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Some remarks on the convergence of stochastic processes

by

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Introduction. Let R be a separable complete metric space. Denote by $M(R)$ the space of all normed σ-measures defined on Borel subsets of R. The sequence $\mu_n \epsilon M(R)$ will be called *weakly convergent to* $\mu \epsilon M(R)$ if for every bounded and continuous function $f(x)$, $x \epsilon R$,

$$(1) \qquad \int_R f(x)\, d\mu_n \to \int_R f(x)\, d\mu .$$

We shall denote the weak convergence by $\Rightarrow$. The space $M(R)$ with the metric L defined by Prokhorov [4] is a separable complete space, and the L-convergence is equivalent to the weak convergence.

For normed measures on the real line such a metric has been given by Lévy; the distance between measures μ_1 and μ_2 has been defined as

$$(2) \qquad \inf_h \{\text{for every } x\colon F_{\mu_1}(x-h)-h \leqslant F_{\mu_2}(x) \leqslant F_{\mu_1}(x+h)+h\},$$

where F_{μ_1} and F_{μ_2} are the distribution functions of the measures μ_1 and μ_2 respectively. This metric may also be interpreted as a metric in the space of all probability distribution functions. The metric L defined by Prokhorov is a simple generalization of the metric defined by (2).

The following conditions are equivalent ([1], § 9):

(i) $\mu_n \Rightarrow \mu$,

(ii) $F_{\mu_n}(x) \to F_\mu(x)$ at every continuity point of $F_\mu(x)$,

(iii) $L(F_{\mu_n}, F_\mu) \to 0$, where $L(F, G)$ is the distance of probability distribution functions defined by (2).

Let R^* be another separable complete metric space. If $\mu \epsilon M(R)$ and f is a μ-almost everywhere continuous function defined on R with values from R^*, then the condition $\mu^f(A) = \mu\{f^{-1}(A)\}$ for μ-measurable $f^{-1}(A) \subset R$ defines the measure $\mu^f \epsilon M(R^*)$. The following theorem holds [4]:

The condition $\mu_n \Rightarrow \mu$ $(\mu_n, \mu \epsilon M(R))$ *holds if and only if for every real μ-almost everywhere continuous function* $f(x), x \epsilon R$: $\mu_n^f \Rightarrow \mu^f$.

Let us denote by $D[0,1]$ the space of all real functions $\xi(t)$, $0 \leqslant t \leqslant 1$, such that there exist limits $\xi(t-0)$ and $\xi(t+0)$ (may be unequal) for each interior point of the interval $[0,1]$, limits $\xi(0+)$ and $\xi(1-0)$ for the points $t=0$ and $t=1$ respectively and for any $t \in [0,1]$ one of the relations $\xi(t) = \xi(t+0)$ or $\xi(t) = \xi(t-0)$ holds.

The space $D[0,1]$ with the metric d defined by Prokhorov [4] is a separable complete space and for the subspace $C[0,1]$ of $D[0,1]$ the d-convergence is equivalent to the uniform one.

Let us take a sequence of finite sequences

(3) $$\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$$

of random variables independent for each n, such that for every $\varepsilon > 0$

(4) $$\lim_{n=\infty} \max_{1\leqslant k\leqslant k_n} P\{|\xi_{nk}| > \varepsilon\} = 0.$$

Write $\zeta_{n0} = 0$, and $\zeta_{nk} = \xi_{n1} + \ldots + \xi_{nk}$, $k = 1, 2, \ldots, k_n$. Take any sequence of partitions $\{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, where $0 = t_{n0} < t_{n1} < \ldots < t_{nk_n} = 1$, such that

(5) $$\lim_{n=\infty} \max_{1\leqslant k\leqslant k_n} (t_{nk} - t_{n,k-1}) = 0.$$

Define the random function $\xi_n(t)$ by the formulas

(6) $$\begin{cases} \xi_n(0) = 0, \\ \xi_n(t) = \zeta_{nk} \quad \text{for} \quad t_{n,k-1} < t \leqslant t_{nk}, \quad k = 1, 2, \ldots, k_n. \end{cases}$$

Denote by P_n the measure in $D[0,1]$ generated by all finite-dimensional [3] distributions $P_n^{t_1,\ldots,t_m}$ of the stochastic process $\xi_n(t)$.

