Conversely, suppose that condition (α) is not satisfied. Then there exist numbers $\varepsilon_0 > 0$ and $\lambda_0 > 0$, a sequence $n_i \to \infty$ and two sequences m_{n_i} and l_{n_i} such that $l_{n_i}/k_{n_i} \to 0$ for $i \to \infty$ and l_{n_i}

$$P\left\{\left|\sum_{j=0}^{l_{n_i}} \xi_{k_i, m_{n_i}+j}\right| > \lambda_0\right\} > \varepsilon_0 .$$

Denote by Δ_i the least interval containing the points $t_{n_i, m_{n_i}}, t_{n_i, m_{n_i}+1}, \ldots, t_{n_i, m_{n_i}+l_{n_i}}$. If limit (17) exists and is a continuous function of t, then $|\Delta_i| \to 0$, and condition (a) of the theorem of Prokhorov is not satisfied. This proves theorem 3.

3. Suppose now that the sequence Ξ^* of random variables

$$\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n} \tag{18}$$

has for each n a common distribution $F_n(x) = P\{\xi_{nk} < x\}$, $k = 1, 2, \ldots, k_n$. From the theorem of Skorohod (see for example [4], § 3.2) and from theorems 2 and 3 we immediately obtain:

THEOREM 4. *The convergence of the sequence of distribution functions*

$$F_{n,k_n}(x) = P\left\{\sum_{j=1}^{k_n} \xi_{nj} < x\right\}$$

for $n \to \infty$ to a (infinitely divisible) limiting distribution $G(x)$ is necessary and sufficient for:

(I) *the compactness of the set of measures $\{P_n(\Xi^*, T)\}$ in the case when the sequence of partitions $T = \{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, belongs to the class K defined by formula* (14);

(II) *the convergence $P_n(\Xi^*, T) \Rightarrow P$ in the case when for the sequence of partitions $T = \{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, limit* (17) *exists.*

In cases (I) *and* (II) *the limiting measures are generated by continuous stochastic processes with independent increments.*

References

[1] Gniedenko i Kołmogorow, *Rozkłady graniczne dla sum niezależnych zmiennych losowych*, Warszawa 1957.

[2] Kimme, *On the convergence of sequences of stochastic processes*, Trans. Amer. Math. Soc. 84 (1957), p. 208.

[3] Kolmogorov, *Foundations of the theory of probability*.

[4] Прохоров, *Сходимость случайных процессов и предельные теоремы теории вероятностей*, Теория вероятностей и ее применения 1.2 (1956).

Reçu par la Rédaction le 15. 2. 1958

STUDIA MATHEMATICA, T. XVII. (1958)

A theorem on distributions integrable with even power

by

K. URBANIK (Wrocław)

1. In this paper we shall consider some spaces of distributions introduced by Schwartz [2]. By $\mathcal{D}_N$ we shall denote the space of all infinitely differentiable complex-valued functions $\varphi = \varphi(x_1, x_2, \ldots, x_N)$ $(-\infty < x_j < \infty,\ j = 1, 2, \ldots, N)$ with compact supports. Put

$$\|\varphi\| = \max_{n_1, x_2, \ldots, x_N} |\varphi(x_1, x_2, \ldots, x_N)| \qquad (\varphi \in \mathcal{D}_N).$$

The convergence in $\mathcal{D}_N$ is defined as follows: $\varphi_j \to 0$ $(\varphi_j \in \mathcal{D}_N,\ j = 1, 2, \ldots)$ if for every system of integers $\langle k_1, k_2, \ldots, k_N\rangle$

$$\left\|\frac{\partial^{k_1+k_2+\ldots+k_N}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_N^{k_N}}\varphi_j\right\| \to 0$$

and the supports of φ_j are contained in a fixed compact.

Let A be an arbitrary subset of the N-dimensional Euclidean space. By $\mathcal{D}_N(A)$ we shall denote the subspace of $\mathcal{D}_N$ consisting of all functions whose supports are contained in A.

The space $\mathcal{D}'_N$ of distributions is the conjugate space of $\mathcal{D}_N$. By (T, φ) we shall denote the value of T at φ $(T \in \mathcal{D}'_N,\ \varphi \in \mathcal{D}_N)$. The conjugate of T is defined by the formula $(\overline{T}, \varphi) = \overline{(T, \overline{\varphi})}$ $(\varphi \in \mathcal{D}_N)$.

