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On the Fredholm alternative in locally convex linear spaces

by

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The present paper is concerned with the Fredholm alternative for linear equations in locally convex spaces and a number of neighboring topics. Such problems have received much attention (1) when the underlying space, or spaces, are Banach. Yet the natural background of the theory are locally convex spaces and this is not only because there are important spaces in applications which are not normable, but also the use of locally convex spaces does allow of a more adequate discussion of adjoints which are closely related to the Fredholm alternative by its very definition. The reader who is familiar with the basic properties of locally convex spaces will find the unifying concept to be that of weak homomorphism or, in other words, that if emphasis is laid on the weak rather than other topologies this will provide a better insight into much of the theory. Thus part of the paper is expository in the sense that it proves, using a uniform approach, results which had been obtained previously for Banach spaces by various methods. We shall not give a detailed account of our results here but refer to the preliminary discussion within the next section and to the four theorems which preferentially may deserve attention.

0. Definitions and notation. By a *linear space* we shall understand a vector space over the real, or complex, scalar field. A linear space is *locally convex* if it carries a locally convex topology (2) which is Hausdorff. If H is a linear subspace of some linear space E, then the codimension of H (in E) is the dimension of the quotient space E/H which is clearly the same as the dimension of any complementary subspace G, i. e. any subspace G such that $E = G+H$ while $G \cap H = 0$. In this case E is called the *algebraical direct sum* of G and H. A topological linear space E is

(1) See the literature quoted at the end of the paper. In the case of locally convex spaces, recent progress is mainly due to Altman [1]-[3], and Grothendieck [1], [2].

(2) For the theory of locally convex spaces we refer to Bourbaki [1], [2], and Köthe [2].

called the *topological direct sum* (or simply, *direct sum*) of G and H if E is the algebraical direct sum of G and H while the linear isomorphism of E onto $G\times H$ is also a homeomorphism (³). We shall denote direct sums by writing $E = G\oplus H$.

Let E be any locally convex space, E' its dual space i. e. the (linear) set of all continuous linear forms (⁴) on E. By $w(E', E)$ (or $m(E', E)$, or $s(E', E)$) we understand the weak, or Mackey, or strong topologies respectively on E', and similarly for E. It is well known that the topology of E is somewhere in between $w(E, E')$ and $m(E, E')$. If E, F are locally convex spaces (or, more generally, topological linear spaces), T a continuous linear mapping on E into F, then we shall say that T is a *homomorphism* if the image under T of any open set in E is open in $T(E)\subset F$. An isomorphism of E onto itself is an automorphism of E. Hence not every continuous linear mapping is a homomorphism, and not every one-to-one homomorphism is an automorphism of E.

DEFINITION 1. Let E, F be locally convex spaces. A homomorphism T on E into F will be called a *σ-transformation* if the null space $N(T)$ of T is finite dimensional while $T(E)$ is a closed subspace of F having finite codimension.

If n and m are the dimensions of $N(T)$ and $F/T(E)$ respectively, then the integer $\varkappa(T) = n-m$ is said to be the *index of* T. Moreover, we shall denote by $\Sigma(E, F)$ the totality of σ-transformations on E into F, while by $\Sigma_w(E, F)$, $\Sigma_m(E, F)$, and $\Sigma_s(E, F)$ we will understand the sets of linear mappings which are σ-transformations under the weak, Mackey, and strong topologies respectively put on both E and F. Further $\Sigma_w'(F', E')$ will mean the set of σ-transformations with respect to $w(F', F)$ and $w(E', E)$ (and similarly for $\Sigma_m'(F', E')$, $\Sigma_s'(F', E')$) while $\Sigma_w(F', E')$ will refer to $w(F', F'')$ and $w(E', E'')$, etc.

Definition 1 and a number of properties shared by σ-transformations may be found in Schaefer [1]. A somewhat more restricted class of linear mappings is characterized by the following definition:

DEFINITION 2. Let E, F be locally convex spaces, E', F' their respective duals, T a weakly continuous linear mapping on E into F, T' its adjoint. Then T is said to be a *Fredholm transformation* if either of the equations

$$Tx = y, \qquad T'y' = x'$$

(³) A subspace of a topological linear space will be understood to carry the topology induced by E unless the contrary is expressly stated.

(⁴) We prefer this term to the more common "linear functional".

is solvable if and only if $y\in N(T')^0$, $x'\in N(T)^0$ respectively(⁵), while the null spaces $N(T)$ and $N(T')$ have the same finite dimension.

More symmetrically, the dual systems $\langle E, E'\rangle$ and $\langle F, F'\rangle$ may be replaced in this definition by any two dual systems but we shall not insist on doing so. It can be seen easily (sec. 2) that T is a Fredholm transformation if and only if it is a weak σ-transformation of index zero. We shall denote the set of all Fredholm transformations on E into F by $\Phi(E, F)$, whereas by $\Phi'(F', E')$, $\Phi(F', E')$ respectively we shall understand the sets of mappings which are Fredholm when the weak (or Mackey), respectively strong topologies are put on both F' and E' (⁶).

For the rest of this paper we shall be concerned with linear mappings on a locally convex space into itself. Then the classes of transformations so far considered will become subsets (and, in fact, semi-groups) of the algebra of linear transformations on E. We next introduce a still simpler class of mappings.

DEFINITION 3. A weak homomorphism T on a locally convex space E will be called a *Riesz transformation* if it satisfies the following conditions:

1. $N(T^n)$ is a fixed finite dimensional subspace of E if $n\geqslant n_0$;
2. $T^m(E)$ is closed and of finite deficiency in E for $m = 1$, and is fixed if $m\geqslant m_0$.

If K is a compact linear transformation on E (⁷), then $I-K$ is a Riesz transformation and it was in fact for this type of mapping that Riesz developed his powerful theory which was carried over to general locally convex spaces by J. Leray [1] and others. Williamson [1] observed that most of the theory holds in any topological linear space.

It turns out that if n_0, m_0 are the smallest non-negative integers such that conditions 1 and 2 of definition 3 are fulfilled, then $m_0 = n_0$. Moreover, T is a Riesz transformation if and only if it is a Fredholm transformation such that condition 1 is satisfied or, equivalently, that the union of null spaces $\bigcup_{n=1}^{\infty} N(T^n)$ is finite dimensional.

