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# The conditional expectations and the ergodic theorem for strictly stationary generalized stochastic processes

by

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**I. Introduction.** In the present note we shall consider generalized stochastic processes defined in [2]. We say that a generalized stochastic process $\Phi(\omega, t)$ is *strictly stationary* if there exists a sequence $\{f_n(\omega, t)\}$ of strictly stationary continuous stochastic processes such that $\Phi(\omega, t) = [f_n(\omega, t)]$. Let $F(\omega, t)$ be a continuous stochastic process and set $\Delta_h F(\omega, t) = F(\omega, t+h) - F(\omega, t)$. Then it is easy to prove the following assertion:

*The generalized process* $d^k F(\omega, t)/dt^k$ $(k \geqslant 1)$ *is strictly stationary if and only if for each* $h_1, h_2, \ldots, h_k$ *the process* $\Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_k} F(\omega, t)$ *is strictly stationary (in the usual sense).*

By $\Xi(t_1, t_2, \ldots, t_k)$ we shall denote the space of all generalized stochastic processes depending on variables $t_1, t_2, \ldots, t_k$. Suppose that $\lambda_{ij}$ $(i, j = 1, 2, \ldots, k)$ are real constants and $\det|\lambda_{ij}| \neq 0$. Let $\Phi(\omega, t_1, t_2, \ldots, t_k) = [f_n(\omega, t_1, t_2, \ldots, t_k)]$. Then the generalized stochastic process $\Phi(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j)$ is defined by the formula

$$\Phi\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big) = \Big[f_n\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big)\Big].$$

It is easy to verify that the convergence $\Phi_T(\omega, t_1, \ldots, t_k) \to \Phi(\omega, t_1, \ldots, t_k)$ when $T \to \infty$ implies the convergence

$$\Phi_T\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big) \to \Phi\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big).$$

(The convergence of generalized stochastic processes is defined in [2]). Hence in particular we obtain the following

LEMMA 1. *Let* $\Phi_T(\omega, t) \in \Xi(t)$. *Then* $\Phi_T(\omega, t_1 + \ldots + t_k) \in \Xi(t_1, \ldots, t_k)$ *and the convergence of* $\Phi_T(\omega, t_1 + \ldots + t_k)$ *when* $T \to \infty$ *implies the convergence of* $\Phi_T(\omega, t_1)$ *(in* $(\Xi t_1, \ldots, t_k)$*).*

**II. Conditional expectations of generalized stochastic processes.** In this paper we assume that the probability measure is complete. Let $\mathcal{F}$ be a $\sigma$-field of measurable $\omega$ sets containing all $\omega$ sets of probability 0. We say that a generalized stochastic process $\Phi(\omega, t)$ is *measurable* with respect to $\mathcal{F}$ if there exist an integer $k$ and a continuous stochastic process $F(\omega, t)$ such that $d^k F(\omega, t)/dt^k = \Phi(\omega, t)$ and for any fixed $t_0$ the random variable $F(\omega, t_0)$ is measurable with respect to $\mathcal{F}$.

Let $f(\omega, t)$ be a continuous stochastic process. By $\mathcal{E}(f(\omega, t)|\mathcal{F})$ we shall denote that version of the conditional expectation of $f(\omega, t)$ relative to $\mathcal{F}$ which is a continuous process, provided that the above-mentioned version exists.

We say that the generalized process $\Psi(\omega, t)$ is *the conditional expectation of* $\Phi(\omega, t)$ *relative to* $\mathcal{F}$ if there are an integer $k$ and a continuous process $F(\omega, t)$ such that the expectation $\mathcal{E}(F(\omega, t)||\mathcal{F})$ exists, the expectation $\mathcal{E}|F(\omega, t)|$ is integrable over every finite interval and

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t), \qquad \frac{d^k}{dt^k} \mathcal{E}(F(\omega, t)|\mathcal{F}) = \Psi(\omega, t).$$

From the definition of the equality of generalized stochastic processes (cf. [2], § 1) it immediately follows that $\Psi(\omega, t)$ does not depend upon the choice of a continuous version of the conditional expectation of $F(\omega, t)$ relative to $\mathcal{F}$.

Now we shall prove that $\psi(\omega, t)$ *does not depend upon the choice of an integer* $k$ *and a continuous process* $F(\omega, t)$. In fact, assume that $F_1(\omega, t), F_2(\omega, t)$ are continuous processes, the conditional expectations $\mathcal{E}(F_1(\omega, t)|\mathcal{F})$, $\mathcal{E}(F_2(\omega, t)|\mathcal{F})$ exist, the expectations $\mathcal{E}|F_1(\omega, t)|$, $\mathcal{E}|F_2(\omega, t)|$ are integrable over every finite interval and for some $k_2 \geqslant k_1$

$$\frac{d^{k_1}}{dt^{k_1}} F_1(\omega, t) = \Phi(\omega, t), \qquad \frac{d^{k_2}}{dt^{k_2}} F_2(\omega, t) = \Phi(\omega, t).$$