We say that a distribution $T \in \mathcal{D}'_N$ is *of order* $\leqslant k_1 + \ldots + k_N$ on A if there is a continuous function f such that

$$(T, \varphi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(x_1, \ldots, x_N)\frac{\partial^{k_1+\ldots+k_N}}{\partial x_1^{k_1}\ldots\partial x_N^{k_N}}\varphi(x_1, \ldots, x_N)\,dx_1\ldots dx_N$$

for each $\varphi \in \mathcal{D}_N(A)$. All the distributions belonging to $\mathcal{D}'_N$ are of finite order on every compact (cf. [2], tome I, chapt. III, § 6).

Let $T \in \mathcal{D}'_1$. By $|T|^{2p}$ $(p = 1, 2, \ldots)$ we shall denote the direct product $\underbrace{T \times T \times \ldots \times T}_{p\text{ times}} \times \underbrace{\overline{T} \times \overline{T} \times \ldots \times \overline{T}}_{p\text{ times}}$, i. e. the distribution belonging to $\mathcal{D}'^{p}_2$

defined by the condition

$$(|T|^{2p}, \varphi) = \prod_{j=1}^{p} (T, \psi_j) \prod_{s=p+1}^{2p} (\overline{T}, \psi_s)$$

for every $\varphi \in \mathcal{D}_{2p}$ of the form $\varphi(x_1, \ldots, x_{2p}) = \prod_{j=1}^{2p} \psi_j(x_j)$. In other words, if

$$(T, \varphi) = \int_{-\infty}^{\infty} f(x) \frac{d^k}{dx^k} \varphi(x)\, dx \quad \text{for} \quad \varphi \in \mathcal{D}_1(A),$$

then

$$(1) \qquad (|T|^{2p}, \varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{j=1}^{p} f(x_j) \prod_{s=p+1}^{2p} \overline{f(x_s)} \frac{\partial^{2pk}}{\partial x_1^k \ldots \partial x_{2p}^k} \varphi(x_1, \ldots, x_{2p})\, dx_1 \ldots dx_{2p}$$

for $\varphi \in \mathcal{D}_{2p}(A \times A \times \ldots \times A)$.

For every pair of real numbers ω_1, ω_2 and for $T \in \mathcal{D}_1'$ we define *the integral* $\int_{\omega_1}^{\omega_2} |T|^{2p} \in \mathcal{D}_{2p}'$ $(p = 1, 2, \ldots)$ by the formula

$$(2) \qquad \Big(\int_{\omega_1}^{\omega_2} |T|^{2p}, \varphi\Big) = (|T|^{2p}, \varphi_{\omega_1, \omega_2}) \quad (\varphi \in \mathcal{D}_{2p}),$$

where $\varphi_{\omega_1, \omega_2}(x_2, \ldots, x_{2p}) = \int_{\omega_1}^{\omega_2} \varphi(x_1 - x, \ldots, x_{2p} - x)\, dx$.

For every number h the transformations $\tau_h^{(j)}$ $(j = 1, 2, \ldots, N)$ of $\mathcal{D}_N$ on $\mathcal{D}_N$ are defined by the formula $\tau_h^{(j)} \varphi(x_1, \ldots, x_N) = \varphi(x_1, \ldots, x_{j-1}, x_j - h, x_{j+1}, \ldots, x_N)$. Further, for every interval I we set $\tau_h I = \{x + h : x \in I\}$.

Let $T \in \mathcal{D}_1'$. If the family $\int_{\omega_1}^{\omega_2} |T|^{2p}$ converges when $\omega_1 \to -\infty$, $\omega_2 \to \infty$, we shall write

$$\int_{-\infty}^{\infty} |T|^{2p} = \lim_{\substack{\omega_1 \to -\infty \\ \omega_2 \to \infty}} \int_{\omega_1}^{\omega_2} |T|^{2p} \quad (p = 1, 2, \ldots).$$

The notion of integral $\int_{-\infty}^{\infty} |T|^2$ was introduced in connection with the study of the filtering of generalized stochastic processes [3]. The distributions $T \in \mathcal{D}_1'$ for which $\int_{-\infty}^{\infty} |T|^2$ exists are weighing distributions in the optimal least-squares prediction of stationary generalized processes.

By L^q $(q > 1)$ we shall denote the space of all measurable complex-valued functions f for which $\int_{-\infty}^{\infty} |f(x)|^q\, dx$ exists. We set

$$\|f\|_{L^q} = \Big(\int_{-\infty}^{\infty} |f(x)|^q\, dx\Big)^{1/q}.$$

By $\mathcal{D}_{L^q}$ we shall denote the space of all infinitely differentiable complex-valued functions $\varphi = \varphi(x)$ $(-\infty < x < \infty)$ for which $d^k\varphi/dx^k \in L^q$ $(k = 0, 1, \ldots)$. The convergence in $\mathcal{D}_{L^q}$ is defined as follows: $\varphi_j \to 0$ $(\varphi \in \mathcal{D}_{L^q}, j = 1, 2, \ldots)$ if, for every non-negative integer k, $\|d^k\varphi_j/dx^k\|_{L^q} \to 0$.