If A is the algebra of continuous linear transformations on some locally convex space E, then the spectrum $\sigma(T)$ of $T\in A$ is defined to be the set of all λ in the scalar field such that $T_\lambda = \lambda I - T$ has no inverse

(⁵) As a rule, polar sets will be denoted by an upper index zero. Hence $N(T)^0$ means the subspace of E' orthogonal to $N(T)$.

(⁶) Notice that in general $\Phi'(F', E')\neq\Phi(F', E')$. When necessary we shall denote by $\Phi_s(E, F)$ the set of Fredholm mappings under $s(E, E')$ and $s(F, F')$.

(⁷) A linear transformation is *compact* if the image under K of some non-empty open set is relatively compact (hence bounded).

while its complement $\varrho(T)$ is usually called the *resolvent set*. If E is a Banach space, then it is well known that $\sigma(T)$ is a closed bounded set, hence $\varrho(T)$ is open and contains the region outside some large enough circle. Now if E is any locally convex space it is by no means clear even whether the resolvent set is still open. However, if K is compact $\sigma(K)$ is known to behave as nicely as it does when E is a Banach space (cf. Leray [1], Altman [1]). To make some further steps in this direction, we introduce the following definition (cf. Schaefer [2]):

DEFINITION 4. Let E be any locally convex space. A linear mapping T on E is said to be *bounded* if there exists some non-empty open set of which the image under T is bounded.

This definition implies that if E is not normable T cannot be a homeomorphism, hence that T cannot have a continuous inverse. It is clear that the set B of all bounded mappings is a two-sided ideal in A which is proper if E is not normable ([8]). If E is normable then $B = A$ while if E is a Montel space then every bounded transformation is compact since bounded sets are relatively compact in such a space. Thus B ranges from the class of all compact linear mappings to the full algebra A according to the type of space considered.

DEFINITION 5. Let T be a continuous linear mapping on E. Then λ is called a *Fredholm (Riesz) point* of T if $T_\lambda = \lambda I - T$ is a Fredholm (Riesz) transformation on E. The set of all Fredholm (Riesz) points of T will be called the *Fredholm (Riesz) domain* of T.

It was shown by the author [2] that for bounded mappings on a complete space $\sigma(T)$ is a closed bounded subset of the complex plane. Using results by Kračkovsky-Goldman [1] valid for Banach spaces we shall show that the Fredholm domain $\varphi(T)$ is open for bounded transformations on complete spaces and, moreover, that $\varphi(T)$ splits into two disjoint classes $\varphi_1(T)$, $\varphi_2(T)$ of components (i. e., maximal connected subsets) such that $\varphi_1(T)$ is the Riesz domain of T, hence $\varphi_2(T) \subset \sigma(T)$.

1. σ-transformations. Let E, F be locally convex linear spaces.

PROPOSITION 1. *In order that a continuous linear mapping T on E into F be a σ-transformation it is necessary and sufficient that there exist two continuous linear mappings on F into E such that*

$$RT = I - L_1, \quad TS = I - L_2, \tag{*}$$

where L_1, L_2 are finite dimensional endomorphisms on E, F respectively ([9]). *Moreover R, S may be chosen so that $R = S$ and L_1, L_2 are (continuous) projections.*

([8]) It would be interesting to know whether B is maximal under some reasonable condition.

([9]) I denoting the identity transformation on each space.

Proof. See Schaefer [1], Satz 12.

COROLLARY 1. *Every σ-transformation keeps this property if the weak, or Mackey, or strong topologies respectively are put on both E and F.*

Since every continuous mapping on one locally convex space into another remains continuous when each corresponding pair of the topologies mentioned is used, it follows that equations (*) continue to hold for these (pairs of) topologies which by prop. 1 implies the assertion.

As another corollary to prop. 1 we state a result which is supplementary to Satz 14 in the author's paper [1] (cf. sec. 3, prop. 9):

COROLLARY 2. *Let A, B denote continuous linear mappings on E into F, and on F into E respectively. Then if both AB and BA are σ-transformations so are A, B.*

Proof. Let $AB = C_1 \in \Sigma(F)$ and $BA = C_2 \in \Sigma(E)$. Then by prop. 1 there are continuous linear mappings R, S on F, E respectively such that $RC_1 = I - L_1$, $C_1 S = I - L_2$, where L_1, L_2 are finite dimensional projections on F, E respectively. Hence we obtain equations

$$(RA)B = I - L_1, \quad B(AS) = I - L_2$$

which by prop. 1 guarantee that $B \in \Sigma(F, E)$. Now by symmetry $A \in \Sigma(E, F)$.

Proposition 1 makes it easy to state some properties of the adjoint T' of a σ-transformation T.

PROPOSITION 2. *If $T \in \Sigma_w(E, F)$, then $T' \in \Sigma_w'(F', E') = \Sigma_m'(F', E') \subset \Sigma_s'(F', E')$.*

In other words, if T is a σ-transformation then so is its adjoint T' if either the weak, or strong, or Mackey topologies are put on both F' and E'.

Proof. By prop. 1, cor. 1 it is sufficient to show that $T' \in \Sigma_w'(F', E')$. Now from (*) we obtain the dual set of equations

$$S'T' = I' - L_2', \quad T'R' = I' - L_1', \tag{*'}$$

and it is clear from Bourbaki [2], p. 103, prop. 6 (corollaire) that all mappings occurring in these equations are continuous for the weak topologies on F' and E'. Again by prop. 1 this proves our assertion.

If again T' denotes the adjoint of T, then an immediate consequence of the preceding proposition is

PROPOSITION 3. *$T \in \Sigma_w(E, F)$ if and only if $T' \in \Sigma_w'(F', E')$.*

It is also obvious that this proposition continues to hold if E', F' are replaced by any two linear spaces in duality with E, F respectively. We conclude the present section by a theorem which is supplementary to prop. 2.

