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A central limit theorem for stochastic processes with independent increments

by

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1. The central limit theorem for sequences $\{Y_k\}$ of random variables (vectors) states, roughly speaking, that whatever are the probability distributions of the particular random variables (vectors), provided that some assumptions are satisfied, the sequence of probability distributions of suitably normed sums ξ_n of Y_k $(k=1,\ldots,n)$ converges as $n\to\infty$ to the corresponding normal probability distribution. The central limit theorem has been generalized in [10] and [5] to random elements in Banach spaces. However, we often have to deal with stochastic processes whose realizations form — by the choice of a convenient distance — a metric non linear function-space. In this case the central limit theorem can be formulated in the following way: Consider a sequence of real stochastic processes $Y_k(t)$ with realizations belonging to some metric, complete and separable function - space $\mathfrak{A}$. Denote by $\xi_n(t)$ a suitably normed sum of $Y_k(t)$ $(k=1,\ldots,n)$ and by $\xi_0(t)$ a Gaussian stochastic process with realizations in $\mathfrak{A}$. Let P^{ξ_n} and P^{ξ_0} denote the probability measures in $\mathfrak{A}$ induced by the finite dimensional distributions ([8], § III,4) of $\xi_n(t)$ and $\xi_0(t)$ respectively. We shall say that the central limit theorem holds if,

$$(1)\qquad P^{\xi_n}\Rightarrow P^{\xi_0},\quad \text{as}\quad n\to\infty.$$

As we know, relation (1) means by definition that for any bounded and continuous function $f(x)$, where $x\in\mathfrak{A}$, the relation

$$\lim_{n\to\infty}\int_{\mathfrak{A}} f(x)\,dP^{\xi_n}=\int_{\mathfrak{A}} f(x)\,dP^{\xi_0}$$

holds.

If we limit ourselves to perfect measures ([6] chap. 1 § 3) then, as Prohorov ([11] Theorem 1.8) has shown, relation (1) holds if and only if for any real function $f(x)$, $x\in\mathfrak{A}$, continuous almost everywhere (P^{ξ_0}) in $\mathfrak{A}$ the sequence of probability distributions of $f[\xi_n(t)]$ converges as

$n \to \infty$ to the probability distribution of $f[\xi_0(t)]$. In this case, whatever the nature of the stochastic processes $Y_k(t)$ under consideration, if some assumptions sufficient for the validity of the central limit theorem are satisfied, the probability distributions of a wide class of functions defined on suitably normed sums $\xi_n(t)$ of $Y_k(t)$ converge as $n \to \infty$ to the probability distribution of the function under consideration defined on the limiting Gaussian stochastic process.

This approach has been used by the author in an earlier paper [4], in which some conditions have been imposed on the moments of $Y_k(t)$ of order 1-4. In the present paper processes with independent increments are considered and only weak assumptions are imposed on the moments of the first and second order.

2. We consider a sequence $\{Y_k(t), 0 \leqslant t \leqslant a\}$ $(k = 1, 2, \ldots)$, where $0 < a < \infty$, of real, separable, independent and equally distributed stochastic processes. The last property means that for an arbitrary finite set of parameter points $t_1, \ldots, t_s$ the finite dimensional distributions of $\{Y_k(t_1), \ldots, Y_k(t_s)\}$ $(k = 1, 2, \ldots)$ are equal. We assume further that the $Y_k(t)$ have independent increments, that the relation

$$P\big(Y_k(0) = 0\big) = 1$$

holds, and that the variance $\sigma^2(t)$ of $Y_k(t)$ satisfies the relation

$$\sigma^2(a) < \infty. \tag{2}$$

It follows that $\sigma^2(t)$ is a non-decreasing and continuous function. Further write

$$\xi_n(t) = \sqrt{n}\left[\frac{1}{n}\sum_{k=1}^{n} Y_k(t) - F(t)\right], \tag{3}$$

where $F(t) = EY_k(t)$. We then have for arbitrary t_1, t_2 $(0 \leqslant t_1 \leqslant t_2 \leqslant a)$

$$E\xi_n(t_1)\,\xi_n(t_2) = EY_k(t_1)Y_k(t_2) - F(t_1)F(t_2) = \sigma^2(t_1). \tag{4}$$

The processes $Y_k(t)$ are also supposed to be centered and without fixed points of discontinuity. The realizations of such processes have ([2], Chap. VIII) with probability 1 left-hand and right-hand limits at each point. We shall finally assume that they are continuous from the right at every t $(0 \leqslant t < a)$ and from the left at $t = a$, and we remark that this assumption is not an essential restriction. In other words, the realizations of the processes considered belong with probability 1 to the space $D\,[0, a]$ of Skorohod-Prohorov ([12] and [11]). This is a metric, complete, separable space with the distance d introduced by Prohorov ([11], p. 228).

THEOREM. *Let $\{Y_k(t), 0 \leqslant t \leqslant a\}$ $(k = 1, 2, \ldots)$, where $0 < a < \infty$, be a sequence of real, separable, centered, independent and equally distributed stochastic processes with independent increments and with no fixed points of discontinuity and let the realizations of $Y_k(t)$ be continuous from the right (at $t = a$ from the left). If relation (2) is satisfied, then relation (1) holds, where $\{\xi_0(t), 0 \leqslant t \leqslant a\}$, is a real, centered, separable, Gaussian stochastic process with independent increments and no fixed points of discontinuity, satisfying the relations*

$$\begin{aligned} &P\big(\xi_0(0) = 0\big) = 1, \\ &E\xi_0(t) = 0, && (0 \leqslant t \leqslant a), \\ &E\xi_0(t_1)\,\xi_0(t_2) = \sigma^2(t_1), && (0 \leqslant t_1 \leqslant t_2 \leqslant a), \end{aligned} \tag{5}$$

and where P^{ξ_n} and P^{ξ_0} are probability measures induced in $D\,[0, a]$ by the finite dimensional distributions of $\xi_n(t)$ and $\xi_0(t)$ respectively.