The space $\mathcal{D}_{L^r}'$ $(r > 1)$ of distributions is the conjugate space of $\mathcal{D}_{L^q}$, where $q = r/(r-1)$. Obviously, for every $r > 1$ the inclusion $\mathcal{D}_{L^r}' \subset \mathcal{D}_1'$ is true.

II. The aim of this paper is to give the following characterization of the space $\mathcal{D}_{L^{2p}}'$ $(p = 1, 2, \ldots)$:

THEOREM. *Let $T \in \mathcal{D}_1'$. Then $T \in \mathcal{D}_{L^{2p}}'$ $(p = 1, 2, \ldots)$ if and only if the integral $\int_{-\infty}^{\infty} |T|^{2p}$ exists.*

Before proving the Theorem we shall prove three Lemmas. In the sequel we shall denote the real line by R.

LEMMA 1. *Let $T \in \mathcal{D}_1'$. If there is a non-empty interval I such that $|T|^{2p}$ $(p = 1, 2, \ldots)$ is of finite order on $\bigcup_{-\infty < h < \infty} \tau_h I \times \tau_h I \times \ldots \times \tau_h I$, then T is of finite order on R.*

Proof. Since $|T|^{2p}$ is of finite order on $\bigcup_{-\infty < h < \infty} \tau_h I \times \tau_h I \times \ldots \times \tau_h I$ there are a continuous function f and an integer r such that the equality

$$(3) \qquad (|T|^{2p}, \varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_{2p}) \frac{\partial^{2pr}}{\partial x_1^r \ldots \partial x_{2p}^r} \varphi(x_1, \ldots, x_{2p})\, dx_1 \ldots dx_{2p}$$

is true for $\varphi \in \mathcal{D}_{2p}(\bigcup_{-\infty < h < \infty} \tau_h I \times \tau_h I \times \ldots \times \tau_h I)$.

To prove the Lemma it is sufficient to show that for every h the distribution T is of order $\leqslant r$ on $\tau_h I$ (cf. [2], tome I, p. 27). Obviously, if $T = 0$ on $\tau_h I$ (for definition see [2], tome I, p. 25, 26), then T is of order $\leqslant r$ on $\tau_h I$. Therefore we may suppose that $T \neq 0$ on $\tau_h I$. Since T is of finite order on $\tau_h I$, there are a continuous function g_0 and an integer k such that the equality

$$(4) \qquad (T, \varphi) = \int_{-\infty}^{\infty} g_0(x) \frac{d^k}{dx^k} \varphi(x)\, dx$$

holds for $\varphi\in\mathcal{D}_1(\tau_h I)$. Without loss of generality we may assume that $k>r$. Let $a_1, a_2, \ldots, a_k$ ($a_i\neq a_j$ for $i\neq j$) be a system of real numbers belonging to $\tau_h I$ and set

$$g(x)=g_0(x)-\sum_{j=1}^{k} g_0(a_j)\frac{(x-a_1)\ldots(x-a_{j-1})(x-a_{j+1})\ldots(x-a_k)}{(a_j-a_1)\ldots(a_j-a_{j-1})(a_j-a_{j+1})\ldots(a_j-a_k)}.$$

Then

(5) $$g(a_n)=0 \qquad (n=1,2,\ldots,k)$$

and, in view of (4),

(6) $$(T,\varphi)=\int_{-\infty}^{\infty} g(x)\frac{d^k}{dx^k}\varphi(x)\,dx \quad \text{for} \quad \varphi\in\mathcal{D}_1(\tau_h I).$$

Hence, taking into account definition (1), we obtain the equality

(7) $$(|T|^{2p},\varphi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{j=1}^{p}g(x_j)\prod_{s=p+1}^{2p}\overline{g(x_s)}\frac{\partial^{2pk}}{\partial x_1^k\ldots\partial x_{2p}^k}\varphi(x_1,\ldots,x_{2p})\,dx_1\ldots dx_{2p}$$

for $\varphi\in\mathcal{D}_{2p}(\tau_h I\times\tau_h I\times\ldots\times\tau_h I)$. From equalities (3) and (7) it follows that there are then functions of $2p-1$ variables $b_{j,s}$ ($j=1,2,\ldots,2p$; $s=0,1,\ldots,k-1$) such that