THEOREM 1. *Consider the following properties, T' denoting the adjoint of T:*

(a) $T \in \Sigma_s(E, F)$, (b) $T \in \Sigma_w(E, F)$, (c) $T' \in \Sigma_w'(F', E')$, (d) $T' \in \Sigma_s'(F', E')$.

Then the following implications are true:

1. (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) *for any locally convex spaces E, F.*
2. (a) $\Leftrightarrow$ (b) *for any disk spaces* (10) *E, F.*
3. (d) $\Rightarrow$ (a) *if E, F are disk spaces such that E is closed in E'' for $s(E'', E')$* (11).

Proof. 1. This follows from prop. 2 since by prop. 3, (b) and (c) are equivalent.

2. Clear from prop. 1, cor. 1 by the definition of disk spaces.

3. From prop. 2 it follows that $T'' \in \Sigma_s'(E'', F'')$ and we have to prove $T \in \Sigma(E, F)$ since E, F are disk spaces. Now as $N(T) \subset N(T'')$ we know that $N(T)$ is finite dimensional, and since so is $N(T')$ the deficiency of the closure $\overline{T(E)}$ is finite as well. Thus there remains to show that T is a homomorphism onto a closed subspace of F. Define $\hat{E} = E + N(T'')$ which is a closed subspace of E'' since E is. We also have $\hat{E} = E \oplus D$ where D is finite dimensional and such that $D \subset N(T'')$. Now because $T'' \in \Sigma_s'(E'', F'')$ we obtain $E'' = N(T'') \oplus E_1$, T'' being an isomorphism of E_1 onto a closed subspace of F''. Further we may write $\hat{E} = N(T'') \oplus E_2$ and choose $E_2 \subset E_1$; e. g., $E_2 = \hat{E} \cap P^{-1}(0)$, where P denotes the projection of E'' onto $N(T'')$ vanishing on E_1. If follows that $T''(E_2)$ is a closed subspace of $T''(E_1)$, hence closed in F'' and this implies that T'' is a homomorphism of $\hat{E}$ onto a closed subspace of F''. But $\hat{E} = E \oplus D$ and $T''(D) = 0$ whence it follows that T'', hence T, is a homomorphism on E onto a closed subspace of F'' which, being a subset of F, is closed in F since F is a disk space and this completes the proof.

It is well known (12) that the abstract theory of σ-transformations has concrete examples in some linear integral transformations involving Cauchy kernels (i. e., kernels of the principal value type). Since the latter are usually handled in Banach spaces it becomes clear from theorem 1 that they still behave reasonably as to their adjoints.

2. Fredholm transformations. Again let E, F be locally convex spaces.

(10) A *disk space* (French *espace tonnellé*) is a locally convex space on which the original and strong topologies coincide. Examples are furnished by reflexive spaces and by spaces of the second category.

(11) This latter condition will be satisfied if, e. g., E is a complete disk space or if E is reflexive.

(12) See the author's paper [1] and the literature therein quoted.

We recall that the index of a σ-transformation T is $\varkappa(T) = n - m$ where n, m are the dimensions of $N(T)$, $F/T(E)$ respectively.

PROPOSITION 4. *A linear mapping on E into F is a Fredholm transformation if and only if it is a weak σ-transformation of index zero* (13).

Proof. Let T be a linear mapping on E into F satisfying definition 2. Then since by hypothesis $T(E) = N(T')^0$ it follows that $T(E)$ is closed in F. Clearly $N(T)$ is finite dimensional and so is $F/T(E)$ which is isomorphic to $N(T')$. It remains to show that T is a weak homomorphism. But this follows from Bourbaki [2], p. 101, prop. 4, as $T'(F')$ is closed in E' because $T'(F') = N(T)^0$ by hypothesis. The converse is clear from definitions 1, 2 and prop. 3.

Now we give another and less trivial description of $\Phi(E, F)$.

THEOREM 2. *Let T be a linear mapping on E into F. If U, V, V_0 (in this order) denote a weak isomorphism of E onto F, a compact, and a finite dimensional weakly continuous mapping on E into F respectively, then the following assertions are equivalent:*

1. *T is a Fredholm transformation,*
2. $T = U + V$,
3. $T = U + V_0$ (14).

Proof. 1 $\Rightarrow$ 3. If $T \in \Phi(E, F)$ then, by prop. 4, $T \in \Sigma_w^0(E, F)$ and there are decompositions

$$E = N(T) \oplus E_1, \quad F = M \oplus T(E)$$

such that $N(T)$ and M have the same finite dimension. That either representation is a direct sum follows from the facts that each finite dimensional subspace of E allows of a topological complement (Bourbaki [1], p. 75, prop. 4, cor. 6), and that any algebraical finite dimensional complement of a closed subspace of F is a topological one (Bourbaki [1], p. 28, prop. 3). Since T is a weak homomorphism of E onto $T(E)$, it is a weak isomorphism from E_1 onto $T(E)$, and as M is isomorphic to $N(T')$ which by hypothesis has the same dimension as $N(T)$, we obtain $\dim N(T) = \dim M$. Thus if P means the projection from E onto $N(T)$ vanishing on E_1, and if A_0 is an isomorphism of $N(T)$ onto M then $U = T(I-P) + A_0 P$ is a weak isomorphism of E onto F and, letting $A_0 P = -V_0$, we have $T = U + V_0$ as was to be proved.

3 $\Rightarrow$ 2. This is obvious since each weakly continuous finite dimensional mapping is compact.

(13) We shall denote this property by writing $T \in \Sigma_w^0(E, F)$. Hence prop. 4 shows that $\Phi(E, F) = \Sigma_w^0(E, F)$.

(14) Of course U does not necessarily denote the same mapping in 2 and 3.

$2 \Rightarrow 1$. If $T = U+V$ then $T = U(I+U^{-1}V)$ where I is the identity transformation on E and $U^{-1}V = V_1$ is a compact mapping on E into itself. Now $I+V_1 \in \Sigma^0(E)$ by the well known Riesz theory. On the other hand $U \in \Sigma^0(E, F)$ so by Schaefer [1], p. 161, Satz 13, we have $T = U(I+V_1) \in \Sigma^0(E, F)$ and even $T \in \Sigma_w^0(E, F)$ since T is weakly continuous. Now by prop. 4 the theorem is proved.

