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Effective processes in the sense of H. Steinhaus

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1. Let E be a Lebesgue measurable subset of the positive half-line. By $|E|$ we shall denote the Lebesgue measure of E. We say that E is *relatively measurable* if the limit

$$|E|_R = \lim_{T\to\infty} \frac{1}{T} |E \cap \{t\colon 0 \leqslant t \leqslant T\}|$$

exists. The number $|E|_R$ is called *the relative measure of* E.

We say that a system of real-valued functions $g_1(t), g_2(t), \ldots, g_k(t)$ defined for $0 \leqslant t < \infty$ is *relatively measurable* if for every $x_1, x_2, \ldots, x_k$ sets $\bigcap_{j=1}^{k} \{t\colon g_j(t) < x_j\}$ are relatively measurable.

Let $f(t)$ be a real-valued function defined for $0 \leqslant t < \infty$. For every interval $I = \{t\colon a_I \leqslant t < b_I\}$ $(0 \leqslant a_I < b_I)$ we shall use the following notation:

$$f^*(I) = f(b_I) - f(a_I), \qquad I+t = \{u+t\colon u \in I\}.$$

We say that $f(t)$ is *an effective process with independent increments* if for every integer k and for every system of disjoint intervals $I_1, I_2, \ldots, I_k$ the system of functions $g_j(t) = f^*(I_j+t)$ $(j = 1, 2, \ldots, k)$ is relatively measurable,

$$(1) \qquad |\bigcap_{j=1}^{k} \{t\colon f^*(I_j+t) < x_j\}|_R = \prod_{j=1}^{k} |\{t\colon f^*(I_j+t) < x_j\}|_R$$

for each $x_1, x_2, \ldots, x_k$ and

$$(2) \qquad D_I(x) = |\{t\colon f^*(I+t) < x\}|_R$$

for every interval I is a distribution function, i. e. is a monotone non-decreasing function continuous on the left, with $D_I(-\infty) = 0$, $D_I(\infty) = 1$. (This notion has been proposed by H. Steinhaus).

We remark that so far for non-degenerate functions (2) there is no effective example of effective processes with independent increments.

In the sequel we shall denote by $P(\Phi)$ the probability of a random event defined by a condition Φ.

In the present note we prove the following

THEOREM. *Let $f(\omega, t)$ be a measurable separable homogeneous stochastic process with independent increments. Then almost all realizations $f(\omega_0, t)$ are effective processes with independent increments. Moreover, for every interval I and for every real number x the equality*

$$|\{t: f^*(\omega_0, I+t) < x\}|_R = P(f^*(\omega, I) < x) \tag{3}$$

is true.

This theorem is an answer to a problem raised by H. Steinhaus and can be regarded as an ergodic theorem for homogeneous stochastic processes with independent increments. A special case of this theorem, when $f(\omega, t)$ is a Brownian movement process or a Poisson process, has been given by C. Ryll-Nardzewski. For compound Poisson processes with a denumerable set of states our assertion is connected to some extent with the work of Khintchine ([4], p. 69).

II. Before proving the theorem we shall give some elementary properties of homogeneous stochastic processes with independent increments.

Let $f(\omega, t)$ be a measurable separable homogeneous stochastic process with independent increments. Then for every $\varepsilon > 0$

$$\lim_{|I|\to 0} P(|f^*(\omega, I)| \geqslant \varepsilon = 0 \tag{4}$$

(cf. [1], p. 117). From the results of Lévy (cf. [3], [2], p. 407) it follows that there is an interval function $g(I)$ for which $f^*(\omega, I)+g(I)\to 0$ with probability 1 if I contracts to a fixed point. The last formula, in view of (4), implies the convergence $f^*(\omega, I)\to 0$ with probability 1 if I contracts to a fixed point. Consequently, for every $\varepsilon > 0$,

$$\lim_{|I|\to 0} P\big(\sup_{J\subset I} |f^*(\omega, J)| \geqslant \varepsilon\big) = 0 . \tag{5}$$

(In virtue of the separability of $f(\omega, t)$, $\sup_{J\subset I}|f^*(\omega, J)|$ is a random variable, i. e. an ω measurable function.) Further, the characteristic function $\varphi_I(z)$ of $f^*(\omega, I)$ is given by the Lévy-Khintchine formula

$$\varphi_I(z) = \exp\left\{i\gamma_f|I|z + |I|\int_{-\infty}^{\infty}\left(e^{iuz} - 1 - \frac{iuz}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG_f(u)\right\}, \tag{6}$$

where γ_f is a real constant and G_f is a monotone non-decreasing bounded function, with $G_f(-\infty) = 0$ (cf. [2], p. 419). Set

$$F_I(x) = P\big(f^*(\omega, I) < x\big). \tag{7}$$

Obviously,

$$\lim_{J\to I} F_J(x) = F_I(x) \tag{8}$$

at all continuity points of the limit function.