Proof. We remark first of all that the realizations of $\xi_0(t)$ are continuous with probability 1. Indeed, taking into account the assumed properties of $\sigma^2(t)$ and relations (5) we conclude that the process $\zeta_0(t) = \xi_0(\tau)$, where $t = \sigma^2(\tau)$, is the Brownian motion process in the interval $[0, \sigma^2(a)]$.

We are now going to use a theorem of Skorohod ([14], Theorem 2.1, and a Remark on it, as well as [13], Theorem 3.2.1, and Remark 3.2.5) concerning the convergence of a sequence of functionals defined on $D[0, a]$ and continuous almost everywhere (P^{ξ_0}) in a special topology J_1 introduced by him ([13], p. 293). In our case, however, there is no necessity to deal with this special topology since the realizations of $\xi_0(t)$ are almost all (P^{ξ_0}) continuous and consequently the continuity of functionals defined on $D[0, a]$ with regard to the topology J_1 reduces to continuity with regard to the usual uniform topology as well as with regard to Prohorov's distance d.

In order to prove our theorem it is sufficient-according to the above-mentioned theorem of Skorohod-to show that:

1. For $s = 1, 2, \ldots$ and arbitrary $t_1, \ldots, t_s$ from $[0, a]$ and arbitrary real $z_1, \ldots, z_s$ the relation

$$\lim_{n\to\infty} P^{\xi_n}\big(\xi_n(t_1) < z_1, \ldots, \xi_n(t_s) < z_s\big) = P^{\xi_0}\big(\xi_0(t_1) < z_1, \ldots, \xi_0(t_s) < z_s\big) \tag{6}$$

holds.

2. For an arbitrary $\varepsilon > 0$

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \sup_{|t_1 - t_2| \leqslant c} P^{\xi_n}\big(|\xi_n(t_2) - \xi_n(t_1)| > \varepsilon\big) = 0. \tag{7}$$

Taking into account relations (3)-(5) we obtain (6) at once from the classical limit theorem. We shall now show that (7) holds.

Let $c > 0$ be a fixed constant. Denote by $\varphi_n(t_1, t_2, u)$, where $0 \leqslant t_1 < t_2 \leqslant a$, the characteristic function of $\xi_n(t_2) - \xi_n(t_1)$. Since

$$E[\xi_n(t_2) - \xi_n(t_1)] = 0, \qquad D^2[\xi_n(t_2) - \xi_n(t_1)] = \sigma^2(t_2) - \sigma^2(t_1), \tag{8}$$

we have

$$\lim_{n\to\infty} \varphi_n(t_1, t_2, u) = \exp\left\{-\frac{\sigma^2(t_2) - \sigma^2(t_1)}{2} u^2\right\}, \tag{9}$$

rand the convergence in (9) is unifo with regard to the variate

$$w = u\sqrt{\sigma^2(t_2) - \sigma^2(t_1)}$$

in every finite interval of w. Taking into account the assumed properties of $\sigma^2(t)$, we conclude that the convergence in (9) is uniform with regard to t_1 and t_2. This implies, by the use of Lévy's uniqueness theorem for characteristic functions ([9], § 14), that the sequence of distribution functions of $\xi_n(t_2) - \xi_n(t_1)$ converges to that of $\xi_0(t_2) - \xi_0(t_1)$ uniformly with regard to t_1 and t_2. Consequently

$$\lim_{n\to\infty} \sup_{|t_2 - t_1| \leqslant c} P^{\xi_n}\big(|\xi_n(t_2) - \xi_n(t_1)| > \varepsilon\big) = \sup_{|t_2 - t_1| \leqslant c} P^{\xi_0}\big(|\xi_0(t_2) - \xi_0(t_1)| > \varepsilon\big). \tag{10}$$

Now since $\xi_0(t)$ has independent increments and is stochastically continuous, it is, as has been shown in [15], uniformly stochastically continuous, and consequently for any $\varepsilon > 0$

$$\lim_{c\to 0} \sup_{|t_2 - t_1| \leqslant c} P^{\xi_0}\big(|\xi_0(t_2) - \xi_0(t_1)| > \varepsilon\big) = 0. \tag{11}$$

Relation (7) follows from relations (10) and (11). Our theorem is thus proved.

The probability distributions of a large number of functionals $f[\zeta_0(t)]$, continuous in the uniform topology, where $\zeta_0(t)$ is a Brownian motion process, are known. Hence using our theorem we can find the limiting distributions of $f[\xi_n(t)]$. For instance, if $\xi_n(t)$ satisfy all assumptions

of our theorem, we have in virtue of Bachelier's [1] result (see also [3] and [7]) for $\lambda > 0$

$$\lim_{n\to\infty} Pr\big(\sup_{0\leqslant t\leqslant a} \xi_n(t) < \lambda\big) = \sqrt{\frac{2}{\pi\sigma^2(a)}} \int_0^\lambda \exp\left(-\frac{x^2}{2\sigma^2(a)}\right) dx,$$

$$\lim_{n\to\infty} Pr\big(\sup_{0\leqslant t\leqslant a} |\xi_n(t)| < \lambda\big) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \exp\left(-\frac{(2j+1)^2\pi^2\sigma^2(a)}{8\lambda^2}\right).$$

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