *Let X and Y be complex manifolds, D a relatively compact open set in Y, and f a function on $X\times\bar{D}$ which satisfies the following conditions:*

(i) *f is locally bounded on $X\times\bar{D}$;*

(ii) *for each $x\in X$, the partial function $f_x: y\to f(x, y)$ belongs to $\mathcal{E}(D)$;*

(iii) *there exists a set $\Lambda\subset\mathcal{E}(D)'$ generating a vector subspace $[\Lambda]$ which is sequentially weakly dense in $\mathcal{E}(D)'$ and such that, for each $L\in\Lambda$, $x\to\langle f_x, L\rangle$ is measurable on X;*

(iv) *there exists a total subset Θ of $\mathcal{E}(D)'$ such that, for each $T\in\Theta$, $x\to\langle f_x, T\rangle$ is holomorphic on X.*

The conclusion is that the mapping $\varphi: X\to\mathcal{E}(D)$ defined by $\varphi(x) = f_x (x\in X)$ is holomorphic; in particular, f is holomorphic on $X\times D$.

Proof. By localisation we may assume that X is a polydisk P. It is clear that $\varphi: P\to\mathcal{E}(D)$ satisfies the conditions of Proposition 2, whence the result.

In view of the remarks preceding Theorem 1 it is clear that, apart from the additional hypothesis (i), Theorem 1 contains numerous analogues and extensions of Hartogs' results ([1], p. 137, Lemma 2; p. 139, Theorem 2 and Lemma 3; p. 140, Theorem 4; p. 141, Theorem 5). Condition (i) may be modified somewhat in accordance with (2.2) adapted to the cases in hand. In the most obvious forms of the analogues, D does not appear explicitly since Theorem 1 is applied to "arbitrarily large" relatively compact domains D.

As another example, we give now an application of Proposition 2 to a theorem of the Hartogs type for mixed real and complex variables. For simplicity we shall suppose that X is a disk in the complex plane and Y an interval of the real axis, but generalizations are easily effected.

THEOREM 2. *Suppose that $f(x, y)$ is defined for complex x satisfying $|x| < R$ and real y satisfying $a < y < b$ ($0 < R \leqslant +\infty$; $-\infty \leqslant a < b \leqslant +\infty$). Suppose also that f satisfies the following conditions:*

(i) *for each x, $y\to f(x, y)$ is C^∞ on (a, b), whilst $\partial^k f/\partial y^k$ is locally bounded in the pair (x, y) for each integer $k\geqslant 0$;*

(ii) *for each y, $x\to\partial^k f/\partial y^k$ is measurable for each integer $k\geqslant 0$;*

(iii) *for each of a total set Θ of distributions T with compact supports in (a, b),*

$$x\to\langle f(x, y), T_{(y)}\rangle$$

is holomorphic for $|x| < R$.

Then f has the form

$$f(x, y) = \sum_{n\geqslant 0} h_n(y)x^n, \tag{3.1}$$

where each $h_n\in C^\infty(a, b)$ and where, for each integer $k\geqslant 0$, the series

$$\sum_{n\geqslant 0}\left(\frac{\partial^k h_n}{\partial y^k}\right)x^n \tag{3.2}$$

converges uniformly on compacts in $|x| < R$, $a < y < b$.

Proof. By localisation, we may assume that for each k the function $\partial^k f/\partial y^k$ is bounded in the pair of variables. We apply Proposition 2, taking $\mathcal{E}$ to be the Fréchet space $C^\infty(a, b)$ equipped with the compact-open topology, and taking for Λ the set of distributions $\partial^k\varepsilon_c/\partial y^k$ ($a < c < b$; $k = 0, 1, 2, \ldots$). It remains only verify that this Λ has the property required in Proposition 2, (a). For this, note that the dual of $C^\infty(a, b)$ is the space of distributions with compact supports in (a, b). Now if T is such a distribution, there is a measure μ with compact support in (a, b) and an integer $k\geqslant 0$ such that $T = \partial^k\mu/\partial y^k$. Since μ is the weak limit of a sequence of finite linear combinations of measures ε_c ($a < c < b$), and since derivation is continuous in the space of distributions, it follows that T is the weak limit of a sequence of finite linear combinations of derivatives $\partial^k\varepsilon_c/\partial y^k$, as required.

The conclusion of Proposition 2 states that the mapping $x\to f_x$ is holomorphic from $|x| < R$ into $C^\infty(a, b)$, and from this (3.1) and (3.2) follow by virtue of the Taylor expansion for holomorphic vector-valued functions (see e. g. [2], Théorème 1).