In the sequel we shall denote by $N_f(I)$ the set of all discontinuity points of $F_I(x)$ and by N_f the union of all sets $N_f(I)$:

$$N_f = \bigcup_{I\neq 0} N_f(I). \tag{9}$$

LEMMA 1. *Let $u_1, u_2, \ldots$ and $a_1, a_2, \ldots$ be two sequences of positive numbers such that*

$$\sum_{k=1}^{\infty} u_k^2 a_k < \infty, \qquad \sum_{k=1}^{\infty} a_k = \infty. \tag{10}$$

Then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \exp\left\{\sum_{k=1}^{\infty}(\cos zu_k - 1)a_k\right\}dz = 0.$$

Proof. For brevity we write

$$m(z) = \sum_{k=1}^{\infty}(1-\cos zu_k)a_k. \tag{11}$$

Put

$$Q_T(x) = \frac{1}{T}\int_0^T e^{-xm(z)}dz. \tag{12}$$

To prove the lemma it is sufficient to show that $\lim_{T\to\infty} Q_T(1) = 0$. Contrary to this statement let us suppose that there is a sequence $T_1, T_2, \ldots \to \infty$ for which

$$\lim_{n\to\infty} Q_{T_n}(1) = q > 0. \tag{13}$$

From (10), (11) and (12) it follows that the function $Q_T(x)$ is differentiable and

$$\frac{d}{dx}Q_T(x) = -\frac{1}{T}\int_0^T m(z)e^{-xm(z)}dz.$$

Since $m(z)\geqslant 0$, we have for some $\tilde{x}$ $(\frac{1}{2}\leqslant \tilde{x}\leqslant 1)$

$$Q_T(\tfrac{1}{2}) - Q_T(1) = \frac{1}{2T}\int_0^T m(z)e^{-\tilde{x}m(z)}dz \geqslant \frac{1}{2T}\int_0^T m(z)e^{-m(z)}dz.$$

Hence, taking into account the inequality $0 \leqslant Q_T(x) \leqslant 1$ for $x \geqslant 0$, we obtain

$$\frac{1}{T}\int_0^T m(z)e^{-m(z)}dz \leqslant 2. \tag{14}$$

Further, from equality (13) it follows that there is then a positive number c such that

$$\limsup_{n\to\infty}\frac{1}{T_n}|E\cap\{z\colon 0\leqslant z\leqslant T_n\}| > 0,$$

where

$$E = \{z\colon e^{-m(z)}\geqslant c\}. \tag{15}$$

By $\chi_E(z)$ we shall denote the indicator of E, i. e. $\chi_E(z) = 1$ or 0 according as z belongs or does not belong to E. We may suppose, without loss of the generality of our considerations, that the following limits exist:

$$0 < c_0 = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\chi_E(z)dz, \tag{16}$$

$$c_k = \lim_{n\to 0}\frac{\sqrt{2}}{T_n}\int_0^{T_n}\chi_E(z)\cos zu_k dz \quad (k = 1, 2, \ldots). \tag{17}$$

Using the well-known formula

$$\frac{1}{T_n}\int_0^{T_n}\chi_E(z)dz = \frac{1}{T_n}\int_0^{T_n}\Big\{\chi_E(z)-\sum_{j=1}^{k}\sqrt{2}\,c_j\cos zu_j\Big\}^2dz+\sum_{j=1}^{k}c_j^2+o(1)$$

we obtain the inequality $\sum_{j=1}^{k} c_j^2 \leqslant 1$ $(k = 1, 2, \ldots)$. There is then an index k_0 such that, according to (16),

$$|c_k| \leqslant \frac{c_0}{\sqrt{2}} \quad \text{for} \quad k \geqslant k_0. \tag{18}$$

