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# On inversion of non-linear transformations

by

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**1.** Let $U$ be a continuous one-to-one transformation of $X$ onto $Y$, where both $X$ and $Y$ are $F$-spaces (that is linear metric spaces, see [1]). It is a well-known fact that if $U$ is only additive, then $U^{-1}$ is continuous (see [1], p. 41, Theorem 5). In the general case, however, it may happen that the inverse transformation $U^{-1}$ is not continuous. This follows from a simple example: we put $X = Y = l^2$ and $U(x) = \varphi(x)x$, where

$$\varphi(x) = \sum_{n=1}^{\infty} \xi_n^2/2, \qquad x = (\xi_n) \in l^2;$$

$U^{-1}$ is not continuous in zero. In fact, $U^{-1}(y) = \varphi(y)^{-1/3}y$ for $y \neq 0$, and thus for $y_k = (k^{-1}\delta_k^n)$ [1]) we have $y_k \to 0$ and $\|U^{-1}(y_k)\| = \sqrt[3]{2^k/k} \to \infty$.

The set of all points of discontinuity of an inverse transformation $U^{-1}$ may be everywhere dense. This can be proved as follows.

If $U_n$ is a continuous one-to-one transformation of $X_n$ onto $Y_n$, where $X_n$, $Y_n$ are $F$-spaces and $n = 1, 2, \ldots$, then the formula $U(x) = U_n\{(x_n)\}$, where $x = \{x_n\}, x_n \in X_n$, defines a one-to-one continuous transformation of the product space $X$ of $X_1, X_2, \ldots$ with the norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x_n\|}{1+\|x_n\|}$$

onto the product space $Y$ of $Y_1, Y_2, \ldots$ with the norm

$$\|y\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|y_n\|}{1+\|y_n\|}.$$

It is easy to see that if every $U_n^{-1}$ is discontinuous at one point at least, say $y_n^0$, then the set of all points of discontinuity of $U^{-1}$ is dense. In fact, $U^{-1}$ is discontinuous at all the points $y = \{y_n\}$ where $y_n = y_n^0$ for some $n$,

[1]) $\delta_k^n = \begin{cases} 0 & \text{for} \quad n \neq k, \\ 1 & \text{for} \quad n = k. \end{cases}$

and the set of all such points is dense. S. Mazur put forward the question whether in general the transformation $U^{-1}$ is of the first Baire class.

The problem of S. Mazur is still open.

This paper concerns similar problems where $U$ satisfies different kinds of continuity conditions.

**2.** It will be useful in the following to prove that a transformation $U$ of an $F$-space $X$ into an $F$-space $Y$ is of the first class of Baire if and only if it is of the first class of Borel.

A metric space $Y$ will be called *retractive* if for every $r>0$ there exists a continuous function $h_r(x,y)$ on the product space $Y\times Y$ with the norm $\|(x,y)\|=\|x\|+\|y\|$ such that $h_r(x,y)\in Y$, $\varrho(h_r(x,y)y)\leqslant r$ for every $x,y\in Y$ and $h_r(x,y)=x$ whenever $\varrho(x,y)\leqslant r$.

As an example of a retractive space we may take an arbitrary $F$-space in which the norm $\|x\|$ is increasing, *i. e.* such that $\|tx\|$ is an increasing function of $t$, $t\geqslant 0$.

In fact, if we put

$$h_r(x,y)=\begin{cases} x & \text{if} \quad \|x-y\|\leqslant r,\\ y+T(x-y), \end{cases}$$

where $T$ is chosen in such a way that $\|T(x-y)\|=r$, for all $x$ such that $\|x-y\|>r$, then the function $h_r(x,y)$ will do.

LEMMA 1. *For each natural $n$ let the operation $U_n$ be a transformation of the first class of Baire which maps a metric space $X$ into a retractive space $Y$. If the sequence $U_n$ converges uniformly in $X$, then the limit tranformation $U$ is also of the first class of Baire.*

Proof. Since $U_n$ converges uniformly, it contains a subsequence $U_{k_n}=W_n$ such that $\varrho(W_n(x),W_{n+1}(x))\leqslant 1/2^n$.