Prokhorov [4] has proved the following theorem:

THEOREM. *Let $\{\xi_n(t)\}$, $n = 1, 2, \ldots$, be a sequence of random functions defined by* (6), *where the ξ_{nk} satisfy* (3) *and* (4) *and the t_{nk} satisfy* (5), *and let P_n be the measure in $D[0,1]$ generated by $\xi_n(t)$. The set $\{P_n\}$ is compact and every limiting measure is generated by a continuous stochastic process $\xi(t)$ with independent increments if and only if for every n*

$$\max_{\Delta, |\Delta| < \delta} P\Big\{\Big|\sum_{t_{nk}\in\Delta} \xi_{nk}\Big| > \lambda\Big\} \leqslant C(\lambda, \delta),$$

where for every n, max is taken over the intervals Δ of the form $[t_{nj}, t)$ and

(a) $C(\lambda, \delta) \to 0$ *for* $\delta \to 0$ *and every fixed* $\lambda > 0$,

(b) $\sup_{0\leqslant\delta\leqslant 1} C(\lambda, \delta) \to 0$ for $\lambda \to \infty$.

In this paper we give other conditions for the convergence $P_n \Rightarrow P$ and for the compactness of the set $\{P_n\}$ defined as above. Moreover, we discuss the conditions for the structure of the set of "tops" of random functions $\xi_n(t)$ necessary for compactness of the set $\{P_n\}$.

1. We shall use the following notation:

$P_n^{t_1,\ldots,t_m}$ for the m-dimensional distribution function of the stochastic process $\xi_n(t)$;

$L(P_n^t, P^t)$ for the distance defined by (2) between probability distribution functions $P_n^t(x) = P\{\xi_n(t) < x\}$ and $P^t(x) = P\{\xi(t) < x\}$.

We prove the following theorem:

THEOREM 1. *If P_n is defined as above for sequence* (3) *with condition* (4), *and P is a measure in $D[0,1]$ generated by a continuous stochastic process $\xi(t)$ with independent increments, then for the condition $P_n \Rightarrow P$ it is necessary and sufficient that the convergence*

(7) $$L(P_n^t, P^t) \to 0$$

hold uniformly with respect to t $(0 \leqslant t \leqslant 1)$.

Proof. It suffices to prove that the uniform convergence of (7) is necessary and sufficient for the conditions (a) and (b) of the theorem of Prokhorov, and for the weak convergence of all finite dimensional distributions $P_n^{t_1,\ldots,t_m}$ to $P^{t_1,\ldots,t_m}$.

Necessity. Note first that for a continuous stochastic process its probability distribution function $P^t(x) = P\{\xi(t) < x\}$ is a continuous function of t. In fact, the continuity of the process means that, for every t, $P\{\lim_{\tau=0} \xi(t+\tau) = \xi(t)\} = 1$. Hence for every fixed t and for exery point of continuity of the function $P^t(x)$ we may write

$$\begin{aligned} P^t(x) &= P\{\xi(t) < x\} = P\{\xi(t) < x\} P\{\lim_{\tau=0} \xi(t+\tau) = \xi(t)\} \\ &= P\{\xi(t) < x; \lim_{\tau=0} \xi(t+\tau) = \xi(t)\} = P\{\lim_{\tau=0} \xi(t+\tau) < x\} \\ &= \lim_{\tau=0} P\{\xi(t+\tau) < x\} = \lim_{\tau=0} P^{t+\tau}(x). \end{aligned}$$

According to the equivalence of conditions (ii) and (iii) the function $L(P^t, P^\tau)$ is a continuous function of t and τ.

To prove the necessity note that the convergence $L(P_n^t, P^t) \to 0$ for every fixed t follows from the finite-dimensional one. Suppose that the convergence of (7) is not uniform. Then there exists a number $\varepsilon_0 > 0$, a sequence $n_i \to \infty$ and a sequence of points τ_i, such that

