Dans le cas de la valeur nous avons, en particulier:

THÉORÈME 2. *Pour que la valeur $T(x_0)$ existe et soit d'ordre $\subset \mathfrak{P}$, il faut et il suffit que dans une voisinage de x_0 la distribution T soit de la forme*

$$T = T(x_0) + \sum_{\mathfrak{P}} D^{p}\sigma_{p}, \quad \text{où} \quad |\sigma_p|(P_\lambda(x_0)) = o(\lambda^{|p|+m}) \tag{8.3.9}$$

(σ_p étant des mesures).

Finalement les théorèmes 1 et 2 donnent les développements

$$(T(x,y), \chi(x,y)) = \left(S, \int \chi(x,y)\,dx\right)_y + \sum_{\mathfrak{S}} \int D_x^p D_y^q \chi(x,y)\,d\sigma_{pq}, \tag{8.3.10}$$

où $|\sigma_{pq}|(P_\lambda(x_0)\times Q) = o(\lambda^{|p|+m})$, pour $\chi \in \mathcal{D}_{\mathcal{E}m_{\times Q}}$, si $S(y) = T(x_0, y)$ et si la fixation est d'ordre $\subset \mathfrak{S}$ sur un ouvert contenant $\bar{Q}$, et

$$\begin{aligned} &(T(x), \varphi(x)) = T(x_0)\int \varphi(x)\,dx + \sum_{\mathfrak{P}} \int D^p \varphi\, d\sigma_p, \\ &|\sigma_p|(P_\lambda(x_0)) = o(\lambda^{|p|+m}), \end{aligned} \tag{8.3.11}$$

si la valeur est d'ordre $\subset \mathfrak{P}$.

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On certain "weak" properties of vector-valued functions

by

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The starting point of this Note is the following theorem of B. J. Pettis ([6], p. 257[1]): a vector-valued[2]) function from a measure space to a Banach space X is (Bochner) measurable if and only if it is almost separably valued[3]) and for every γ belonging to a norming set of functionals the function $\gamma x(t)$ is measurable. The subset Γ of the space Ξ, conjugate to X, is called *norming* if there are two positive constants A and B such that

$$\sup\{A|\gamma x| : \gamma \in \Gamma, \|\gamma\| \leqslant B\} \geqslant \|x\|$$

for every x. In this Note we prove that the set Γ in the above statement may be replaced by any total subset of Ξ (the set Γ is *total* if $\gamma x = 0$ for any $\gamma \in \Gamma$, implies $x = 0$). Every norming set is necessarily total; the converse, however, is not true, as is shown by the following example of Mazurkiewicz [7]. Suppose that the set of all pairs (i, k) of positive integers is arranged into a single sequence, and let $\nu(i, k)$ denote the place occupied there by (i, k). Then in the space c_0 of the sequences $x = \{x_n\}$, convergent to zero, consider the set of all the functionals

$$\xi_{ik}(x) = \frac{x_1}{2^1} + \ldots + \frac{x_{2i+1}}{2^{2i+1}} + i x_{2\nu(i,k)}$$

where $i, k = 1, 2, \ldots$; the linear span Γ of this set is linear, total but not norming.

1. Let X be a separable (real or complex) Banach space, let Ξ be the conjugate space, and let Γ be a linear subset of Ξ. It is well known that the set Γ is total if and only if $\bar{\Gamma}$, its closure in the $\sigma(\Xi, X)$ topo-

[1]) Numbers in brackets refer to the bibliography at the end of this paper.

[2]) In the sequel all Banach-space-valued functions will be called simply *vector-valued*. Numerically valued functions will be called *functions*.