Since

$$m(z) \geqslant \sum_{k=k_0}^{\infty}(1-\cos zu_k)a_k,$$

we have, according to (15), (16), (17) and (18),

$$\liminf_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}m(z)e^{-m(z)}dz \geqslant \liminf_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\chi_E(z)m(z)e^{-m(z)}dz$$

$$\geqslant c\liminf_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\chi_E(z)m(z)dz \geqslant c\liminf_{n\to\infty}\sum_{k=k_0}^{\infty}\frac{1}{T_n}\int_0^{T_n}\chi_E(z)(1-\cos zu_k)a_k dz$$

$$\geqslant c\sum_{k=k_0}^{\infty}a_k\Big(c_0-\frac{c_k}{\sqrt{2}}\Big) \geqslant \frac{cc_0}{2}\sum_{k=k_0}^{\infty}a_k = \infty,$$

which contradicts inequality (14). The lemma is thus proved.

LEMMA 2. *Let $f(\omega, t)$ be a measurable separable homogeneous stochastic process with independent increments, satisfying the condition*

$$\int_{-1}^{1}\frac{1}{u^2}dG_f(u) = \infty. \tag{19}$$

Then the equality $N_f = 0$ holds.

Proof. Since

$$F_I(x+0)-F_I(x-0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-ixz}\varphi_I(z)dz,$$

to prove the lemma it is sufficient to show that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\varphi_I(z)|dz = 0 \tag{20}$$

for every non-empty interval I. Setting $H(u) = \frac{1}{2}(G_f(u)-G_f(-u))$, we have, according to (6) and (19),

$$|\varphi_I(z)| = \exp\Big\{|I|\int_{-\infty}^{\infty}(\cos zu-1)\frac{1+u^2}{u^2}dH(u)\Big\}, \tag{21}$$

$$\int_0^1\frac{1}{u^2}dH(u) = \infty. \tag{22}$$

Let H_c be the continuous component of H and let H_j be the jump component:

(23) $$H(u) = H_c(u)+H_j(u).$$

First we assume that

(24) $$\int_0^1 \frac{1}{u^2}\,dH_c(u) = \infty.$$

Given an arbitrary $\varepsilon > 0$, we consider independent random variables ξ and η with the common distribution function

(25) $$P(\xi < x) = \sum_{k=0}^{\infty} \frac{V_k(x)}{k!} \exp\left\{-\frac{|I|}{2}\int_\varepsilon^\infty \frac{1+u^2}{u^2}\,dH_c(u)\right\},$$

where

$$V_0(x) = \begin{cases} 0 & \text{if} \quad x \leqslant 0, \\ 1 & \text{if} \quad x > 0, \end{cases}$$

$$V_1(x) = \begin{cases} 0 & \text{if} \quad x \leqslant \varepsilon, \\ \dfrac{|I|}{2}\displaystyle\int_\varepsilon^x \frac{1+u^2}{u^2}\,dH_c(u) & \text{if} \quad x > \varepsilon, \end{cases}$$

$$V_{k+1}(x) = \int_0^x V_k(x-y)\,dV_1(y) \qquad (k = 1, 2, \ldots).$$

From (25) immediately follows the equality

$$P(\xi = x) = \begin{cases} 0 & \text{if} \quad x \neq 0, \\ \exp\left\{-\dfrac{|I|}{2}\displaystyle\int_\varepsilon^\infty \frac{1+u^2}{u^2}\,dH_c(u)\right\} & \text{if} \quad x = 0. \end{cases}$$

Consequently,

(26) $$P(\xi-\eta = 0) = \exp\left\{-|I|\int_\varepsilon^\infty \frac{1+u^2}{u^2}\,dH_c(u)\right\}.$$

Let $\psi(z)$ be the characteristic function of $\xi-\eta$. It is easy to verify, in view of (25), that

$$\psi(z) = \exp\left\{-|I|\int_\varepsilon^\infty (\cos zu-1)\frac{1+u^2}{u^2}\,dH_c(u)\right\}.$$

Hence, according to (21) and (23), $|\varphi_I(z)| \leqslant \psi(z)$. Since

$$P(\xi-\eta = 0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \psi(z)\,dz,$$

the last inequality and formula (26) imply

$$\limsup_{T\to\infty} \frac{1}{T}\int |\varphi_I(z)|\,dz \leqslant \exp\left\{-|I|\int_\varepsilon^\infty \frac{1+u^2}{u^2}\,dH_c(u)\right\}.$$

Hence, in virtue of (24), for $\varepsilon \to 0$ we obtain equality (20).