The transformations $W_n$ being of the first class of Baire, there exists for each $n$ such a sequence of continuous transformations $W_n^k$ that

$$\lim_k W_n^k(x)=W_n(x).$$

We define by induction the sequences $V_n^k(x)$:

$$V_1^k(x)=W_1^k(x);\qquad V_{n+1}^k(x)=h_{1/2^n}(W_{n+1}^k(x),V_n^k(x)).$$

From the properties of $h_r(x,y)$ it follows that every $V_n^k(x)$ is continuous and

$$\varrho(V_{n+1}^k(x),V_n^k(x))\leqslant 1/2^n$$

for every $x$, and for each $x$ and $n$ there exists such a $k_n$ that for every $k>k_n$ we have $V_n^k(x)=W_n^k(x)$.



Now we are going to show that

$$\lim_n V_n^n(x)=U(x).$$

We take an arbitrary point $x\in X$ and a real number $\varepsilon>0$. Since $W_n(x)$ tends uniformly to $U(x)$, there exists such an $n_0$ that for $n>n_0$

$$\varrho(W_n(x),U(x))<\varepsilon/3 \quad \text{and} \quad 2^{n-1}<\varepsilon/3$$

On the other hand, for arbitrary $n>n_0$ there is such a $k_n$ that

$$W_n^{k_n}(x)=V_n^{k_n}(x)\quad \text{and} \quad \varrho(W_n^{k_n}(x),W_n(x))<\varepsilon/3.$$

Then

$$\varrho(V_n^n(x),U(x))\leqslant\varrho(V_n^n(x),V_n^{k_n}(x))+\varrho(V_n^{k_n}(x),W_n(x))+\varrho(W_n(x),U(x))$$
$$=\varepsilon/3+\varepsilon/3+2^{n-1}<\varepsilon.$$

Thus $\lim_n V_n^n(x)=U(x)$ and $U(x)$ is of the first class of Baire.

LEMMA 2. *If a transformation $U$ maps a metric space $X$ into a separable metric arcwise connected space $Y$ and the set of its values is isolated, then $U$ is of the first class of Baire.*

Proof. Since $Y$ is separable, it follows that the set of the values of $U$ is at most countable, and we can form from them a sequence $\{y_n\}$. The set $F_n=\{x: U(x)=y_n\}$ is an $F_\sigma$ as an inverse image of an open set. Hence $F_n=\bigcup_k F_n^k$, where $F_n^k$ are closed sets and $F_n^k\subset F_n^{k+1}$.

There are continuous transformations $U_n$ such that $U_n(x)=y_i$ for $x\in F_i^n$, $i\leqslant 1\leqslant n$ (see [2], lemma 8). It is easy to see that if $x\in F_n^k$ and $m>\max(n,k)$, then $U_m(x)=U(x)$, whence $\lim_n U_n(x)=U(x)$, and thus $U$ is of the first class of Baire.

LEMMA 3. *In order that a transformation $U$ of a metric space $X$ into a retractive separable arcwise connected space $Y$ be of the first class of Baire, it is necessary and sufficient that $U$ be of the first class of Borel.*

Proof. Necessity follows from the well known fact that the limit of transformation of the $a$ Borel class is of $a+1$ Borel class (see [4], p. 293).

To prove sufficiency we note that every transformation $U$ of the first Borel class is a limit of a uniformly convergent sequence of transformations of the first Borel class and thus for each $n$ the set of values of $U_n$ is isolated (see [2], lemma 4). Accordingly, by lemma 2 we see that each $U_n$ is of the first class Baire and by lemma 1 so is $U$.

Since the definitions of the class of Baire as well as of Borel are invariant under homeomorphisms, it follows that when $Y$ is an arbitrary space homeomorphic with a certain retractive separable arcwise connected space, the equivalence of these classes still holds. In particular, since in every $F$-space we can introduce an increasing norm (see [3]) it follows that a transformation $U$ of a metric space $X$ into a separable $F$-space $Y$ is of the first class of Baire if and only if it is of the first class of Borel.

3. Now we turn back to the problem of inversion of transformations. In what follows sequences convergent in the norm topology will be called *n-convergent*, weakly convergent sequences of elements will be called *e-convergent*, and weakly convergent sequences of functionals will be called *f-convergent*.