Remark. There is an analogous theorem in which $C^\infty(a, b)$ is replaced throughout by $C^p(a, b)$ where p is an integer $\geqslant 0$. Then k can be restricted to the unique value $k = p$.

It is perhaps worth observing also that each h_n figuring in the expansion (3.1) depends continuously of f in the sense that, for any continuous seminorm p on $C^\infty(a, b)$ and any number r satisfying $0 < r < R$, one has the inequality

$$p(h_n)\leqslant r^{-n}p(f_x), \tag{3.3}$$

f_x being the element of $C^\infty(a, b)$ defined by $y \to f(x, y)$. This is so because h_n is simply $\varphi^{(n)}(0)/n!$, where φ denotes the holomorphic vector-valued function $x \to f_x$; thus

$$h_n = (2\pi i)^{-1} \int_{|x|=r} \varphi(x) x^{-n-1} dx,$$

and (3.3) follows at once, just as for the Cauchy inequalities for a scalar-valued holomorphic function.

4. Other extensions. Theorems 1 and 2 admit extensions in which the function f takes its values in a separable Fréchet space F. $\mathcal{E}(D)$ would be replaced by the space $\mathcal{E}(D, F)$ of functions from $\overline{D}$ into F which are continuous on D and holomorphic on D; and $C^\infty(a, b)$ would be modified in like manner. $\mathcal{E}(D, F)$ will be a Fréchet space when equipped with the topology defined by the seminorms

$$\sup_{y \in D} p_n(g(y)),$$

where the p_n $(n = 1, 2, \ldots)$ are seminorms defining the topology of F. It is easily seen that $\mathcal{E}(D, F)$ will be separable whenever F has this property (cf. [4], p. 58, Proposition 5). (One might weaken continuity on $\overline{V}$ to weak continuity on $\overline{V}$, together with separability conditions on the function involved, but this would have little advantage from the point of view of applications.) Similar remarks apply to the space of vector-valued C^∞ functions.

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The two-norm spaces and their conjugate spaces

by

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In this paper we continue our investigations on the two-norm spaces, presented in the papers [2], [3], [5].

A two-norm space is a linear space X provided with two norms: $\| \; \|$ and a coarser [1] one $\| \; \|^*$; these two norms lead to the following notion of limit: the sequence x_n is termed *γ-convergent* to x_0 (written $x_n \xrightarrow{\gamma} x_0$) if $\sup_{n=1,2,\ldots} \|x_n\| < \infty$ and $\lim_{n\to\infty} \|x_n - x_0\|^* = 0$. Thus, as regards the distributive functionals, three classes arise in a natural way: the spaces $\langle \Xi, \| \; \| \rangle$ and $\langle \Xi^*, \| \; \|^* \rangle$ conjugate to the normed spaces $\langle X, \| \; \| \rangle$ and $\langle X, \| \; \|^* \rangle$, respectively, and the space Ξ_γ of functionals sequentially continuous with respect to the convergence γ. Obviously $\Xi^* \subset \Xi_\gamma \subset \Xi$.

The triplet $\langle X, \| \; \|, \| \; \|^* \rangle$ is called the *two-norm space*. The space $\langle \Xi^*, \| \; \|^*, \| \; \| \rangle$ [2] seems to be the natural two-norm space conjugate to $\langle X, \| \; \|, \| \; \|^* \rangle$. We show that, analogously to the Banach space case, every two-norm space may be *canonically* embedded into its biconjugate two-norm space, with the preservation of both norms. The canonical mapping enables us to embed any two-norm space into a two-norm space sequentially complete with respect to the convergence γ; this process will be called the *γ-completion*.

The main purpose of this paper is the study of the interrelations of the two-norm spaces and of the concepts arising in connection with them. Some pages are devoted to the *γ-reflexive* spaces, i. e. such which are canonically embedded onto the biconjugate two-norm space; a characterization similar to the Banach space case is derived. We study also the *γ-compact* spaces, i. e. such that each γ-bounded sequence contains a subsequence which is γ-convergent (to an element); a detailed study is devoted

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[1] The norm $\| \; \|^*$ is called *coarser* than $\| \; \|$ (or $\| \; \|$ is called *finer* than $\| \; \|^*$) if $\|x_n\| \to 0$ implies $\|x_n\|^* \to 0$.

[2] In the triplet-notation for a two-norm space the finer norm will always precede the coarser one; so in this case the norm $\| \; \|$ is coarser than $\| \; \|^*$.