The last equalities imply

$$(1) \qquad F_2(\omega, t) = \begin{cases} \dfrac{1}{(k_2-k_1-1)!}\displaystyle\int_0^t (t-u)^{k_2-k_1-1} F_1(\omega, u)\,du + \displaystyle\sum_{j=0}^{k_2-1} a_j(\omega) t^j & \text{if} \quad k_2 > k_1 \\ F_1(\omega, t) + \displaystyle\sum_{j=0}^{k_2-1} a_j(\omega) t^j & \text{if} \quad k_2 = k_1, \end{cases}$$



where $a_j(\omega)$ $(j = 0, 1, \ldots, k_2-1)$ are random variables. Put

$$A_t = \{\omega : \mathcal{E}(F_1(\omega, t)|\mathcal{F}) \geqslant 0\}, \qquad B_t = \{\omega : \mathcal{E}(F_1(\omega, t)|\mathcal{F}) < 0\}.$$

Obviously,

$$(2) \qquad \int_\Omega |\mathcal{E}(F_1(\omega, t)|\mathcal{F})|\,d\omega = \int_{A_t} \mathcal{E}(F_1(\omega, t)|\mathcal{F})\,d\omega - \int_{B_t} \mathcal{E}(F_1(\omega, t)|\mathcal{F})\,d\omega,$$

where $\Omega$ denotes the space of points $\omega$. Since $A_t, B_t \in \mathcal{F}$, we have

$$\int_{A_t} \mathcal{E}(F_1(\omega, t)|\mathcal{F})\,d\omega = \int_{A_t} F_1(\omega, t)\,d\omega \leqslant \mathcal{E}|F_1(\omega, t)|,$$

$$-\int_{B_t} \mathcal{E}(F_1(\omega, t)|\mathcal{F})\,d\omega = -\int_{B_t} F_1(\omega, t)\,d\omega \leqslant \mathcal{E}|F_1(\omega, t)|.$$

Hence and from (2) it follows that $\int_\Omega |\mathcal{E}(F_1(\omega, t)|\mathcal{F})|\,d\omega$ is integrable over every finite interval. Since the value of an absolutely convergent iterated integral is independent of the order of integration, we obtain for $k_2 > k_1$

$$\mathcal{E}\left(\frac{1}{(k_2-k_1-1)!}\int_0^t (t-u)^{k_2-k_1-1} F_1(\omega, u)\,du \Big| \mathcal{F}\right) = \frac{1}{(k_2-k_1-1)!}\int_0^t (t-u)^{k_2-k_1-1} \mathcal{E}(F_1(\omega, u)|\mathcal{F})\,du.$$

Hence and from (1) we infer that

$$\mathcal{E}\Big(\sum_{j=0}^{k_2-1} a_j(\omega) t^j \Big| \mathcal{F}\Big)$$

exists. Consequently $\mathcal{E}(a_j(\omega)|\mathcal{F})$ $(j = 0, 1, \ldots, k_2-1)$ exist, and the following equality holds:

$$\mathcal{E}(F_2(\omega, t)|\mathcal{F}) = \begin{cases} \dfrac{1}{(k_2-k_1-1)!}\displaystyle\int_0^t (t-u)^{k_2-k_1-1} \mathcal{E}(F_1(\omega, u)|\mathcal{F})\,du + \displaystyle\sum_{j=1}^{k_2-1} \mathcal{E}(a_j(\omega)|\mathcal{F}) t^j & \text{if} \quad k_2 > k_1, \\ \mathcal{E}(F_1(\omega, t)|\mathcal{F}) + \displaystyle\sum_{j=0}^{k_2-1} \mathcal{E}(a_j(\omega)|\mathcal{F}) t^j & \text{if} \quad k_2 = k_1. \end{cases}$$

Thus

$$\frac{d^{k_1}}{dt^{k_1}} \mathcal{E}(F_1(\omega, t)|\mathcal{F}) = \frac{d^{k_2}}{dt^{k_2}} \mathcal{E}(F_2(\omega, t)|\mathcal{F}),$$

q. e. d.

The conditional expectation of $\Phi(\omega, t)$ relative to $\mathcal{F}$ we shall denote by $E(\Phi(\omega, t)|\mathcal{F})$.