(8) $$L(P_{n_i}^{\tau_i}, P^{\tau_i}) > \varepsilon_0.$$

Without loss of generality we can suppose that $\tau_i \to \tau$, and $\tau_1 < \tau_2 < \ldots < \tau$. From the continuity of the process if follows that there exists such a number $\delta > 0$ that for every τ' for which $|\tau - \tau'| < \delta$

$$L(P^{\tau}, P^{\tau'}) < \varepsilon_0/4. \tag{9}$$

Let us take such an N that, for $n_i > N$, we have $\tau - \tau_i < \delta$ and

$$L(P^{\tau}_{n_i}, P^{\tau}) < \varepsilon_0/4. \tag{10}$$

For $n_i > N$ conditions (9) and (10), according to definition (2), may be rewritten in the form:

$$(9')\quad P\left\{\xi(\tau_i) < x - \frac{\varepsilon_0}{4}\right\} - \frac{\varepsilon_0}{4} \leqslant P\{\xi(\tau) < x\} \leqslant P\left\{\xi(\tau_i) < x + \frac{\varepsilon_0}{4}\right\} + \frac{\varepsilon_0}{4};$$

$$(10')\quad P\left\{\xi_{n_i}(\tau) < x - \frac{\varepsilon_0}{4}\right\} - \frac{\varepsilon_0}{4} \leqslant P\{\xi(\tau) < x\} \leqslant P\left\{\xi_{n_i}(\tau) < x + \frac{\varepsilon_0}{4}\right\} + \frac{\varepsilon_0}{4}.$$

for every x.

If (8) is satisfied, then there exists such a sequence $\{x_i\}$ that

$$(8')\quad \begin{cases} P\{\xi_{n_i}(\tau_i) < x_i - \varepsilon_0\} - \varepsilon_0 > P\{\xi(\tau_i) < x_i\} \\ \text{or} \\ P\{\xi_{n_i}(\tau_i) < x_i + \varepsilon_0\} + \varepsilon_0 < P\{\xi(\tau_i) < x_i\}. \end{cases}$$

For example let the first inequality (8′) hold. Then for $x = x_i - \varepsilon_0/4$ we obtain from (10′), (9′) and (8′)

$$P\left\{\xi_{n_i}(\tau) < x_i - \frac{\varepsilon_0}{2}\right\} - \frac{\varepsilon_0}{4} \leqslant P\left\{\xi(\tau) < x_i - \frac{\varepsilon_0}{4}\right\} \leqslant P\left\{\xi(\tau_i) < x_i\right\} + \frac{\varepsilon_0}{4}$$
$$< P\{\xi_{n_i}(\tau_i) < x_i - \varepsilon_0\} - \varepsilon_0 + \frac{\varepsilon_0}{4}.$$

The left side of the last inequality may be estimated as follows:

$$P\left\{\xi_{n_i}(\tau) < x_i - \frac{\varepsilon_0}{2}\right\} = P\left\{\sum_{t_{n_ik} \leqslant \tau} \xi_{n_ik} < x_i - \frac{\varepsilon_0}{2}\right\}$$
$$= P\left\{\sum_{t_{n_ik} < \tau_i} \xi_{n_ik} + \sum_{t_{n_ik} \in \Delta_i} \xi_{n_ik} < x_i - \frac{\varepsilon_0}{2}\right\}$$
$$\geqslant P\left\{\sum_{t_{n_ik} < \tau_i} \xi_{n_ik} < x_i - \varepsilon_0;\ \left|\sum_{t_{n_ik} \in \Delta_i} \xi_{n_ik}\right| < \frac{\varepsilon_0}{2}\right\}$$
$$= P\{\xi_{n_i}(\tau_i) < x_i - \varepsilon_0\} \cdot P\left\{\left|\sum_{t_{n_ik} \in \Delta_i} \xi_{n_ik}\right| < \frac{\varepsilon_0}{2}\right\},$$

where Δ_i denotes the interval $[\tau_i, \tau)$. Hence we have

$$P\{\xi_{n_i}(\tau_i) < x_i - \varepsilon_0\} \cdot P\left\{\left|\sum_{t_{n_ik} \in \Delta_i} \xi_{n_ik}\right| < \frac{\varepsilon_0}{2}\right\} < P\{\xi_{n_i}(\tau_i) < x_i - \varepsilon_0\} - \frac{\varepsilon_0}{2}.$$

Passing to the limit with $i \to \infty$ we see that condition (a) of the theorem of Prokhorov is not satisfied, which proves the necessity of the uniform convergence of (7).