A transformation $U$ of an $F$-space $X$ into $F$-space is called *$\tau$-$\sigma$-continuous* if it transforms every $\tau$-convergent sequence into a $\sigma$-convergent one[2]). An $F$-space $X$ will be called *locally $\tau$-compact* if every ball $K\subset X$ is $\tau$-compact in itself[3]).

As examples of locally $\tau$-compact spaces we can take the spaces conjugate to separable $B$-spaces with $f$-convergence (see [1], p. 123, theorem 3) as well as $H^{\mu}$-spaces of functions $x(t)$ on $[0,1]$ satisfying Hölder's condition with exponent $\mu$, where

$$\|x\| = \sup_{0<t<1}|x(t)| + \sup_{0<t<t<1}\frac{|x(t)-x(t')|}{|t-t'|^{\mu}}$$

and the $\tau$-convergence is the ordinary uniform convergence.

THEOREM. *If $U$ is a one-to-one $\tau$-$\sigma$-continuous transformation of a separable locally $\tau$-compact $F$-space $X$ into an $F$-space $Y$ in which every n-convergent sequence is $\tau$-convergent, then the inverse transformation $U^{-1}$ is of the first class of Baire (according to n-convergence).*

Proof. We take a number $\varepsilon>0$, and then cover $X$ with a countable family of bells of radius $\varepsilon$. Since these bells are $\tau$-compact, their images are $\sigma$-compact, whence $\sigma$-closed, whence closed in the norm topology, the norm convergence being stronger than $\sigma$-convergence. It follows that the range of $U^{-1}$ can be covered by a countable family of sets whose diameters do not exceed $2\varepsilon$, and whose inverse images are $F_{\sigma}$. Thus $U^{-1}$ is of the first class of Borel, and because of the equivalence of the first class of Baire and that of Borel in separable $F$-spaces, we find that $U^{-1}$ is of the first class of Baire.

COROLLARY 1. *If a one-to-one $\tau$-f-continuous transformation maps a locally $\tau$-compact separable space $X$ into $Y$, where $Y$ is a conjugate to a certain separable $B$-space* **Y**, *then $U^{-1}$ is a limit of a sequence of transformations $(U^{-1})_n$ which are f-n-continuous.*

[2]) $\tau$-convergence, $\sigma$-convergence can be arbitrary 1*-convergence.

[3]) $K$ is $\tau$-compact in itself if in every sequence $x_n \epsilon K$ we can find a subsequence $\tau$-convergent to $x_0 \epsilon K$.



Proof. We introduce in $Y$ another norm as follows:

$$\|y\| = \sum_{n=1}^{\infty}\frac{1}{2^n}|y(\eta_n)|,$$

where the sequence $\eta_n$ is dense in **Y**, the convergence induced by this norm being called *$n_0$-convergence*. Every $f$-convergent sequence is $n_0$-convergent, and thus $U^{-1}$ is of the first class of Baire, whence it is a limit of a sequence $(U^{-1})_n$ of $n_0$-$n$-continuous transformations, which are also *f-n-continuous*.

COROLLARY 2. *If $U$ is a one-to-one $\tau$-e-continuous transformation of a locally $\tau$-compact separable $B$-space $X$ into a separable $B$-space $Y$, then the inverse transformation $U^{-1}$ is a limit of a sequence $(U^{-1})_n$ of transformations which are e-n-continuous.*

Proof. We take a sequence $y$ which is dense in $Y$, and we introduce a new norm in $Y$

$$\|y\| = \sum_{n=1}^{\infty}\frac{1}{2}|F_n(y)|,$$

where $F_n$ are functionals such that $\|F_n\|=1$ and $|F_n(y_n)|=1$ (see [1], p. 51).

Then we proceed as in the preceding proof.

## References

[1] S. Banach, *Théorie des opérations linéaires*, New York 1949.

[2] — *Über analytisch darstellbare Operationen in abstrakten Räumen*, Fund. Math. 17 (1931), p. 283-295.

[3] M. Eidelheit und S. Mazur, *Bemerkung über die Räume vom Typus (F)*, Studia Math. 7 (1938), p. 159-161.

[4] C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1948.

[5] — *Topologie II*, Warszawa 1952.

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