Now we assume that $\int_0^1 u^{-2}dH_c(u)$ is finite. Then, in view of (22) and (23), the equality

(27) $$\int_0^1 \frac{1}{u^2}\,dH_j(u) = \infty$$

is true. From (21) and (22) immediately follows

$$|\varphi_I(z)| \leqslant \exp\left\{|I|\int_0^1 (\cos zu-1)\frac{1}{u^2}\,dH_j(u)\right\}.$$

The discontinuity points of $H(u)$ $(0 < u < 1)$ will be denoted by $u_1, u_2, \ldots$ Then

$$|\varphi_I(z)| \leqslant \exp\left\{\sum_{k=1}^\infty (\cos zu_k-1)a_k\right\},$$

where

$$a_k = \frac{|I|}{u_k^2}\big(H_j(u_k+0)-H_j(u_k-0)\big) > 0 \qquad (k = 1, 2, \ldots).$$

Obviously, $\sum_{k=1}^\infty u_k^2 a_k < \infty$ and, in view of (27), $\sum_{k=1}^\infty a_k = \infty$. Hence, in virtue of lemma 1, we obtain equality (20). The lemma is thus proved.

LEMMA 3. *Let $f(\omega, t)$ be a measurable separable homogeneous stochastic process with independent increments for which $N_f \neq 0$. There is then a real constant β_f such that setting $f_0(\omega, t) = f(\omega, t)-\beta_f t$*

(28) $$\lim_{|I|\to 0} P\big(\sup_{J\subset I} |f_0^*(\omega, J)| = 0\big) = 1$$

and for every x

(29) $$\lim_{J\to I} P\big(f_0^*(\omega, J) < x\big) = P\big(f_0^*(\omega, I) < x\big).$$

Moreover, N_{f_0} is a countable set.

Proof. From the assumption $N_f \neq 0$, in virtue of lemma 2, it follows that

$$\int_{-1}^{1} \frac{1}{u^2}\, dG_f(u) < \infty .$$

Setting

$$\beta_f = \gamma_f - \int_{-\infty}^{\infty} \frac{1}{u}\, dG_f(u), \qquad H_f(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2}\, dG_f(u)$$

we have, according to (6),

$$\varphi_I(z) = \exp\left\{i\beta_f|I|z + |I| \int_{-\infty}^{\infty} (e^{izu}-1)\, dH_f(u)\right\}.$$

Taking into account the last equality it is easy to verify that for every Borel subset E the equality

$$P\big(f_0^*(\omega, I)\,\epsilon\, E\big) = \sum_{k=0}^{\infty} \frac{|I|^k}{k!} \int_E dH_f^{*k}(u) \exp\big(-|I|H_f(\infty)\big) \tag{30}$$

holds, where

$$H_f^{*0}(x) = \begin{cases} 0 & \text{if} \quad x \leqslant 0, \\ 1 & \text{if} \quad x > 0, \end{cases}$$

$$H_f^{*1}(x) = H_f(x), \qquad H_f^{*k+1}(x) = \int_{-\infty}^{\infty} H_f^{*k}(x-y)\, dH_f(y) \quad (k = 1, 2, \ldots).$$

The following equality is a direct consequence of the last formula $N_{f_0}(I) = N_{f_0}(J)$ for each $I, J \neq 0$. Consequently, if we take into account the definition (9), N_{f_0} is a countable set. Moreover, from (30) immediately follows assertion (29). Equality (30) also implies the well-known formula

$$P\big(\sup_{J\subset I} |f_0^*(\omega, J)| = 0\big) = e^{-|I|h},$$

where

$$h = \lim_{|I|\to 0} \frac{1-P(f_0^*(\omega, I) = 0)}{|I|} = H_f(\infty) - H_f(+0) + H_f(-0)$$

(cf. [2], p. 259). Hence we obtain assertion (28). The lemma is thus proved.