To prove the sufficiency note first that from convergence (7) follows the finite-dimensional one. In fact, let us take any m points $t_1, t_2, \ldots, t_m \in [0,1]$. For example let $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1$. If convergence (7) holds, and if $(a_1, \ldots, a_m)$ is a continuity point of the distribution function of the m-dimensional random variable $\{\xi(t_i) - \xi(t_{i-1}),\ i = 1, 2, \ldots, m\}$, then we may write

$$P\left\{\bigcap_{i=1}^{m} \xi(t_i) - \xi(t_{i-1}) < a_i\right\} = \prod_{i=1}^{m} P\{\xi(t_i) - \xi(t_{i-1}) < a_i\}$$
$$= \lim_{n=\infty} \prod_{i=1}^{m} P\{\xi_n(t_i) - \xi_n(t_{i-1}) < a_i\}$$
$$= \lim_{n=\infty} P\left\{\bigcap_{i=1}^{m} \xi_n(t_i) - \xi_n(t_{i-1}) < a_i\right\}.$$

Now suppose that condition (a) of the theorem of Prokhorov is not satisfied. Then there exist numbers $\varepsilon_0 > 0$ and $\lambda_0 > 0$, a sequence $n_i \to \infty$ and a sequence of intervals $\Delta_i = [t_i', t_i'')$ such that $|\Delta_i| \to 0$ and

$$P\left\{\left|\sum_{t_{n_ik} \in \Delta_i} \xi_{n_ik}\right| > \lambda_0\right\} > \varepsilon_0. \tag{11}$$

Without loss of generality we may suppose that $t_i' \to \tau$ and $t_i'' \to \tau$. Let us take any number $\delta > 0$ and denote by Δ_δ the interval $[\tau - \delta, \tau + \delta]$. For all $\Delta_i \subset \Delta_\delta$ we may write

$$P\{V_{\xi(t)}(\Delta_\delta) > \lambda_0\} \geqslant P\{|\xi(t_i'') - \xi(t_i)| > \lambda_0\},$$

where $V_{\xi(t)}(\Delta) = \sup_{t_1, t_2 \in \Delta} |\xi(t_1) - \xi(t_2)|$. If λ_0 and $-\lambda_0$ are the continuity points of the distribution functions of the variables $\xi(t_i'') - \xi(t_i')$, $i = 1, 2, \ldots$ (which may always be assumed), and if convergence of (7) is uniform, then, uniformly with respect to i,

$$P\{V_{\xi(t)}(\Delta_\delta) > \lambda_0\} \geqslant \lim_{k=\infty} P\{|\xi_{n_k}(t_i'') - \xi_{n_k}(t_i')| > \lambda_0\},$$

and passing to the limit with $i \to \infty$ we obtain using (11):

$$P\{V_{\xi(t)}(\Delta_\delta) > \lambda_0\} \geqslant \lim_{i=\infty} \lim_{k=\infty} P\{|\xi_{n_k}(t_i'') - \xi_{n_k}(t_i')| > \lambda_0\}$$
$$\geqslant \lim_{j=\infty} P\{|\xi_{n_j}(t_j'') - \xi_{n_j}(t_j')| > \lambda_0\} > \varepsilon_0.$$

Since $\delta > 0$ is arbitrary, the process is not continuous at the point τ.

Suppose now that condition (b) of the theorem of Prokhorov is not satisfied. It means that there exist a number $\varepsilon_0 > 0$, a sequence $n_i \to \infty$ and $\lambda_i \to \infty$ and a sequence of intervals $\Delta_i = [t_i', t_i'')$ such that

$$(12) \qquad P\Big\{\Big|\sum_{t_{n_i k} \in \Delta_i} \xi_{n_i k}\Big| > \lambda_i\Big\} > \varepsilon_0 .$$

From condition (a) proved above it follows that $|\Delta_i| \not\to 0$. Hence there exists such a number $a > 0$ that $|\Delta_i| > a$, except at most a finite number of intervals. Let us take a point τ^*, which belongs to a infinite number of intervals Δ_i; for example let $\tau^* \in \Delta_{i_\nu}$, $\nu = 1, 2, \ldots$, and let Δ be such an interval that $\Delta_{i_\nu} \subset \Delta$, $\nu = 1, 2, \ldots$ Then for any $A > 0$ we may write

$$P\{V_{\xi(t)}(\Delta) > A\} \geqslant P\{|\xi(t_{i_\nu}'') - \xi(t_{i_\nu}')| > A\}.$$