(8) $$[(k-r-1)!]^{-2p}\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_{2p}}\prod_{j=1}^{2p}(x_j-u_j)^{k-r-1}f(u_1,\ldots,u_{2p})\,du_1\ldots du_{2p}$$
$$=\prod_{j=1}^{p}g(x_j)\prod_{s=p+1}^{2p}\overline{g(x_s)}-\sum_{j=1}^{2p}\sum_{s=0}^{k-1}b_{j,s}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{2p})\;x_j^s,$$

for $\langle x_1,x_2,\ldots,x_{2p}\rangle\in\tau_h I\times\tau_h I\times\ldots\times\tau_h I$ (cf. [1], § 10). Substituting in the last formula $x_{j_0}=a_n$ ($n=1,2,\ldots,k$; $j_0=1,2,\ldots,2p$) and taking into account equality (5), we obtain the linear equations for the functions $b_{j_0,s}$ ($s=0,1,\ldots,k-1$):

(9) $$\sum_{s=0}^{k-1}b_{j_0,s}(x_1,\ldots,x_{j_0-1},x_{j_0+1},\ldots,x_{2p})a_n^s$$
$$=B_{j_0,n}(x_1,\ldots,x_{2p})-\sum_{j\neq j_0}\sum_{s=0}^{k-1}b_{j,s}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{j_0-1},a_n,x_{j_0+1},\ldots,x_{2p})x_j^s$$
$$(n=1,2,\ldots,k),$$

where the $k-r$-th derivatives of $B_{j_0,n}$ ($n=1,2,\ldots,k$) are continuous. Hence it follows that $b_{j_0,s}$ ($s=0,1,\ldots,k-1$) are linear combinations

of the right-hand side of equations (9). Consequently, taking into account formula (8), we have the equality

$$\prod_{j=1}^{p}g(x_j)\prod_{s=p+1}^{2p}\overline{g(x_s)}=C(x_1,\ldots,x_{2p})+$$
$$+\sum_{1\leqslant j_1<j_2\leqslant 2p}\sum_{s_1,s_2=0}^{k-1}c_{j_1,j_2,s_1,s_2}(x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_{j_2-1},x_{j_2+1},\ldots,x_{2p})x_{j_1}^{s_1}x_{j_2}^{s_2}$$

for $\langle x_1,x_2,\ldots,x_{2p}\rangle\in\tau_h I\times\tau_h I\times\ldots\times\tau_h I$, where all the $k-r$-th derivatives of $C(x_1,\ldots,x_{2p})$ are continuous. By iteration of this procedure we finally reach the equality

(10) $$\prod_{j=1}^{p}g(x_j)\prod_{s=p+1}^{2p}\overline{g(x_s)}=D(x_1,\ldots,x_{2p})+\sum_{0\leqslant s_1,s_2,\ldots,s_{2p}\leqslant k-1}d_{s_1,\ldots,s_{2p}}x_1^{s_1}\ldots x_{2p}^{s_{2p}}$$

for $\langle x_1,x_2,\ldots,x_{2p}\rangle\in\tau_h I\times\tau_h I\times\ldots\times\tau_h I$, where all the $k-r$-th derivatives of $D(x_1,\ldots,x_{2p})$ are continuous and $d_{s_1,\ldots,s_{2p}}$ ($0\leqslant s_1,\ldots,s_{2p}\leqslant k-1$) are constants.

Since $T\neq 0$ on $\tau_h I$, there is, according to (6), a number $a\in\tau_h I$ such that $g(a)\neq 0$. Formula (10) implies the equality

$$g(x)=[\overline{g(a)}]^{-p}[g(a)]^{-p+1}\Big\{D(x,a,\ldots,a)+$$
$$+\sum_{0\leqslant s_1,s_2,\ldots,s_{2p}\leqslant k-1}d_{s_1,\ldots,s_{2p}}x^{s_1}a^{s_2+\ldots+s_{2p}}\Big\} \quad \text{for} \quad x\in\tau_h I.$$

Thus the $k-r$-th derivative of $g(x)$ is continuous in $\tau_h I$. Setting

$$h(x)=(-1)^{k-r}\frac{d^{k-r}}{dx^{k-r}}g(x),$$

we have, according to (6),

$$(T,\varphi)=\int_{-\infty}^{\infty}h(x)\frac{d^r}{dx^r}\varphi(x)\,dx \quad \text{for} \quad \varphi\in\mathcal{D}_1(\tau_h I).$$

Consequently, T is of order $\leqslant r$ on $\tau_h I$. The lemma is thus proved.