Remark. The above conditions on U, V, and V_0 may be replaced by the corresponding ones with respect to the Mackey topologies $m(E, E')$ and $m(F, F')$.

This is because every weak isomorphism of E onto F is also an isomorphism for the Mackey topologies and conversely, and similarly a finite dimensional mapping is weakly continuous if and only if it is continuous for $m(E, E')$. Finally, if V is a compact linear transformation on E into F for the Mackey topologies, then the $2 \Rightarrow 1$ argument in the foregoing proof shows that $T \in \Sigma_m^0(E, F)$ which by prop. 1, cor. 1, implies $T \in \Sigma_w^0(E, F) = \Phi(E, F)$.

We now turn to the question for what topologies other than the weak ones a Fredholm transformation T and its adjoint T' are homomorphisms.

Proposition 5. *Every Fredholm transformation T is a homomorphism for the Mackey and strong topologies and so is its adjoint whence, in particular, $\Phi'(F', E') \subset \Phi(F', E')$.*

Proof. Since we know (prop. 4) that $T \in \Sigma_w^0(E, F)$ it follows from prop. 1, cor. 1 that T is a strong σ-transformation which implies by definition 1 that T is a strong homomorphism. The corresponding statement for T' may be inferred in the same way from prop. 2.

We have just seen that $\Phi'(F', E') \subset \Phi(F', E')$. The question arises quite naturally to what extent the converse assertion holds, i. e., if $T' \in \Phi(F', E')$ and T' is the adjoint of some T on E into F (15) when is it true that $T \in \Phi(E, F)$? We give the following condition which is rather satisfactory:

Proposition 6. *Let E, F be disk spaces such that E is closed in E'' (for $s(E'', E')$), T a linear mapping on E with adjoint T'. Then if T' is a Fredholm transformation so is T or, more precisely, $T' \in \Phi(F', E')$ implies $T \in \Phi(E, F)$.*

Proof. As $\Phi(F', E') = \Sigma_w^0(F', E')$ we know from prop. 2 that $T'' \in \Sigma_s'(E'', F'')$. But as is easily checked the third part of the proof to th. 1 is based exactly upon this assumption, in addition to the requirements on E, F which are here the same. Thus $T \in \Sigma(E, F)$ which implies $T \in \Sigma_w(E, F)$ (prop. 1, cor. 1) and as clearly $\varkappa(T) = -\varkappa(T') = 0$, we obtain $T \in \Sigma_w^0(E, F)$ which by prop. 4 completes the proof.

(15) This is equivalent to the condition that T' be continuous for the topologies $w(F', F)$ and $w(E', E)$.

Corollary. *Let E, F be Fréchet spaces. Then a linear mapping T on E into F is Fredholm if and only if its strong adjoint is.*

If E, F are Banach spaces, K a linear transformation on E into F, then by a well known theorem due to Schauder K is compact if and only if K' is. As is shown in a paper by Köthe [1] the situation is entirely different even if E, F are F-spaces. Now let, e. g., $E = F$ be an F-space, K a linear mapping on E such that K' is (strongly) compact on E'. Then $T' = I'+K'$ is Fredholm and so is $T = I+K$ by the above corollary. Hence, by th. 2, $T = U+V$ where U is a strong automorphism of E while V may even be assumed finite dimensional. Curious as it is, it cannot be inferred from $I+K = U+V$ that K is compact.

3. Riesz transformations. From now on we restrict our attention to linear mappings on a locally convex space E into itself.

Proposition 7. *A linear mapping T on E is a Riesz transformation if and only if it is a Fredholm transformation such that the union of null spaces $\bigcup_{n=1}^{\infty} N(T^n)$ is finite dimensional.*

Proof. If T is a Riesz transformation on E then, by definition 3, T is a weak homomorphism of E onto a closed subspace $T(E)$ such that both $N(T)$ and $E/T(E)$ are finite dimensional. Hence $T \in \Sigma_w(E)$ and by prop. 4 there remains to show that T has index zero. Now $T^m \in \Sigma_w(E)$ and $\varkappa(T^m) = n\varkappa(T)$ by Schaefer [1], Satz 13, and since, by definition 3, $|\varkappa(T^m)| < c$ for all n, it follows that $\varkappa(T) = 0$. If on the other hand T is Fredholm on E and such that $\bigcup_{n=1}^{\infty} N(T^n)$ is of finite dimension, then by def. 3 we have only to show that the inclusions $T^{m+1}(E) \subset T^m(E)$ will cease to be proper at some $m = m_0$. But since this is true for the inclusions $N(T^m) \subset N(T^{m+1})$ and as $\varkappa(T^m) = n\varkappa(T) = 0$ by hypothesis, the two chains will become stationary at exactly the same integer. The proof is finished.

Corollary. *If n_0, m_0 are the smallest non-negative integers satisfying conditions 1, 2 of definition 3 respectively, then $n_0 = m_0$.*

We now establish another property which th. 3 will show to be characteristic of Riesz transformations.

Proposition 8. *If T is a Riesz transformation on E, there is a decomposition $E = N \oplus \hat{E}$ such that T is a weak automorphism on $\hat{E}$ while N is finite dimensional and $T(N) \subset N$.*

Proof. By prop. 7 we know that $T^m \in \Sigma_w^0(E)$, so for each integer $k \geqslant 1$ there are two decompositions of E, viz.