If A and $-A$ are the continuity points of the distribution functions of the random variables $\xi(t_{i_\nu}'') - \xi(t_{i_\nu}')$, $\nu = 1, 2, \ldots$, and if the convergence of (7) is uniform, then

$$P\{V_{\xi(t)}(\Delta) > A\} \geqslant \lim_{k=\infty} P\{|\xi_{n_{i_k}}(t_{i_\nu}'') - \xi_{n_{i_k}}(t_{i_\nu}')| > A\}$$

uniformly with respect to ν. Passing to the limit with $\nu \to \infty$ and using (12) we obtain

$$P\{V_{\xi(t)}(\Delta) > A\} \geqslant \lim_{\nu=\infty} \lim_{k=\infty} P\{|\xi_{n_{i_k}}(t_{i_\nu}'') - \xi_{n_{i_k}}(t_{i_\nu}')| > A\}$$
$$\geqslant \lim_{j=\infty} P\{|\xi_{n_{i_j}}(t_{i_j}'') - \xi_{n_{i_j}}(t_{i_j}')| > A\} > \varepsilon_0 .$$

Thus, since $A > 0$ is arbitrary, $P\{V_{\xi(t)}(\Delta) = \infty\} > \varepsilon_0$.

Divide the interval Δ into two equal closed intervals, and denote by $\Delta^{(1)}$ the part for which $P\{V_{\xi(t)}(\Delta^{(1)}) = \infty\} > \varepsilon_0$. Then divide the interval $\Delta^{(1)}$ into two equal closed intervals and denote by $\Delta^{(2)}$ the part for which $P\{V_{\xi(t)}(\Delta^{(2)}) = \infty\} > \varepsilon_0$. In this way we obtain a sequence of closed intervals $\Delta^{(1)} \supset \Delta^{(2)} \supset \ldots$, for which $|\Delta^{(n)}| \to 0$ and

$$P\{V_{\xi(t)}(\Delta^{(n)}) = \infty\} > \varepsilon_0, \qquad n = 1, 2, \ldots$$

At the point $\tau = \Delta^{(1)} \cap \Delta^{(2)} \cap \ldots$ the limiting distribution will not exist; this proves theorem 1.

Theorems 4, 5, 7, 8, 9 and 10 of [2] are simple consequences of theorem 1 proved above.

2. Now we will show that the convergence $P_n \Rightarrow P$ depends upon the manner in which the partitions $0 = t_{n0} < t_{n1} < \ldots < t_{nk_n} = 1$ are constructed. Let us take for example the following sequence of series $\{\xi_{nk}\}$, $k = 1, 2, \ldots, n$:

$$(13) \qquad P\Big\{\xi_{nk} = \frac{1}{n}\Big\} = 1, \qquad k = 1, 2, \ldots, n.$$

For the sequence of partitions $t_{nk} = k/n$, $k = 0, 1, \ldots, n$ the limiting distribution for sequence (13) will be of the form

$$P\{\xi(t) \equiv t\} = 1, \qquad 0 \leqslant t \leqslant 1.$$

Now let us take another sequence of partitions. Divide for every $n > 1$ the interval [0,1] into three parts: $I_1^n = [0, \frac{1}{2} - 1/2n)$, $I_2^n = [\frac{1}{2} - 1/2n, \frac{1}{2} + 1/2n]$, $I_3^n = (\frac{1}{2} + 1{,}2n, 1]$, and divide the intervals I_1^n and I_3^n into $(\frac{1}{2})[\log n]$ equal parts, and I_2^n into $n - (\frac{1}{2})[\log n]$ equal parts. Then for sequence (13) the limiting distribution will be of the form

$$P\{\xi(t) \equiv 0\} = 1 \quad \text{for} \quad t \in [0, \tfrac{1}{2}),$$
$$P\{\xi(\tfrac{1}{2}) = \tfrac{1}{2}\} = 1,$$
$$P\{\xi(t) \equiv 1\} = 1 \quad \text{for} \quad t \in (\tfrac{1}{2}, 1].$$

For the functional $f(\xi(t)) = \max_{0 \leqslant t < \frac{1}{4}} \xi(t)$ we shall have $P\{f(\xi(t)) = 0\} = 1$. and for every $n > 1$: $P\{f(\xi_n(t)) > \frac{1}{4}\} = 1$; thus, according to definition (1) condition $P_n \Rightarrow P$ is not satisfied.