LEMMA 2. *Let $T\in\mathcal{D}_1'$. If the integral $\int_{-\infty}^{\infty}|T|^{2p}$ ($p=1,2,\ldots$) exists, then T is of finite order on R.*

Proof. Since the integral $\int_{-\infty}^{\infty}|T|^{2p}$ exists, the inequality

$$\sup_{-\infty<\omega_1,\omega_2<\infty}\left|\Big(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\Big)\right|<\infty$$

holds for each $\varphi\in\mathcal{D}_{2p}$. Consequently, for every non-empty interval I there are a constant M and a system of integers $\langle k_1,k_2,\ldots,k_{2p}\rangle$ such that the inequality

$$\sup_{-\infty>\omega_1,\omega_2<\infty}\left|\Big(\int_{\omega_1}^{\omega_2}T^{2p},\varphi\Big)\right|\leqslant M\left\|\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2p}^{k_{2p}}}\varphi\right\| \tag{11}$$

holds for $\varphi\in\mathcal{D}_{2p}(I\times I\times\ldots\times I)$ (cf. [1], § 6).

Let h be an arbitrary real number and $\varphi\in\mathcal{D}_{2p}(\tau_hI\times\tau_hI\times\ldots\times\tau_hI)$. Then $\tau_h^{(1)}\ldots\tau_h^{(2p)}\varphi\in\mathcal{D}_{2p}(I\times I\times\ldots\times I)$,

$$\left\|\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2p}^{k_{2p}}}\varphi\right\|=\left\|\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2p}^{k_{2p}}}\tau_h^{(1)}\ldots\tau_h^{(2p)}\varphi\right\|$$

and, according to (2),

$$\Big(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\Big)=\Big(\int_{\omega_1+h}^{\omega_2+h}|T|^{2p},\tau_h^{(1)}\ldots\tau_h^{(2p)}\varphi\Big).$$

Hence, in virtue of (11), for every h we have the inequality

$$\sup_{-\infty<\omega_1,\omega_2<\infty}\left|\Big(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\Big)\right|\leqslant M\left\|\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2p}^{k_{2p}}}\varphi\right\| \tag{12}$$

if $\varphi\in\mathcal{D}_{2p}(\tau_hI\times\tau_hI\times\ldots\times\tau_hI)$. Put

$$\varphi_j(x_1,\ldots,x_{2p})=\frac{\partial}{\partial x_j}\varphi(x_1,\ldots,x_{2p})\qquad(j=1,2,\ldots,2p).$$

Then the well-known equality

$$\int_{\omega_1}^{\omega_2}\sum_{j=1}^{2p}\varphi_j(x_1-x,\ldots,x_{2p}-x)\,dx=\varphi(x_1-\omega_1,\ldots,x_{2p}-\omega_1)-\varphi(x_1-\omega_2,\ldots,x_{2p}-\omega_2)$$

is true. Consequently, according to (2),

$$\lim_{\omega_2\to\infty}\Big(\int_0^{\omega_2}|T|^{2p},\sum_{j=1}^{2p}\frac{\partial}{\partial x_j}\varphi\Big)=(|T|^{2p},\varphi)$$

for $\varphi\in\mathcal{D}_{2p}$. Hence, in view of (12), the inequality

$$|(|T|^{2p},\varphi)|\leqslant\sup_{-\infty<\omega_1,\omega_2<\infty}\left|\Big(\int_{\omega_1}^{\omega_2}|T|^{2p},\sum_{j=1}^{2p}\frac{\partial}{\partial x_j}\varphi\Big)\right|$$

$$\leqslant M\left\|\sum_{j=1}^{2p}\frac{\partial}{\partial x_j}\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2k}^{k_{2p}}}\varphi\right\|\leqslant C\left\|\frac{\partial^{k_1+\ldots+k_{2p}+2p}}{\partial x_1^{k_1+1}\ldots\partial x_{2p}^{k_{2p}+1}}\varphi\right\|$$

holds for $\varphi\in\mathcal{D}_{2p}(\tau_hI\times\tau_hI\times\ldots\times\tau_hI)$, where C is a constant. Consequently, $|T|^{2p}$ is of order $\leqslant k_1+\ldots+k_{2p}+4p$ on $\tau_hI\times\tau_hI\times\ldots\times\tau_hI$ for every h (cf. [1], § 5). This implies that $|T|^{2p}$ is of finite order on $\bigcup_{-\infty<h<\infty}\tau_hI\times\tau_hI\times\ldots\times\tau_hI$. Hence, in virtue of Lemma 1, T is of finite order on R. The lemma is thus proved.