III. Proof of theorem. Without loss of generality we may suppose that, in the case $N_f \neq 0$, the constant β_f determined by lemma 3 is equal to 0. In other words

$$f(\omega, t) = f_0(\omega, t) \quad \text{if} \quad N_f \neq 0 . \tag{31}$$

For brevity we write $\tilde{f}(\omega, I)$ instead of $\sup_{J\subset I} f^*(\omega, J)$ and

$$\tilde{F}_I(x) = P\big(\tilde{f}(\omega, I) < x\big). \tag{32}$$

From the assumption of the separability and the measurability of $f(\omega, t)$ it follows that for every interval I the process $\tilde{f}(\omega, I+t)$ is measurable. Set

$$g_{I,v}(\omega, t) = \begin{cases} 0 & \text{if} \quad f^*(\omega, I+t) \geqslant v, \\ 1 & \text{if} \quad f^*(\omega, I+t) < v, \end{cases} \tag{33}$$

$$\tilde{g}_{I,v}(\omega, t) = \begin{cases} 0 & \text{if} \quad \tilde{f}(\omega, I+t) \geqslant v, \\ 1 & \text{if} \quad \tilde{f}(\omega, I+t) < v, \end{cases} \tag{34}$$

$$\tilde{g}_I(\omega, t) = \begin{cases} 0 & \text{if} \quad \tilde{f}(\omega, I+t) \neq 0, \\ 1 & \text{if} \quad \tilde{f}(\omega, I+t) = 0. \end{cases} \tag{35}$$

Obviously, the processes (33), (34) and (25) are measurable. From the homogeneity of $f(\omega, t)$ we infer that all the processes

$$\prod_{j=1}^{k} g_{I_j,v_j}(\omega, t), \qquad \tilde{g}_{I,v}(\omega, t), \qquad \tilde{g}_I(\omega, t) \tag{36}$$

are strictly stationary. It is well known that the homogeneous stochastic process $f(\omega, t)$ with independent increments is metrically transitive relatively to the difference field, i. e. relatively to the smallest Borel field of ω sets with respect to which all the increments $f^*(\omega, I)$ are measurable (cf. [2], p. 512). Consequently, all the processes (36) are metrically transitive. Moreover, the expectations of processes (36) are finite.

Let us denote by R the set of all rational numbers. The set of all non-empty intervals with rational endpoints will be denoted by $\mathfrak{R}$. From lemma 3, in virtue of assumption (31), we infer that the set $R \cup N_f$ is denumerable. There is then, in view of Birkhoff's ergodic theorem (cf. [2], p. 515), a ω set Ω_0, with $P(\Omega_0) = 1$, such that all realizations $f(\omega_0, t)$ $(\omega_0 \epsilon\, \Omega_0)$ are Lebesgue measurable functions and for each $U, U_1, U_2, \ldots, U_k \epsilon \mathfrak{R}$, $v, v_1, v_2, \ldots, v_k \,\epsilon\, \mathfrak{R} \cup N_f$ $(k = 1, 2, \ldots)$ the following limits exist:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \prod_{j=1}^{k} g_{U_j,v_j}(\omega_0, t)\, dt = E \prod_{j=1}^{k} g_{U_j,v_j}(\omega, 0), \tag{37}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{g}_{U,v}(\omega_0, t)\, dt = E\, \tilde{g}_{U,v}(\omega, 0), \tag{38}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{g}_U(\omega_0, t)\, dt = E\, \tilde{g}_U(\omega, 0). \tag{39}$$

From (7), (32), (33), (34) and (35) we obtain the following equalities:

$$\int_0^T \prod_{j=1}^k \tilde g_{U_j,v_j}(\omega_0, t)\,dt = \Big|\bigcap_{j=1}^k \{t: f^*(\omega_0, U_j+t) < v_j,\ 0 \leqslant t \leqslant T\}\Big|,$$

$$\int_0^T \tilde g_{U,v}(\omega_0, t)\,dt = |\{t: \tilde f(\omega_0, U+t) < v,\ 0 \leqslant t \leqslant T\}|,$$

$$\int_0^T \tilde g_U(\omega_0, t)\,dt = |\{t: \tilde f(\omega_0, U+t) = 0,\ 0 \leqslant t \leqslant T\}|,$$

$$E\tilde g_{U,v}(\omega, 0) = \tilde F_U(v), \qquad E\tilde g_U(\omega, 0) = \tilde F_U(+0) - \tilde F_U(-0) = \tilde F_U(+0),$$

and for disjoint intervals $U_1, U_2, \ldots, U_k$

$$E\prod_{j=1}^k g_{U_j,v_j}(\omega, 0) = \prod_{j=1}^k F_{U_j}(v_j).$$

Hence, according to (37), (38) and (39), we have the following assertion: for every $\omega_0 \epsilon \Omega_0$, and every system of disjoint intervals $U, U_1, U_2, \ldots, U_k \epsilon \mathfrak{R}$, $v, v_1, v_2, \ldots, v_k \epsilon R \cup N_f$ and $k = 1, 2, \ldots$ the equalities

$$(40)\qquad \Big|\bigcap_{j=1}^k \{t: f^*(\omega_0, U_j+t) < v_j\}\Big|_R = \prod_{j=1}^k F_{U_j}(v_j),$$

$$(41)\qquad |\{t: \tilde f(\omega_0, U+t) < v\}|_R = \tilde F_U(v),$$

$$(42)\qquad |\{t: \tilde f(\omega_0, U+t) = 0\}|_R = \tilde F_U(+0)$$

are true.