Two theorems which we shall now prove will concern the question of the possible types of partitions for which the weak convergence $P_n \Rightarrow P$ holds.

Denote by K the class of those sequences of partitions $0 = t_{n0} < t_{n1} < \ldots < t_{nk_n} = 1$ for which condition (5) holds, and

$$(14) \qquad \limsup_{n=\infty} \frac{\max\limits_{1 \leqslant k \leqslant k_n} (t_{nk} - t_{n,k-1})}{\min\limits_{1 \leqslant k \leqslant k_n} (t_{nk} - t_{n,k-1})} < \infty .$$

Denote by T a sequence of partitions satisfying condition (5), by Ξ a sequence of series (3) satisfying condition (4), and by $P_n(\Xi, T)$ the measure generated in $D[0,1]$ by all finite-dimensional distributions of the stochastic process $\xi_n(t)$ defined by formulas (6) for the sequence Ξ and partitions T.

The following theorem holds:

THEOREM 2. *If for a certain sequence Ξ and for some $T^0 \in K$ the set of measures $\{P_n(\Xi, T^0)\}$ is compact and every limiting measure corresponds to a continuous stochastic process with independent increments, then for any $T \in K$ the same property holds for the set of measures $\{P_n(\Xi, T)\}$.*

Proof. Let us take any sequence of series $\Xi = \{\xi_{nk}\}$, $k = 1, \ldots, k_n$ and any two sequences of partitions $T = \{t_{nk}\}$, and $T' = \{t'_{nk}\}$, $k = 0, 1, \ldots, k_n$, from the class K. Then there exists a number $A < \infty$ such that

$$(15) \qquad \limsup_{n=\infty} \frac{\max_k (t_{nk} - t_{n,k-1})}{\min_k (t_{nk} - t_{n,k-1})} < A, \qquad \limsup_{n=\infty} \frac{\max_k (t'_{nk} - t'_{n,k-1})}{\min_k (t'_{nk} - t'_{n,k-1})} < A.$$

Suppose that for the sequence T the set $\{P_n(\Xi, T)\}$ is not compact. Then the conditions of the theorem of Prokhorov are not satisfied. Since condition (b) is independent of the sequence of partitions, we may assume, that condition (a) is not satisfied. It means that there exist numbers $\varepsilon_0 > 0$ and $\lambda_0 > 0$, a sequence of numbers $n_i \to \infty$ and a sequence of intervals Δ_i of the form $[t_{n_i k}, t_{n_i l})$ such that $|\Delta_i| \to 0$ and

$$P\Big\{\Big|\sum_{t_{n_i j} \in \Delta_i} \xi_{n_i j}\Big| > \lambda_0\Big\} > \varepsilon_0.$$

Denote by Δ'_i the least interval containing those $t'_{n_i j}$ for which $t_{n_i j} \in \Delta_i$. Then obviously

$$P\Big\{\Big|\sum_{t'_{n_i j} \in \Delta'_i} \xi_{n_i j}\Big| > \lambda_0\Big\} > \varepsilon_0,$$

and it suffices to prove that $|\Delta'_i| \to 0$.

From condition (15) it follows that, for sufficiently large n,

$$\max_k (t_{nk} - t_{n,k-1}) < A \min_k (t_{nk} - t_{n,k-1});$$

using the obvious relation

$$\min_k (t_{nk} - t_{n,k-1}) \leqslant 1/k_n \leqslant \max_k (t_{nk} - t_{n,k-1})$$

we obtain

$$1/Ak_n \leqslant (1/A) \cdot \max_k (t_{nk} - t_{n,k-1}) \leqslant \min_k (t_{nk} - t_{n,k-1}) \leqslant 1/k_n$$
$$\leqslant \max_k (t_{nk} - t_{n,k-1}) \leqslant A \min_k (t_{nk} - t_{n,k-1}) \leqslant A/k_n,$$

and the same relation holds for the sequence $\{t'_{nk}\}$. Let Q_i be the number of those intervals of the form $[t_{n_i k}, t_{n_i,k-1}]$ which are $\subset \Delta_i$. Then