By Δ_hf we shall denote the difference $f(t+h)-f(t)$. We shall use the notation $f*g(x)=\int_{-\infty}^{\infty}f(x-u)g(u)\,du$, provided that this convolution exists.

LEMMA 3. *Let $\varphi\in\mathcal{D}_1$ and let f be a continuous function such that for $|h|\leqslant c$ $(c>0)$ $\Delta_hf\in L^r$ $(r>1)$ and the functions $\|\Delta_hf*d^s\varphi/dx^s\|_{L^r}$ $(s=0,1,2)$ are integrable (with respect to h, $|h|\leqslant c$). Then for every pair $h_1<h_2$ $(|h_1|<c,\ |h_2|<c)$ of Lebesgue points of $\|\Delta_hf*\varphi\|_{L^r}$ the inequality*

$$\left\|f*\frac{d}{dx}\varphi\right\|_{L^r}\leqslant(h_2-h_1)^{-1}\left\{\|\Delta_{h_1}f*\varphi\|_{L^r}+\|\Delta_{h_2}f*\varphi\|_{L^r}+\int_{h_1}^{h_2}\left\|\Delta_hf*\frac{d}{dx}\varphi\right\|_{L^r}dh\right\}$$

is true. Consequently, $f*d\varphi/dx\in L^r$.

Proof. It is easy to verify the following equality:

$$\Delta_{h+v}f*\varphi=\Delta_hf*\Delta_v\varphi+f*\Delta_v\varphi+\Delta_hf*\varphi. \tag{13}$$

Let $h_1<h_2$ $(|h_1|<c,\ |h_2|<c)$ be a pair of Lebesgue points of $\|\Delta_hf*\varphi\|_{L^r}$. Then the equality

$$\lim_{v\to0}\frac{1}{v}\int_{h_j}^{h_j+v}\|\Delta_hf*\varphi\|_{L^r}dh=\|\Delta_{h_j}f*\varphi\|_{L^r}\qquad(j=1,2) \tag{14}$$

is true. By integration of (13) with respect to h we obtain the equality

$$(h_2-h_1)f*\Delta_v\varphi=\int_{h_2}^{h_2+v}\Delta_hf*\varphi\,dh-\int_{h_1}^{h_1+v}\Delta_hf*\varphi\,dh-\int_{h_1}^{h_2}\Delta_hf*\Delta_v\varphi\,dh.$$

Consequently, for a sufficiently small positive number v the inequality

$$(15)\qquad (h_2-h_1)\left\|f*\frac{1}{v}\Delta_v\varphi\right\|_{L^r}\leqslant\frac{1}{v}\int_{h_1}^{h_1+v}\|\Delta_h f*\varphi\|_{L^r}dh+ +\frac{1}{v}\int_{h_2}^{h_2+v}\left\|\Delta_h f*\varphi\right\|_{L^r}dh+\int_{h_1}^{h_2}\left\|\Delta_h f*\frac{1}{v}\Delta_v\varphi\right\|_{L^r}dh$$

holds. Since

$$\frac{1}{v}\Delta_h\varphi(x)-\frac{d}{dx}\varphi(x)=\frac{1}{v}\int_0^v\frac{d^2}{du^2}\varphi(u+x)(v-u)\,du,$$

we have

$$\int_{h_1}^{h_2}\left\|\Delta_h f*\left(\frac{1}{v}\Delta_v\varphi-\frac{d}{dx}\varphi\right)\right\|_{L^r}\leqslant\frac{1}{2}v\int_{h_1}^{h_2}\left\|\Delta_h f*\frac{d^2}{dx^2}\varphi\right\|_{L^r}dh.$$

Hence follows the convergence

$$(16)\qquad \lim_{v\to 0}\int_{h_1}^{h_2}\left\|\Delta_h f*\frac{1}{v}\Delta_v\varphi\right\|_{L^r}dh=\int_{h_1}^{h_2}\left\|\Delta_h f*\frac{d}{dx}\varphi\right\|_{L^r}dh.$$

Further, according to Fatou's Lemma,

$$\liminf_{v\to 0}\left\|f*\frac{1}{v}\Delta_v\varphi\right\|_{L^r}\geqslant\left\|f*\frac{d}{dx}\varphi\right\|_{L^r}.$$

Hence, in view of (14), (15) and (16), we obtain the inequality

$$(h_2-h_1)\left\|f*\frac{d}{dx}\varphi\right\|_{L^r}\leqslant\|\Delta_{h_1}f*\varphi\|_{L^r}+\|\Delta_{h_2}f*\varphi\|_{L^r}+\int_{h_1}^{h_2}\left\|\Delta_h f*\frac{d}{dx}\varphi\right\|_{L^r}dh,$$

q. e. d.