$$E = N_k \oplus E_k, \qquad E = U_k \oplus E_k$$

such that $N_k = N(T^k)$ and U_k are of the same finite dimension while $\hat{E}_k = T^k(E)$ and T^k is a weak isomorphism of E_k onto $\hat{E}_k$. Further we know that for some (smallest) integer n_0, $N_k = N$ and $\hat{E}_k = \hat{E}$ are independent of k if $k \geqslant n_0$. We first prove that $\hat{E} \cap N = 0$. Otherwise there would be some non-zero $x_0 \in \hat{E}$ such that $T^{n_0} x_0 = 0$. Now since T^{n_0} is an isomorphism of E_{n_0} onto $\hat{E}_{n_0} = \hat{E}$ there were a unique non-zero $z_0 \in E_{n_0}$ with $T^{n_0} z_0 = x_0$. Hence $T^{2n_0} z_0 = 0$ which implies $z_0 \in N_{2n_0} = N_{n_0} = N$ and this is contradictory. On the other hand, $\dim U_{n_0} = \dim N$ so N must be a topological complement of $\hat{E}$ since $\hat{E}$ is closed in E. Thus $E = N \oplus \hat{E}$ which is the desired representation of E. It remains to prove that T is a weak automorphism of $\hat{E}$ and $T(N) \subset N$. Now by definition $N = N(T^k)$ for $k \geqslant n_0$ and it is clear that $Tx \in N$ if $x \in N$. Further $T(\hat{E}) = T^{n_0+1}(E_{n_0}) = \hat{E}_{n_0+1} = \hat{E}$ and, as we know that $\hat{E} \cap N(T) = 0$, T is one-to-one on $\hat{E}$ to itself. Since by hypothesis T is a weak homomorphism on E it follows that T is a weak automorphism of $\hat{E}$ and the proof is complete.

The two foregoing propositions make it easy to set up the following characterization of Riesz mappings on any locally convex space E:

THEOREM 3. *Let E be any locally convex space, T a linear mapping on E into itself. Then the following propositions are equivalent:*

1. *T is a Riesz transformation.*

2. *T is a Fredholm transformation such that the union of null spaces $\bigcup_{n=1}^{\infty} N(T^n)$ is finite dimensional.*

3. *There exists a decomposition $E = N \oplus \hat{E}$ such that N is finite dimensional with $T(N) \subset N$ while T is a weak automorphism of $\hat{E}$.*

4. *$T = U + V_0$ where U is a weak automorphism of E and V_0 a weakly continuous finite dimensional mapping such that $UV_0 = V_0 U$.*

5. *$T = U + V$ where U is a weak automorphism of E while V is compact and such that $UV = VU$.*

Proof. $1 \Rightarrow 2$ and $2 \Rightarrow 3$ follow immediately from propositions 7 and 8.

$3 \Rightarrow 4$. If T_0 denotes the restriction to N of T then since $T_0(N) \subset N$ it is well known that $T_0 = U_0 + A_0$ where $U_0 A_0 = A_0 U_0$ and U_0 is an automorphism of N. Now if P is the projection of E onto N vanishing on $\hat{E}$, it is easy to see that $U = U_0 P + T(I-P)$ is a weak automorphism of E. Further $T = U + A_0 P$ and, since $UA_0 P = U_0 A_0 P = A_0 U_0 P = A_0 PU$, 4 is proved.

$4 \Rightarrow 5$. Clear, as each weakly continuous finite dimensional mapping is compact.

$5 \Rightarrow 1$. As $T = U + V = U(I + U^{-1}V)$ and $I + U^{-1}V \in \Sigma_w^0(E)$ by the Riesz theory, it follows from Schaefer [1], Satz 13, that $T \in \Sigma_w^0(E)$. Thus by propositions 4 and 7 there remains to show that the union of null spaces $\bigcup_{n=1}^{\infty} N(T^n)$ is finite dimensional. But since U and V commute we obtain $T^n = U^n(I + U^{-1}V)^n$, so the null spaces of T^n are identical with those of $(I + U^{-1}V)^n$, whose union, again by the Riesz theory, is finite dimensional.

It is not hard to realize that propositions 5 and 6 of the preceding section apply to Riesz instead of Fredholm transformations with only slight modifications in proof. We now state a result which is, in fact, a special case of corollary 2 to proposition 1. Yet it is of particular interest and still a generalization of a theorem which Altman proved in his paper [3] in case E is an F-space.

PROPOSITION 9. *Let E be any locally convex space, A, B continuous linear mappings on E. If $AB = BA$ is Riesz then so are A, B.*

Proof. By prop. 1, cor. 2, we know that $A \in \Sigma_w(E)$. Let $AB = BA = C$. Then since $B^n A^n = C^n$ and C is a Riesz transformation, the union of null spaces $\bigcup_{n=1}^{\infty} N(A^n)$ must be finite dimensional. Similarly, because of $A^n B^n = C^n$ the codimension of $A^n(E)$ must be bounded as $n \to \infty$. Hence, by definition 3, A is a Riesz transformation and, by symmetry, the same applies to B.

COROLLARY. *If U is a continuous linear mapping on E such that U^n is compact for some n, then $I - U$ is a Riesz transformation.*

It is sufficient to apply prop. 9 to $I - U^n = (I-U)(I + U + \ldots + U^{n-1})$.

4. Bounded linear transformations on locally convex spaces [16].

Let E be a locally convex linear space.

PROPOSITION 10. *If T is a bounded linear transformation on E, then its adjoint T' is strongly bounded. If E is a disk space, the converse is also true.*

Proof. By definition 4 there exists a neighborhood U of zero such that $T(U) = B$ is bounded. Now the polar set U^0 of U is weakly compact, hence strongly bounded in E', while B^0 is an $s(E', E)$-neighborhood of zero and $T'(B^0) \subset U^0$, so T' is bounded for the strong topology on E'.

([16]) A general approach to the theory of bounded operators may be found in Grothendieck [1], chap. V.

To prove the converse let T' be the adjoint of some (necessarily continuous) linear mapping T on E. Then by hypothesis $T'(U_1) = B_1$ where $U_1 = B^0$ is a strong neighborhood of zero in E' which we may assume to be the polar of some convex, circled, closed bounded subset of E. Since E is a disk space we have $B_1 \subset U^0$ for some convex, circled, closed neighborhood of zero in E. As $T'(B^0) \subset U^0$ implies $T(U) \subset B$ (Bourbaki [2], p. 101, prop. 2) the proof is complete.