Now we shall prove that all realizations $f(\omega_0, t)$ $(\omega_0 \epsilon \Omega_0)$ are effective processes in the sense of Steinhaus. Suppose that we are given an arbitrary system of disjoint intervals $I_1, I_2, \ldots, I_k$ and an arbitrary system of real numbers $x_1, x_2, \ldots, x_k$. Let U_{jn} $(j = 1, 2, \ldots, k;\ n = 1, 2, \ldots)$ be a sequence of intervals belonging to $\mathfrak{R}$ such that

$$(43)\qquad U_{jn} \subset I_j, \quad \lim_{n\to\infty} U_{jn} = I_j \quad (j = 1, 2, \ldots, k)$$

Obviously, the set $I_j \setminus U_{jn}$ is the union of two disjoint intervals: $I_j \setminus U_{jn} = I'_{jn} \cup I''_{jn}$. Then there are intervals U'_{jn}, U''_{jn} belonging to $\mathfrak{R}$ such that $I'_{jn} \subset U'_{jn}$, $I''_{jn} \subset U''_{jn}$ and

$$(44)\qquad \lim_{n\to\infty} |U'_{jn}| = 0 = \lim_{n\to\infty} |U''_{jn}| \quad (j = 1, 2, \ldots, k).$$

Let $v_{1m}, v_{2m}, \ldots, v_{km}$ $(m = 1, 2, \ldots)$ be a sequence of numbers belonging to $R \cup N_f$ and satisfying the conditions

$$(45)\qquad v_{jm} + \frac{2}{m} < x_j \quad (j = 1, 2, \ldots, k;\ m = 1, 2, \ldots),$$

$$(46)\qquad \lim_{m\to\infty} v_{jm} = x_j \quad (j = 1, 2, \ldots, k).$$

Since for $j = 1, 2, \ldots, k$ and $n = 1, 2, \ldots$

$$(47)\qquad f^*(\omega_0, I_j) = f^*(\omega_0, U_{jn}) + f^*(\omega_0, I'_{jn}) + f^*(\omega_0, I''_{jn}),$$

$$(48)\qquad f^*(\omega_0, I'_{jn}) \leqslant \tilde f(\omega_0, U'_{jn}), \qquad f^*(\omega_0, I''_{jn}) \leqslant \tilde f(\omega_0, U''_{jn}),$$

we have, in view of (45), the inclusion

$$\{t: f^*(\omega_0, I_j+t) < x_j\}$$
$$\supset \{t: f^*(\omega_0, U_{jn}+t) < v_{jm}\} \setminus \Big(\Big\{t: f^*(\omega_0, I'_{jn}+t) \geqslant \frac{1}{m}\Big\} \cup$$
$$\cup \Big\{t: f^*(\omega_0, I''_{jn}+t) \geqslant \frac{1}{m}\Big\}\Big) \supset \{t: f^*(\omega_0, U_{jn}+t) < v_{jm}\} \setminus$$
$$\setminus \Big(\Big\{t: \tilde f(\omega_0, U'_{jn}+t) \geqslant \frac{1}{m}\Big\} \cup \Big\{t: \tilde f(\omega_0, U''_{jn}+t) \geqslant \frac{1}{m}\Big\}\Big)$$

holds for $j = 1, 2, \ldots, k$ $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ Consequently,

$$\bigcap_{j=1}^k \{t: f^*(\omega_0, I_j+t) < x_j\} \supset \bigcup_{j=1}^k \{t: f^*(\omega_0, U_{jn}+t) < v_{jm}\} \setminus$$
$$\setminus \bigcup_{j=1}^k \Big(\Big\{t: \tilde f(\omega_0, U'_{jn}+t) \geqslant \frac{1}{m}\Big\} \cup \Big\{t: \tilde f(\omega_0, U''_{jn}+t) \geqslant \frac{1}{m}\Big\}\Big).$$