By iteration of the last lemma we obtain the following

COROLLARY. *Let $\varphi\in\mathcal{D}_1$ and let f be a continuous function such that, for $h_1, h_1, \ldots, h_k$ $(|h_j|\leqslant c;\ c>0;\ j=1,2,\ldots,k)$, $\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k}f\in L^r$ $(r>1)$ and the functions*

$$\left\|\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k}f*\frac{d^s}{dx^s}\varphi\right\|_{L^r}\qquad(s=0,1,\ldots,k+1)$$

are integrable (with respect to $h_1, h_2, \ldots, h_k$; $|h_j|\leqslant c$, $j=1,2,\ldots,k$). Then $f*d^k\varphi/dx^k\in L^r$.

Proof of the theorem. Sufficiency. Let us suppose that $T\in\mathcal{D}_1'$ and that the integral $\int_{-\infty}^{\infty}|T|^{2p}$ exists. Then, in virtue of Lemma 2, T is of finite order on R. There are then a continuous function f and an integer k such that

$$(17)\qquad T=\frac{d^k}{dx^k}f.$$

Let I be a non-empty interval containing the point 0. There are then a family of continuous functions $g_{\langle\omega_1,\omega_2\rangle}=g_{\langle\omega_1,\omega_2\rangle}(x_1,\ldots,x_{2p})$ and a system of integers $\langle k_1,\ldots,k_{2p}\rangle$ such that

$$(18)\qquad \left(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}g_{\langle\omega_1,\omega_2\rangle}(x_1,\ldots,x_{2p})\frac{\partial^{k_1+\ldots+k_{2p}}}{\partial x_1^{k_1}\ldots\partial x_{2p}^{k_{2p}}}\times \times\varphi(x_1,\ldots,x_{2p})\,dx_1\ldots dx_{2p}$$

for $\varphi\in\mathcal{D}_{2p}(I\times I\times\ldots\times I)$ and the family $g_{\langle\omega_1,\omega_2\rangle}$ converges uniformly on $I\times I\times\ldots\times I$ when $\omega_1\to-\infty$, $\omega_2\to\infty$ (see [1], § 10). Without loss of generality we may suppose that

$$(19)\qquad k_1=k_2=\ldots=k_{2p}=k.$$

Further, according to (1), (2) and (17), we obtain the equality

$$\left(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\int_{\omega_1}^{\omega_2}\prod_{j=1}^{p}f(x_j+x)\prod_{s=p+1}^{2p}\overline{f(x_s+x)}\,dx\times \times\frac{\partial^{2pk}}{\partial x_1^k\ldots\partial x_{2p}^k}\varphi(x_1,\ldots,x_{2p})\,dx_1\ldots dx_{2p}$$

for $\varphi\in\mathcal{D}_{2p}$. Hence, taking into account (18) and (19), for sufficiently small $h_1^{(j)},\ldots,h_k^{(j)}, x_j$ $(j=1,2,\ldots,2p)$ we have the equality

$$\int_{\omega_1}^{\omega_2}\prod_{j=1}^{p}\Delta_{h_1^{(j)}}\ldots\Delta_{h_k^{(j)}}f(x_j+x)\prod_{s=p+1}^{2p}\Delta_{h_1^{(s)}}\ldots\Delta_{h_k^{(s)}}\overline{f(x_s+x)}\,dx =\Delta_{h_1^{(1)}}\ldots\Delta_{h_k^{(1)}}\ldots\Delta_{h_1^{(2p)}}\ldots\Delta_{h_k^{(2p)}}g_{\langle\omega_1,\omega_2\rangle}(x_1,\ldots,x_{2p}).$$

Hence, taking into account the convergence of the family $g_{\langle\omega_1,\omega_2\rangle}$ we infer that the family $\int_{\omega_1}^{\omega_2}|\Delta_{h_1}\ldots\Delta_{h_k}f(x)|^{2p}dx$ converges when $\omega_1\to-\infty$, $\omega_2\to\infty$ uniformly for $|h_j|\leqslant c$ $(j=1,2,\ldots,k)$, where c is a positive constant. Consequently, for every $h_1,\ldots,h_k$ $(|h_j|\leqslant c,\ j=1,2,\ldots,k)$,