Now for the study of bounded transformations we shall employ the following device. If T is bounded then there clearly exists a convex, circled, closed neighborhood of zero such that $T(U) = B$ is bounded. Let p be the (continuous) semi-norm on E for which $U = \{x: p(x) \leqslant 1\}$. If V denotes the null space of U, i. e., $V = \{x: p(x) = 0\}$, then the quotient space $\mathfrak{B} = E/V$ becomes a normed space by letting $\|\mathfrak{x}\| = p(x)$ for some (and in fact, any) $x \in \mathfrak{x} \in \mathfrak{B}$. Notice that the quotient topology on $\mathfrak{B}$ is finer than the norm topology, thus the natural mapping χ of E onto $\mathfrak{B}$ is continuous though, in general, not a homomorphism. As T is bounded on U it must vanish on V. Hence T gives rise to a linear mapping $\mathfrak{T}$ on $\mathfrak{B}$ by letting $\mathfrak{y} = \mathfrak{T}\mathfrak{x}$ if $y = Tx$, $\mathfrak{x} = \chi(x)$, $\mathfrak{y} = \chi(y)$. Further if $\{p_a\}$ is a set of seminorms generating the topology of E, by the boundedness of T there must be constants C_a such that

$$p_a(Tx) \leqslant C_a p(x)$$

for any $x \in E$ and, in particular, $p(Tx) \leqslant Cp(x)$ for some C. This implies $\|\mathfrak{T}\mathfrak{x}\| \leqslant C\|\mathfrak{x}\|$ so that $\mathfrak{T}$ is a bounded linear mapping on $\mathfrak{B}$ ([17]). The dual space $\mathfrak{B}'$ of $\mathfrak{B}$ is clearly (isomorphic to) the set of linear forms on E that are bounded on U. Hence $\mathfrak{B}'$ may be identified with a subspace of the space $V^0 \subset E'$ orthogonal to V. So if $x' = \mathfrak{x}' \in \mathfrak{B}'$ and $x \in \mathfrak{x} \in \mathfrak{B}$ we obtain

$$\langle \mathfrak{x}, \mathfrak{x}' \rangle = \langle x, x' \rangle$$

for the canonical bilinear forms on $\mathfrak{B} \times \mathfrak{B}'$ and $E \times E'$ respectively or, more precisely, for the restriction to $E \times \mathfrak{B}'$ of $\langle x, x' \rangle$. It should be noted that the weak topology on $\mathfrak{B}'$, $w(\mathfrak{B}', \mathfrak{B})$, coincides with the one induced by $w(E', E)$ (Bourbaki [2], p. 54, prop. 6), while the norm topology on $\mathfrak{B}'$ is finer, in general, than the topology induced on $\mathfrak{B}'$ by $s(E', E)$.

In the propositions to follow $\mathfrak{T}$ will always denote the mapping on $\mathfrak{B} = E/V$ generated by some bounded transformation T on E through the device we have just explained.

PROPOSITION 11. *If $\mathfrak{B}'$ is considered a subset of E' then the adjoint $\mathfrak{T}'$ of $\mathfrak{T}$ is the restriction to $\mathfrak{B}'$ of the adjoint T' of T. Moreover $T'(E') \subset \mathfrak{B}'$.*

([17]) Unless the contrary is expressly stated $\mathfrak{B}$ will be understood to carry the norm topology just introduced.

Proof. We first prove $T'(E') \subset \mathfrak{B}'$. In fact, for any $x' \in E'$ we obtain

$$\sup_{x \in U} |\langle x, T'x' \rangle| = \sup_{x \in U} |\langle Tx, x' \rangle| < +\infty$$

since T is bounded on U. Now let $x' = \mathfrak{x}' \in \mathfrak{B}'$ and $x \in \mathfrak{x} \in \mathfrak{B}$. Then by $\langle Tx, x' \rangle = \langle x, T'x' \rangle$, $\langle \mathfrak{T}\mathfrak{x}, \mathfrak{x}' \rangle = \langle \mathfrak{x}, \mathfrak{T}'\mathfrak{x}' \rangle$, and $\langle Tx, x' \rangle = \langle \mathfrak{T}\mathfrak{x}, \mathfrak{x}' \rangle$ we have $\langle x, T'x' \rangle = \langle \mathfrak{x}, \mathfrak{T}'\mathfrak{x}' \rangle$ for all $x \in E$. So by the identification indicated we obtain $T'x' = \mathfrak{T}'\mathfrak{x}'$ for any $x' = \mathfrak{x}' \in \mathfrak{B}'$.

We shall now prove two propositions concerning the mutual relation between Fredholm (Riesz) points of T and $\mathfrak{T}$ respectively.

PROPOSITION 12. *Let E be any locally convex space. Then if $\lambda \neq 0$ is a Fredholm (Riesz) point of T the same is true of $\mathfrak{T}$.*

Proof. We go back to definition 2 and will show first that $\mathfrak{T}_\lambda$ is a Fredholm transformation if T_λ is. Consider the equations

$$\mathfrak{T}_\lambda \mathfrak{x} = \mathfrak{y}, \quad \mathfrak{T}_\lambda' \mathfrak{y}' = \mathfrak{x}'.$$

Now let $\langle \mathfrak{y}_0, \mathfrak{x}' \rangle = 0$ for any solution of $\mathfrak{T}_\lambda' \mathfrak{x}' = 0$. Then if $y \in \mathfrak{y}_0$ we have $\langle y, x' \rangle = 0$ for any solution of $T_\lambda' x' = 0$ since by $N(T_\lambda') \subset \mathfrak{B}'$ (which, in turn, is a consequence of $T'(E') \subset \mathfrak{B}'$ (prop. 11) and $\lambda \neq 0$) there holds $\langle y, x' \rangle = \langle y, \mathfrak{x}' \rangle = \langle \mathfrak{y}_0, \mathfrak{x}' \rangle = 0$ for any $x' \in N(T_\lambda')$. Thus as T_λ is supposed to be Fredholm, $T_\lambda x = y$ is solvable if $y \in \mathfrak{y}_0$, which implies that $\mathfrak{T}_\lambda \mathfrak{x} = \mathfrak{y}_0$ has a solution. On the other hand, in case $\langle \mathfrak{x}, \mathfrak{x}_0' \rangle = 0$ for any $\mathfrak{x} \in N(\mathfrak{T}_\lambda)$ then $\langle x, x_0' \rangle = 0$ for any $x \in N(T_\lambda)$ if $x_0' = \mathfrak{x}_0'$. Consequently $T_\lambda' y' = x_0'$ has a solution and as each solution y' belongs to $\mathfrak{B}'$ we know that $\mathfrak{T}_\lambda' \mathfrak{y}' = \mathfrak{x}_0'$ is solvable.