$$|\Delta'_i| \leqslant Q_i \max_k (t'_{n_i k} - t'_{n_i,k-1}) \leqslant Q_i A \min_k (t'_{n_i k} - t'_{n_i,k-1})$$
$$\leqslant Q_i A/k_{n_i} = A^2 Q_i/Ak_{n_i} \leqslant A^2 \min_k (t_{n_i k} - t_{n_i,k-1}) \leqslant A^2 |\Delta_i|,$$

and if $|\Delta_i| \to 0$ then $|\Delta'_i| \to 0$ as asserted.

Now we prove

THEOREM 3. *For every sequence of partitions $T = \{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, the following conditions are equivalent:*

(A) *for every sequence of series $\Xi = \{\xi_{nk}\}$, $k = 1, 2, \ldots, k_n$, the set $\{P_n(\Xi, T)\}$ is compact and every limiting distribution corresponds to a continuous stochastic process with independent increments if and only if*

(α) *for every fixed $\lambda > 0$*

$$\max_k P\Big\{\Big|\sum_{j=0}^{l} \xi_{n,k+j}\Big| > \lambda\Big\} \to 0$$

uniformly with respect to n when $l/k_n \to 0$,

$$(\beta) \qquad \max_k \max_l P\Big\{\Big|\sum_{j=0}^{l} \xi_{n,k+j}\Big| > \lambda\Big\} \to 0$$

uniformly with respect to n when $\lambda \to \infty$.

(B) *For the sequence of series $\Xi^0 = \{\xi^0_{nk}\}$, $k = 1, 2, \ldots, k_n$, defined by formula*

$$(16) \qquad P\{\xi^0_{nk} = 1/k_n\} = 1, \qquad k = 1, 2, \ldots, k_n,$$

the set $\{P_n(\Xi^0, T)\}$ is compact and every limiting distribution is of the form $P\{\xi(t) = \varphi(t)\} = 1$, where $\varphi(t)$ is a continuous function for $0 \leqslant t \leqslant 1$ (it is evident that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(t)$ is a non-decreasing function).

Proof. It is evident that (A) $\to$ (B), since if (B) is not satisfied, then (16) represents a sequence for which the equivalence (A) is not satisfied. To prove (B) $\to$ (A) suppose for simplicity that for series (16) the limiting distribution exists and is of the form $P\{\xi(t) = \varphi(t)\} = 1$, where $\varphi(t)$ is a continuous function. Then the limit

$$(17) \qquad \lim_{n=\infty} \frac{\varphi_n(t)}{k_n},$$

where $\varphi_n(t)$ denotes the number of those t_{nk} from the partition T which are less then t, exists and equals $\equiv \varphi(t)$.

Note first that condition (β) coincides with condition (b) of Prokhorov's theorem. Suppose that condition (a) of the theorem of Prokhorov is not satisfied; then there exist numbers $\varepsilon_0 > 0$ and $\lambda_0 > 0$, a sequence $n_i \to \infty$ and a sequence of intervals Δ_i such that $|\Delta_i| \to 0$ and

$$P\Big\{\Big|\sum_{t_{n_i k} \in \Delta_i} \xi_{n_i k}\Big| > \lambda_0\Big\} > \varepsilon_0.$$

Denote by $t_{n_i, m_{n_i}}$ and $t_{n_i, l_{n_i}}$ the smallest and the largest point of the form $t_{n_i k}$ of the interval Δ_i. From the existence and continuity of limit (17) it follows that $(l_{n_i} - m_{n_i})/k_{n_i} \to 0$ for $i \to \infty$; hence (α) is not satisfied.

Conversely, suppose that condition (α) is not satisfied. Then there exist numbers $\varepsilon_0 > 0$ and $\lambda_0 > 0$, a sequence $n_i \to \infty$ and two sequences m_{n_i} and l_{n_i} such that $l_{n_i}/k_{n_i} \to 0$ for $i \to \infty$ and l_{n_i}

$$P\Big\{\Big|\sum_{j=0}^{l_{n_i}} \xi_{k_i, m_{n_i}+j}\Big| > \lambda_0\Big\} > \varepsilon_0 .$$

Denote by Δ_i the least interval containing the points $t_{n_i,m_{n_i}}, t_{n_i,m_{n_i}+1}, \ldots, t_{n_i,m_{n_i}+l_{n_i}}$. If limit (17) exists and is a continuous function of t, then $|\Delta_i| \to 0$, and condition (a) of the theorem of Prokhorov is not satisfied. This proves theorem 3.