[3]) *i. e.*, there exists a subset N of measure zero such that the set $\{y : y = x(t), t \operatorname{non} \epsilon N\}$ is separable.

logy[4]), is equal to Ξ. After Banach ([2], p. 213) we denote by Γ^1 the weak sequential closure of Γ (*i. e.*, $\gamma \in \Gamma^1$ if and only if there exists a sequence γ_n of elements of Γ such that $\gamma_n(x) \to \gamma(x)$ for every $x \in X$). Then for every ordinal $\varphi < \Omega$ we define $\Gamma^\varphi = (\bigcup_{\alpha<\varphi} \Gamma^\alpha)^1$. Banach has shown (partly published in [2]) that for every $\varphi < \Omega$ there exists a linear set Γ such that $\Gamma^\varphi \neq \Gamma^{\varphi+1}$; on the other hand, for any linear set Γ there exists a $\varphi < \Omega$ such that $\Gamma^\varphi = \Gamma^{\varphi+1}$ (for the set defined by Mazurkiewicz we have $\Gamma \neq \Gamma^1 \neq \Gamma^2 = \Xi$).

Let φ be the smallest ordinal such that $\Gamma^\varphi = \Gamma^{\varphi+1}$; then $\Gamma^\varphi = \overline{\Gamma}$.

Indeed, by a theorem of Banach ([2], p. 124, théorème 5) the set Γ^φ is regularly closed, which is equivalent to the closedness in the $\sigma(\Xi, X)$ topology. Evidently $\Gamma \subset \Gamma^\varphi \subset \overline{\Gamma}$ and since $\overline{\Gamma}$ is the smallest weakly ($=$ regularly) closed set containing Γ, we have $\overline{\Gamma} \subset \Gamma^\varphi$.

Denote by Σ_1 the sphere: $\|\xi\| \leqslant 1$. Then $\overline{\Gamma} = \Gamma^\varphi$ implies $\overline{\Gamma} \frown \Sigma_1 = (\Gamma \frown \Sigma_1)^\varphi$.

2. Now let $\mathfrak{R}$ be a family of functions defined in a set D, and let $\mathfrak{R}$ have the following property:

(1) the limit of any pointwise convergent sequence of functions of $\mathfrak{R}$ belongs to $\mathfrak{R}$.

THEOREM 1. *Suppose that $x(t)$ is a function from D to X, X being separable. Let Γ be a linear total subset of Ξ. If $\gamma x(t)$ is in $\mathfrak{R}$ for every $\gamma \in \Gamma$ (for every $\gamma \in \Gamma \frown \Sigma_1$), then $\xi x(t)$ is in $\mathfrak{R}$ ($\xi x(t)$ is in $a_\xi \mathfrak{R}$, a_ξ being a constant) for every $\xi \in \Xi$.*

Proof. Let $\mathfrak{L}_\varphi$ be the family of all functions $\gamma x(t)$ with $\gamma \in \Gamma^\varphi$; in virtue of (1) we verify by transfinite induction that $\mathfrak{L}_\varphi \subset \mathfrak{R}$ for every $\varphi < \Omega$. There exists a φ such that $\Gamma^\varphi = \overline{\Gamma} = \Xi$.

The proof of the alternative part of Theorem 1 is similar.

As applications we get:

THEOREM 2. *A vector-valued function $x(t)$ is measurable[6]) if and only if it is almost separably valued and $\gamma x(t)$ is measurable for every γ in a total subset Γ of Ξ.*

Proof. The necessity being trivial, we prove the sufficiency. We may freely suppose that the space X is separable itself and that the set Γ is linear. Taking as $\mathfrak{R}$ the family of all measurable functions we see that $\xi x(t)$ is measurable for every $\xi \in \Xi$, whence theorem follows by a theorem of Pettis ([5], p. 278).

[4]) This is the weakest locally convex topology in Ξ for which the functionals $f(\xi) = \xi x, x \in X$ are continuous; the basis of neighbourhoods of the null element is formed by the sets $\bigcap_{i=1}^{n} \{\xi : |\xi(x_i)| \leqslant 1\}$ where $x_1, \ldots, x_n$ are arbitrary elements of X; the notation $\sigma(\Xi, X)$ is due to J. Dieudonné [3].

[5]) $a\mathfrak{R} = \{g : g = ah, h \in \mathfrak{R}\}$.

[6]) in the sense of Bochner, with respect to a σ-measure.