Further, the pairs of null spaces $N(T_\lambda)$, $N(\mathfrak{T}_\lambda)$ and $N(T_\lambda')$, $N(\mathfrak{T}_0')$ respectively are each in one-to-one correspondence hence isomorphic. This applies to $N(T_\lambda')$, $N(\mathfrak{T}_\lambda')$ by prop. 11 since $N(T_\lambda') \subset \mathfrak{B}'$ as we have just observed. For $N(T_\lambda)$, $N(\mathfrak{T}_\lambda)$ the assertion is true because, owing to $\lambda \neq 0$, $N(T_\lambda) \cap V = 0$ so that the natural mapping χ is an isomorphism from $N(T_\lambda)$ onto $N(\mathfrak{T}_\lambda)$. Clearly $\mathfrak{T}_\lambda$ is weakly continuous on $\mathfrak{B}$ and the proof is done, at least as far as Fredholm points are concerned. But as each Riesz point is a Fredholm point, to prove the remainder of the proposition it will do by prop. 7 to know that $\bigcup_{n=1}^{\infty} N(\mathfrak{T}_\lambda^n)$ is finite dimensional. Now since to T_λ^n there corresponds $\mathfrak{T}_\lambda^n$ and for a Riesz point of T, $\bigcup_{n=1}^{\infty} N(T_\lambda^n)$ is finite dimensional by hypothesis, the desired conclusion is easily drawn from the fact that $N(T_\lambda^n) \cap V = 0$ for all n.

PROPOSITION 13. *Let E be any locally convex space. Then if $\lambda \neq 0$ is a Fredholm (Riesz) point of $\mathfrak{T}$ the same applies to T.*

Proof. Since by the preceding proof the null spaces of T_λ, $\mathfrak{T}_\lambda$ and T_λ', $\mathfrak{T}_\lambda'$ (and even of their consecutive powers) respectively are isomorphic, we have to show only that T_λ is a weak homomorphism onto a closed subspace of E (cf. propositions 4.7). Now T_λ is weakly continuous; it is a weak homomorphism if and only if $T_\lambda'(E')$ is closed in E'. Let $y' \in E'$ be such that $\langle x_0, y'\rangle = 0$ for any $x_0 \in N(T_\lambda)$. Then let z' be a solution of $T_\lambda' z' = T'y'/\lambda$. (Such a solution exists as $\langle x_0, T'y'\rangle = 0$, $T'y' \in \mathfrak{B}'$, and $\mathfrak{T}_\lambda' \mathfrak{z}' = T'y'/\lambda$ is solvable because $\mathfrak{T}_\lambda'(\mathfrak{B}')$ is closed in $\mathfrak{B}'$, $\mathfrak{T}_\lambda$ being a homomorphism by hypothesis). Letting $x' = (y'+z')/\lambda$, we have $T_\lambda' x' = y'$ so $T_\lambda'(E')$ is closed. To show that $T_\lambda(E)$ is closed consider the equation $T_\lambda x = y_0$. Then from $\langle y_0, x'\rangle = 0$ for any $x' \in N(T_\lambda')$ it follows that $\langle \mathfrak{y}_0, \mathfrak{x}'\rangle = 0$ if $\mathfrak{x}' \in N(\mathfrak{T}_\lambda')$, $\mathfrak{y}_0 = \chi(y_0)$. This implies, by hypothesis, that $\mathfrak{T}_\lambda \mathfrak{x} = \mathfrak{y}_0$ has a solution so $T_\lambda x = y_0$ is solvable and the proof is complete.

Now we establish a proposition which, in the proof of our final theorem, will overcome the difficulty arising from the fact that the normed space $\mathfrak{B}$ is, in general, incomplete.

Proposition 14. *Let E be a sequentially complete locally convex space, T a bounded mapping on E. Then if $\tilde{\mathfrak{T}}$ is the extension of $\mathfrak{T}$ to the completion $\tilde{\mathfrak{B}}$ of $\mathfrak{B}$, the set of non-zero Fredholm (Riesz) points will be the same for $\mathfrak{T}$ and $\tilde{\mathfrak{T}}$.*

Proof. If λ is a Fredholm point of $\mathfrak{T}$ then, by th. 2, $\mathfrak{T}_\lambda = \mathfrak{A} + \mathfrak{a}$ where $\mathfrak{A}$ is a weak isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}$, $\mathfrak{a}$ being finite dimensional. Now $\mathfrak{A}$ clearly extends to a weak isomorphism $\tilde{\mathfrak{A}}$ of $\tilde{\mathfrak{B}}$ onto $\tilde{\mathfrak{B}}$ while the extension $\tilde{\mathfrak{a}}$ to $\tilde{\mathfrak{B}}$ of $\mathfrak{a}$ is still finite dimensional. Hence, by th. 2, $\tilde{\mathfrak{T}}_\lambda = \tilde{\mathfrak{A}} + \tilde{\mathfrak{a}}$ is a Fredholm mapping and this is true for any locally convex space E instead of $\mathfrak{B}$. It is obvious by th. 3 that if $\mathfrak{T}_\lambda$ is a Riesz transformation then so is $\tilde{\mathfrak{T}}_\lambda$.

To prove the converse we first observe that, by the sequential completeness of E, $\tilde{\mathfrak{T}}(\tilde{\mathfrak{B}}) \subset \mathfrak{B}$. For letting $\tilde{\mathfrak{x}} \in \tilde{\mathfrak{B}}$ there is a sequence $\{\mathfrak{x}_n\} \subset \mathfrak{B}$ such that $\mathfrak{x}_n \to \tilde{\mathfrak{x}}$. Hence if $x_n \in \mathfrak{x}_n$ we have $p(x_m - x_n) \to 0$ as $m, n \to \infty$ so by $p_a(Tx) \leqslant C_a p(x)$, $z_n = Tx_n$ turns out to be a Cauchy sequence which by hypothesis converges to some $z \in E$. If $\mathfrak{z} = \chi(z)$ then clearly $\mathfrak{z} = \tilde{\mathfrak{T}}\tilde{\mathfrak{x}} = \lim_{n\to\infty} \mathfrak{T}\mathfrak{x}_n$ as was to be shown.