3. Suppose now that the sequence Ξ^* of random variables

$$\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n} \tag{18}$$

has for each n a common distribution $F_n(x) = P\{\xi_{nk} < x\}$, $k = 1, 2, \ldots, k_n$. From the theorem of Skorohod (see for example [4], § 3.2) and from theorems 2 and 3 we immediately obtain:

THEOREM 4. *The convergence of the sequence of distribution functions*

$$F_{n,k_n}(x) = P\Big\{\sum_{j=1}^{k_n} \xi_{nj} < x\Big\}$$

for $n \to \infty$ to a (infinitely divisible) limiting distribution $G(x)$ is necessary and sufficient for:

(I) *the compactness of the set of measures $\{P_n(\Xi^*, T)\}$ in the case when the sequence of partitions $T = \{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, belongs to the class K defined by formula* (14);

(II) *the convergence $P_n(\Xi^*, T) \Rightarrow P$ in the case when for the sequence of partitions $T = \{t_{nk}\}$, $k = 0, 1, \ldots, k_n$, limit* (17) *exists.*

In cases (I) *and* (II) *the limiting measures are generated by continuous stochastic processes with independent increments.*

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A theorem on distributions integrable with even power

by

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1. In this paper we shall consider some spaces of distributions introduced by Schwartz [2]. By $\mathcal{D}_N$ we shall denote the space of all infinitely differentiable complex-valued functions $\varphi = \varphi(x_1, x_2, \ldots, x_N)$ $(-\infty < x_j < \infty,\ j = 1, 2, \ldots, N)$ with compact supports. Put

$$\|\varphi\| = \max_{x_1, x_2, \ldots, x_N} |\varphi(x_1, x_2, \ldots, x_N)| \qquad (\varphi \in \mathcal{D}_N).$$

The convergence in $\mathcal{D}_N$ is defined as follows: $\varphi_j \to 0$ $(\varphi_j \in \mathcal{D}_N,\ j = 1, 2, \ldots)$ if for every system of integers $\langle k_1, k_2, \ldots, k_N\rangle$

$$\left\| \frac{\partial^{k_1+k_2+\ldots+k_N}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_N^{k_N}} \varphi_j \right\| \to 0$$

and the supports of φ_j are contained in a fixed compact.

Let A be an arbitrary subset of the N-dimensional Euclidean space. By $\mathcal{D}_N(A)$ we shall denote the subspace of $\mathcal{D}_N$ consisting of all functions whose supports are contained in A.

The space $\mathcal{D}_N'$ of distributions is the conjugate space of $\mathcal{D}_N$. By (T, φ) we shall denote the value of T at φ $(T \in \mathcal{D}_N',\ \varphi \in \mathcal{D}_N)$. The conjugate of T is defined by the formula $(\overline{T}, \varphi) = \overline{(T, \overline{\varphi})}$ $(\varphi \in \mathcal{D}_N)$.

We say that a distribution $T \in \mathcal{D}_N'$ is *of order* $\leqslant k_1 + \ldots + k_N$ on A if there is a continuous function f such that

$$(T, \varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) \frac{\partial^{k_1+\ldots+k_N}}{\partial x_1^{k_1} \ldots \partial x_N^{k_N}} \varphi(x_1, \ldots, x_N)\, dx_1 \ldots dx_N$$

for each $\varphi \in \mathcal{D}_N(A)$. All the distributions belonging to $\mathcal{D}_N'$ are of finite order on every compact (cf. [2], tome I, chapt. III, § 6).

Let $T \in \mathcal{D}_1'$. By $|T|^{2p}$ $(p = 1, 2, \ldots)$ we shall denote the direct product $\underbrace{T \times T \times \ldots \times T}_{p \text{ times}} \times \underbrace{\overline{T} \times \overline{T} \times \ldots \times \overline{T}}_{p \text{ times}}$, i. e. the distribution belonging to $\mathcal{D}_2'^{p}$