$\Delta_{h_1}\dots\Delta_{h_k}f\in L^{2p}$ and the norm $\|\Delta_{h_1}\dots\Delta_{h_k}f\|_{L^{2p}}$ is continuous with respect to $h_1,\dots,h_k$. Moreover, for every $\varphi\in\mathcal{D}_1$ the norms $\|\Delta_{h_1}\dots\Delta_{h_k}f*d^s_{\varphi}\varphi/dx^s\|_{L^{2p}}$ $(s=0,1,\dots,k+1)$ are integrable with respect to $h_1,\dots,h_k$ $(|h_j|\leqslant c,$ $j=1,2,\dots,k)$. Hence, in virtue of the Corollary to Lemma 3,

$$f*\frac{d^k}{dx^k}\varphi\in L^{2p}\qquad\text{if}\qquad\varphi\in\mathcal{D}_1. \tag{20}$$

Since the support of $\varphi(\varphi\in\mathcal{D}_1)$ is compact, $T*\varphi$ exists. (The convolution of distributions is defined in [2], tome II, chapter VI). Moreover, from equality (17) it follows that $T*\varphi=f*d^k\varphi/dx^k$. Hence, in view of (20), $T*\varphi\in L^{2p}$ for each $\varphi\in\mathcal{D}_1$. Thus, according to a theorem of Schwartz ([2], tome II, p. 57) $T\in\mathcal{D}'_{L^{2p}}$. The sufficiency of the condition of the theorem is thus proved.

Necessity. Let $T\in\mathcal{D}'_{L^{2p}}$. There is then, according to a theorem of Schwartz ([2], tome II, p. 57), a system of functions $f_0,f_1,\dots,f_n$ belonging to L^{2p} such that

$$T=\sum_{r=0}^{u}\frac{d^r}{dx^r}f_r.$$

Let $g,g_0,g_1,\dots,g_n$ be a system of continuous functions such that

$$g=\sum_{r=0}^{n}g_r, \tag{21}$$

$$\frac{d^{k-r}}{dx^{k-r}}g_r=f_r\qquad(r=0,1,\dots,n) \tag{22}$$

and, consequently, $T=d^kg/dx^k$. Hence, in virtue of (1) and (2),

$$\Big(\int_{\omega_1}^{\omega_2}|T|^{2p},\varphi\Big)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\int_{\omega_1}^{\omega_2}\prod_{j=1}^{p}g(x_j+x)\prod_{s=p+1}^{2p}\overline{g(x_s+x)}\,dx\,\frac{\partial^{2pk}}{\partial x_1^k\dots\partial x_{2p}^k}\times$$
$$\times\varphi(x_1,\dots,x_{2p})\,dx_1\dots dx_{2p}\qquad(\varphi\in\mathcal{D}_{2p}).$$

This implies, according to (21) and (22), the following equality:

$$\Big(\int_{\omega_1}^{\omega_2}|T|^{\,p},\varphi\Big)$$
$$=\sum_{0\leqslant s_1,\dots,s_{2p}\leqslant n}\pm\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\int_{\omega_1}^{\omega_2}\prod_{j-1}^{p}f_{s_j}(x_j+x)\prod_{r-p+1}^{2p}\overline{f_{s_r}(x_r+x)}\,dx\,\frac{\partial^{s_1+\dots+s_{2p}}}{\partial x_1^{s_1}\dots\partial x_{2p}^{s_{2p}}}\times$$
$$\times\varphi(x_1,\dots,x_{2p})\,dx_1\dots dx_{2p}\qquad(\varphi\in\mathcal{D}_{2p})$$

Consequently, to prove that $\int_{-\infty}^{\infty}|T|^{2p}$ exists it is sufficient to show that for every system $0\leqslant s_1,\dots,s_{2p}\leqslant n$

$$\sup_{x_1,\dots,x_{2p}}\int_{-\infty}^{\infty}\prod_{j=1}^{2p}|f_{s_j}(x_j+x)|\,dx<\infty. \tag{23}$$

From the inequality

$$\prod_{j=1}^{2p}|f_{s_j}(x_j+x)|\leqslant\sum_{j=1}^{2p}|f_{s_j}(x_j+x)|^{2p}$$

it follows that

$$\int_{-\infty}^{\infty}\prod_{j=1}^{2p}|f_{s_j}(x_j+x)|\,dx\leqslant\sum_{j=1}^{2p}\|f_{s_j}\|^{2p}_{L^{2p}},$$

which implies formula (23). Thus $\int_{-\infty}^{\infty}|T|^{2p}$ exists, which was to be proved.

References

[1] I. Halperin, *Introduction to the theory of distributions*, Toronto 1952.

[2] L. Schwartz, *Théorie des distributions*, Paris, tome I, 1950; tome II, 1951.

[3] K. Urbanik, *Filtering of stationary generalized stochastic processes*, Science Record 1958.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 24. 2. 1958