Now assume that $\lambda \neq 0$ and $\tilde{\mathfrak{T}}_\lambda = \lambda I - \tilde{\mathfrak{T}}$ is Fredholm on $\tilde{\mathfrak{B}}$. It follows from $\tilde{\mathfrak{T}}(\tilde{\mathfrak{B}}) \subset \mathfrak{B}$ that if $\tilde{\mathfrak{T}}_\lambda \mathfrak{x} = \mathfrak{y} \in \mathfrak{B}$ then $\mathfrak{x} \in \mathfrak{B}$, so the null space of $\tilde{\mathfrak{T}}_\lambda$ is a subspace of $\mathfrak{B}$, hence coincides with $N(\mathfrak{T}_\lambda)$; since the latter is finite dimensional we have $\mathfrak{B} = N(\mathfrak{T}_\lambda) \oplus \mathfrak{B}_1$ which implies $\tilde{\mathfrak{B}} = N(\mathfrak{T}_\lambda) \oplus \tilde{\mathfrak{B}}_1$ if $\tilde{\mathfrak{B}}_1$ denotes the completion of $\mathfrak{B}_1$. As $\tilde{\mathfrak{T}}_\lambda$ is an isomorphism of $\tilde{\mathfrak{B}}_1$ onto $\tilde{\mathfrak{T}}_\lambda(\tilde{\mathfrak{B}})$, so is its restriction to $\mathfrak{B}_1$, $\mathfrak{T}_\lambda$, which maps $\mathfrak{B}_1$ onto $\mathfrak{T}_\lambda(\mathfrak{B})$. This implies that $\mathfrak{T}_\lambda$ is a strong and weak homomorphism on $\mathfrak{B}$. We next show that $\mathfrak{T}_\lambda(\mathfrak{B})$ is closed in $\mathfrak{B}$. By the above remark, the equations $\mathfrak{T}_\lambda \mathfrak{x} = \mathfrak{y}$ and $\tilde{\mathfrak{T}}_\lambda \tilde{\mathfrak{x}} = \mathfrak{y}$ are equivalent if $\mathfrak{y} \in \mathfrak{B}$. Hence the former is solvable if and only if $\mathfrak{y} \in \mathfrak{B} \cap N(\tilde{\mathfrak{T}}_\lambda')^0$ which shows that $\mathfrak{T}_\lambda(\mathfrak{B})$ is closed in $\mathfrak{B}$ (since it is the intersection with $\mathfrak{B}$ of a closed subspace of $\tilde{\mathfrak{B}}$). Finally, the dimension of $\mathfrak{B}/\mathfrak{T}_\lambda(\mathfrak{B})$ equals that of $N(\mathfrak{T}_\lambda')$ which in turn is equal to $\dim N(\tilde{\mathfrak{T}}_\lambda)$ as $\mathfrak{T}_\lambda' = \tilde{\mathfrak{T}}_\lambda'$. But we know that $N(\tilde{\mathfrak{T}}_\lambda) = N(\mathfrak{T}_\lambda)$ and, as the case of Riesz points needs no further comment, the proof is complete.

We are now in a position to formulate the main theorem of this section.

Theorem 4. *Let T be a bounded linear mapping on a sequentially complete locally convex space E. Then the spectrum $\sigma(T)$ is a closed bounded subset of the complex plane while the set $\varphi(T)$ of all Fredholm points of T is an open set which splits into two disjoint classes $\varphi_1(T)$, $\varphi_2(T)$ of components such that $\varphi_1(T)$ consists of all Riesz points of T whence $\varphi_2(T) \subset \sigma(T)$. Moreover, the points of $\varphi_1(T) \cap \sigma(T)$ are isolated.*

Proof. From propositions 12 and 13 it follows that the sets of Fredholm and Riesz points of T, except perhaps for $\lambda = 0$, each coincide with the corresponding sets of a bounded mapping $\mathfrak{T}$ on a normed space $\mathfrak{B}$ which, by prop. 14, may be supposed complete. It may happen that $0 \in \sigma(T)$ while it is regular for $\mathfrak{T}$, and if E is not normable then certainly $0 \in \sigma(T)$, as we have seen ([18]).

Now since the resolvent set of T, being a subset of $\varphi(T)$, coincides with that of $\mathfrak{T}$ (except possibly for $\lambda = 0$), $\sigma(T)$ is a closed bounded set. Secondly, by Kračkovsky-Goldman [1] the set of non-zero Fredholm points of $\mathfrak{T}$, $\varphi(\mathfrak{T}) - \{0\}$, is known to have the properties stated in th. 4. Hence the same applies to $\varphi(T) - \{0\}$ and it is easily seen that the statements on $\varphi(T) - \{0\}$ remain unaltered upon adjunction of $\lambda = 0$ if necessary (i. e., if $0 \in \varphi(T)$). The proof is finished.

Corollary. *If there are points $\lambda \neq 0$ such that T_λ is not a Riesz transformation then there are non-zero points in $\sigma(T)$ which are not Fredholm points of T.*

Proof. Clearly if $\lambda \neq 0$ is not Riesz then $\lambda \in \sigma(T)$. Since $\lambda \notin \varphi_1(T)$, λ is either of the type wanted or else $\lambda \in \varphi_2(T)$, which shows that in this case $\varphi_2(T)$ is not empty. Now the boundary of $\varphi_2(T)$ cannot consist of 0

([18]) We might have avoided the exceptional role of $\lambda = 0$ by writing $I - \lambda T$ but prefer to follow the common use in spectral theory.

alone as the spectrum $\sigma(T)$ is bounded. But any boundary point of $\varphi_2(T)$ is not in either $\varphi_1(T)$ or $\varphi_2(T)$ as these are open sets while it must be in $\sigma(T)$ because $\varphi_2(T)\subset\sigma(T)$ and $\sigma(T)$ is closed. Hence the corollary is proved.

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