Hence,

$$\Big|\bigcap_{j=1}^k \{t: f^*(\omega_0, I_j+t) < x_j,\ 0 \leqslant t \leqslant T\}\Big|$$
$$\geqslant \Big|\bigcap_{j=1}^k \{t: f^*(\omega_0, U_{jn}+t) < v_{jm},\ 0 \leqslant t \leqslant T\}\Big| -$$
$$- \sum_{j=1}^k \Big(T - \Big|\Big\{t: \tilde f(\omega_0, U''_{jn}+t) < \frac{1}{m},\ 0 \leqslant t \leqslant T\Big\}\Big| -$$
$$- \sum_{j=1}^k \Big(T - \Big|\Big\{t: \tilde f(\omega_0, U'_{jn}+t) < \frac{1}{m},\ 0 \leqslant t \leqslant T\Big\}\Big|.$$

Thus, taking into account relations (40) and (41), we have the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, I_j+t) < x_j,\ 0\leqslant t\leqslant T\}\Big|$$
$$\geqslant \prod_{j=k}^{k} F_{U_{jn}}(v_{jm}) - \sum_{j=1}^{k}\left(1-\tilde F_{U'_{jn}}\left(\frac{1}{m}\right)\right) - \sum_{j=1}^{k}\left(1-\tilde F_{U''_{jn}}\left(\frac{1}{m}\right)\right).$$

From lemma 3 and from formula (8) it follows that $F_I(x)$ is a continuous interval function for $I\neq 0$. Consequently, in virtue of (5), (32), (43) and (44), the last inequality implies for $n\to\infty$:

$$\liminf_{T\to\infty}\frac{1}{T}\Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, I_j+t) < x_j,\ 0\leqslant t\leqslant T\}\Big| \geqslant \prod_{j=1}^{k} F_{I_j}(v_{jm}).$$

Hence, according to (45), (46) and according to the continuity on the left of $F_I(x)$, we obtain for $m\to\infty$ the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, I_j+t) < x_j,\ 0\leqslant t\leqslant T\}\Big| \geqslant \prod_{j=1}^{k} F_{I_j}(x_j). \tag{49}$$

Further, we may suppose that $x_1, x_2, \ldots, x_r$ non ϵN_f and $x_{r-1}, \ldots, x_k \epsilon N_f$. Let $w_{1m}, w_{2m}, \ldots, w_{km}$ $(m = 1, 2, \ldots)$ be a sequence of numbers belonging to $R\cup N_f$ and satisfying the conditions

$$w_{jm} > x_j + \frac{2}{m} \qquad (j = 1, 2, \ldots, r;\ m = 1, 2, \ldots) \tag{50}$$

$$w_{jm} = x_j \qquad (j = r+1, \ldots, k;\ m = 1, 2, \ldots) \tag{51}$$

$$\lim_{m\to\infty} w_{jm} = x_j \qquad (j = 1, 2, \ldots, k). \tag{52}$$

From (47), (48), (50) and (51) we obtain the following inclusions for $j = 1, 2, \ldots, r$, $n = 1, 2, \ldots, m = 1, 2, \ldots$:

$$\{t: f^*(\omega_0, I_j+t) < x_j\}$$
$$\subset \{t: f^*(\omega_0, U_{jn}+t) < w_{jm}\} \cup \left\{t: f^*(\omega_0, I'_{jn}+t) \geqslant \frac{1}{m}\right\} \cup$$
$$\cup \left\{t: f^*(\omega_0, I''_{jn}+t) \geqslant \frac{1}{m}\right\} \subset \{t: f^*(\omega_0, U_{jn}+t) < w_{jm}\} \cup$$
$$\cup \left\{t: \tilde f(\omega_0, U'_{jn}+t) \geqslant \frac{1}{m}\right\} \cup \left\{t: \tilde f(\omega_0, U''_{jn}+t) \geqslant \frac{1}{m}\right\}$$

and for $j = 1, \ldots, k;\ n = 1, 2, \ldots;\ m = 1, 2, \ldots$

$$\{t: f^*(\omega_0, I_j+t) < x_j\} \subset \{t: f^*(\omega_0, U_{jn}+t) < w_{jm}\} \cup$$
$$\cup \{t: f^*(\omega_0, I'_{jn}+t) \neq 0\} \cup \{t: f^*(\omega_0, I''_{jn}+t) \neq 0\} \subset \{t: f^*(\omega_0, U_{jn}+t) < w_{jm}\} \cup$$
$$\cup \{t: \tilde f(\omega_0, U'_{jn}+t) \neq 0\} \cup \{t: \tilde f(\omega_0, U''_{jn}+t) \neq 0\}.$$