The following result generalizes a theorem of Dunford:

THEOREM 3. *A vector valued function $x(\zeta)$ from a domain D of the complex plane to a complex Banach space X is holomorphic in D if and only if it is separably valued, almost uniformly bounded[7]), and $\gamma x(\zeta)$ is holomorphic in D for every γ in a total subset of Ξ.*

Proof. We may suppose again that the space X is separable and that the set Γ is linear. We prove the necessity only. Let C_n be compact subsets of D such that $D = \bigcup_{n=1}^{\infty} C_n$; then $A_n = \sup\{\|x(\zeta)\| : \zeta \in C_n\} < \infty$. Applying the alternative part of Theorem 1 to the family $\mathfrak{R}$ of holomorphic functions g in D satisfying the inequality $\sup\{|g(\zeta)| : \zeta \in C_n\} \leqslant A_n$, we infer that for every $\xi \in \Xi$ the function $\xi x(\zeta)$ is in $a_\xi \mathfrak{R}$ (a_ξ — a constant depending on ξ), whence it is holomorphic. We conclude the proof by applying the theorem of Dunford.

Let us now consider vector valued functions from a metric space T. If $x(t)$ is separably valued, then $x(t)$ is of Baire's α-th class[8]) if and only if for every open set $G \subset Y$ the set $\{t : x(t) \in G\}$ is of additive α-th class of Borel.

THEOREM 4. *Let the function $x(t)$ be separably valued and let $\gamma x(t)$ be a Baire function for every γ in a total subset of Ξ. Then $x(t)$ is a function of Baire.*

Proof. Using the device applied above we may show that for every $\xi \in \Xi$ the numeric function $\xi x(t)$ is in a Baire's class, in B^{α_ξ}, say. By a theorem of Banach ([2], p. 124) there exists a sequence ξ_n of linear functionals such that for every $\xi \in \Xi$ there exists a sequence n_k such that $\xi_{n_k}(x) \to \xi(x)$ for every $x \in X$. Set $\varphi = \sup_n \alpha_{\xi_n}$, then $\xi x(t)$ is of class at most $B^{\varphi+1}$, whence by a theorem of the author and Orlicz ([1], p. 108) $x(t)$ is at most of class $B^{\varphi+2}$.

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[7]) *i. e.*, is bounded in every compact subset of D.

[8]) We adopt here the "analytic" classification: the continuous functions form the 0-th class B^0; B^α is the class of all functions which are limits of pointwise convergent sequences of functions of classes less than B^α. The functions of Baire are the functions of $\bigcup_{\alpha<\Omega} B^\alpha$.

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Addition to the paper "On some theorems of S. Saks"

by

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Mr. C. Ryll-Nardzewski has pointed out in a review[1]) that theorem 3 of my paper *On some theorems of S. Saks*[2]) must be corrected, for the number ϱ in this theorem depends on ε. Indeed, the number ϱ is not preceeded there by a quantifier operating on it, and it is obvious that this must be the existential one. Thus the correct formulation is as follows:

THEOREM 3. *Under the hypotheses of theorem 2 there exists for every* $\varepsilon > 0$ *a decomposition* $T = A+B+C$, *a* $\varrho > 0$, *and a residual set* Z *such that*

(a) *the series* $\sum_{n=0}^{\infty} V_n(x,t)\zeta^n$ *converges for any* x *and every* $|\zeta| < \varrho$ *a. e. in* A,

(b) *the series* $\sum_{n=0}^{\infty} V_n(x,t)\zeta^n$ *diverges for every* $x \in Z$ *and every* $|\zeta| > 0$ *a. e. in* B,

(c) $\mu(C) < \varepsilon$.

On the other hand, the following theorem is easily deduced by the general argument:

THEOREM 3'. *Under the hypotheses of Theorem 2 there exists for every* $\varrho > 0$ *a decomposition* $T = A+B$ *and a residual set* Z *such that*

(a) *for every* x *the series* $\sum_{n=0}^{\infty} V_n(x,t)\zeta^n$ *has a. e. in* A *the radius of convergence at least equal to* ϱ,

(b) *for every* $x \in Z$ *the series* $\sum_{n=0}^{\infty} V_n(x,t)\zeta^n$ *has a. e. in* B *the radius of convergence less than* ϱ.

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[1]) Polska Bibliografia analityczna, Matematyka (1956), review 220.
[2]) Studia Mathematica 13 (1953), p. 18-29.