Hence, similarly to the preceding considerations, we obtain the inequality

$$\Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, I_j+t) < x_j,\ 0\leqslant t\leqslant T\}\Big|$$
$$\leqslant \Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, U_{jn}+t) < w_{jm},\ 0\leqslant t\leqslant T\}\Big| +$$
$$+ \sum_{j=1}^{r}\left(T - \left|\left\{t: \tilde f(\omega_0, U'_{jn}+t) < \frac{1}{m},\ 0\leqslant t\leqslant T\right\}\right|\right) +$$
$$+ \sum_{j=1}^{r}\left(T - \left|\left\{t: \tilde f(\omega_0, U''_{jn}+t) < \frac{1}{m},\ 0\leqslant t\leqslant T\right\}\right|\right) +$$
$$+ \sum_{j=r+1}^{k}\left(T - |\{t: \tilde f(\omega_0, U'_{jn}+t) = 0,\ 0\leqslant t\leqslant T\}|\right) +$$
$$+ \sum_{j=r+1}^{k}\left(T - |\{t: \tilde f(\omega_0, U''_{jn}+t) = 0,\ 0\leqslant t\leqslant T\}|\right).$$

Thus, taking into account the relations (40), (41) and (42), we get the inequality

$$\limsup_{T\to\infty}\frac{1}{T}\Big|\bigcap_{j=1}^{k}\{t: f^*(\omega_0, I_j+t) < x_j,\ 0\leqslant t\leqslant T\}\Big|$$
$$\leqslant \prod_{j=1}^{k} F_{U_{jn}}(w_{jm}) + \sum_{j=1}^{r}\left(1-\tilde F_{U'_{jn}}\left(\frac{1}{m}\right)\right) + \sum_{j=1}^{r}\left(1-\tilde F_{U''_{jn}}\left(\frac{1}{m}\right)\right) +$$
$$+ \sum_{j=r+1}^{k}\left(1-\tilde F_{U'_{jn}}(+0)\right) + \sum_{j=r+1}^{k}\left(1-\tilde F_{U''_{jn}}(+0)\right).$$

If $r < k$, then $N_f \neq 0$. Therefore, in view of (32), (44) and lemma 3, we have

$$\lim_{n\to\infty}\tilde F_{U'_{jn}}(+0) = 1 = \lim_{n\to\infty}\tilde F_{U''_{jn}}(+0).$$

Thus, taking into account formulas (4), (43), (44), (51) and the continuity of the interval function $F_I(x)$ $(I \neq 0)$, we obtain for $n \to \infty$;

$$\limsup_{T\to\infty} \frac{1}{T} \Big| \bigcap_{j=1}^{k} \{t: f^*(\omega_0, I_j+t) < x_j,\ 0 \leqslant t \leqslant T\} \Big| \leqslant \prod_{j=1}^{r} F_{I_j}(w_{jm}) \prod_{j=r+1}^{k} F_{I_j}(x_j).$$

Since w_{jm} non ϵN_j $(j = 1, 2, \ldots, r;\ m = 1, 2, \ldots)$, according to (52), the last inequality implies for m

$$\limsup_{T\to\infty} \frac{1}{T} \Big| \bigcap_{j=1}^{k} \{t: f^*(\omega_0, I_j+t) < x_j,\ 0 \leqslant t \leqslant T\} \Big| \leqslant \prod_{j=1}^{k} F_{I_j}(x_j).$$

Thus, if we take into account inequality (49), for every system of disjoint intervals $I_1, I_2, \ldots, I_k$ and for every system of real numbers $x_1, x_2, \ldots, x_k$ the relative measure $\big|\bigcap_{j=1}^{k} \{t: f^*(\omega_0, I_j+t) < x_j\}\big|_R$ exists. Moreover, the equality

$$\Big|\bigcap_{j=1}^{k} \{t: f^*(\omega_0, I_j+t) < x_j\}\Big|_R = \prod_{j=1}^{k} F_{I_j}(x_j)$$

is true. This implies equalities (1) and (3). In other words, $f(\omega_0, t)$ $(\omega_0 \epsilon \Omega_0)$ is an effective process satisfying condition (3). The theorem is thus proved.

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