The following statements are direct consequences of the definition of conditional expectations of generalized processes:

(a) *$E(\Phi(\omega, t)|\mathcal{F})$ is measurable with respect to $\mathcal{F}$.*

(b) *If $E(\Phi_j(\omega, t)|\mathcal{F})$ $(j = 1, 2, \ldots, m)$ exist and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are constants, then $E\left(\sum_{j=1}^{m} \lambda_j \Phi_j(\omega, t)|\mathcal{F}\right)$ exists and*

$$E\left(\sum_{j=1}^{m} \lambda_j \Phi_j(\omega, t)|\mathcal{F}\right) = \sum_{j=1}^{m} \lambda_j E(\Phi_j(\omega, t)|\mathcal{F}).$$

(c) *If $E(\Phi(\omega, t)|\mathcal{F})$ exists, then $E\left(\frac{d}{dt}\Phi(\omega, t)|\mathcal{F}\right)$ exists and*

$$E\left(\frac{d}{dt}\Phi(\omega, t)|\mathcal{F}\right) = \frac{d}{dt}E(\Phi(\omega, t)|\mathcal{F}).$$

(d) $E\big(E(\Phi(\omega, t)|\mathcal{F})\big) = E(\Phi(\omega, t))$.

(The expectation of generalized stochastic processes is defined in [2], § I.)

(e) *If $\mathcal{F}$ is the σ-field of all sets having probability 0 or 1, then $E(\Phi(\omega, t)|\mathcal{F}) = E(\Phi(\omega, t))$.*

(f) *If $\mathcal{F}_1 \subset \mathcal{F}_2$, then*

$$E\big(E(\Phi(\omega, t)|\mathcal{F}_2)|\mathcal{F}_1\big) = E(\Phi(\omega, t)|\mathcal{F}_1).$$

Now we shall prove the following assertion:

(g) *If $\Phi(\omega, t)$ is measurable with respect to $\mathcal{F}$ and if $E(\Phi(\omega, t))$ exists, then $E(\Phi(\omega, t)|\mathcal{F}) = \Phi(\omega, t)$.*

Proof. From the assumption it follows that there are continuous processes $f(\omega, t)$, $g(\omega, t)$ and an integer $k$ such that

(3) $$\frac{d^k}{dt^k}f(\omega, t) = \Phi(\omega, t) = \frac{d^k}{dt^k}g(\omega, t),$$

the expectation $\mathcal{E}|g(\omega, t)|$ is bounded in every finite interval and $f(\omega, t)$ is measurable with respect to $\mathcal{F}$. Put

$$h(\omega, t) = f(\omega, t) - \sum_{j=1}^{k} f(\omega, x_j)\frac{(t-x_1)\ldots(t-x_{j-1})\ldots(t-x_{j+1})\ldots(t-x_k)}{(x_j-x_1)\ldots(x_j-x_{j-1})\ldots(x_j-x_{j+1})\ldots(x_j-x_k)}$$



where $x_1, x_2, \ldots, x_k$ are constants and $x_i \neq x_j$ for $i \neq j$. Evidently $h(\omega, t)$ is measurable with respect to $\mathcal{F}$,

(4) $$h(\omega, x_j) = 0 \quad (j = 1, 2, \ldots, k)$$

and

(5) $$\frac{d^k}{dt^k}h(\omega, t) = \Phi(\omega, t).$$

Hence, according to (3), we obtain the equality

(6) $$h(\omega, t) = g(\omega, t) + \sum_{s=0}^{k-1} a_s(\omega)t^s,$$

where $a_s(\omega)$ $(s = 0, 1, \ldots, k-1)$ are random variables. From the last equality and from (4) it follows that

$$\sum_{s=0}^{k-1} a_s(\omega)x_j^s = -g(\omega, x_j) \quad (j = 1, 2, \ldots, k).$$

Since $\mathcal{E}|g(\omega, t)| < \infty$, the last equalities imply $\mathcal{E}|a_s(\omega)| < \infty$ $(s = 0, 1, \ldots, k-1)$ and consequently, in view of (6), $\mathcal{E}|h(\omega, t)|$ is bounded in every finite interval. Therefore $h(\omega, t)$ is a continuous version of the conditional expectation of $h(\omega, t)$ relative to $\mathcal{F}$: $h(\omega, t) = \mathcal{E}(h(\omega, t)|\mathcal{F})$. Hence, taking into account equality (5), we obtain $\Phi(\omega, t) = E(\Phi(\omega, t)|\mathcal{F})$. Assertion (g) is thus proved.

From the assertions (a), (f) and (g) it follows that

(h) *If $\mathcal{F}_1 \subset \mathcal{F}_2$ and if $E(\Phi(\omega, t)|\mathcal{F}_2)$ is measurable with respect to $\mathcal{F}_1$, then*

$$E(\Phi(\omega, t)|\mathcal{F}_1) = E(\Phi(\omega, t)|\mathcal{F}_2).$$

THEOREM 1. *If the expectation $E(\Phi(\omega, t))$ exists, then also the conditional expectation $E(\Phi(\omega, t)|\mathcal{F})$ exists.*

Proof. From the assumption it follows that there are a continuous process $F(\omega, t)$ and an integer $k$ such that $\mathcal{E}|F(\omega, t)|$ is integrable over every finite interval and

(7) $$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t), \quad \frac{d^k}{dt^k}\mathcal{E}(F(\omega, t)) = E(\Phi(\omega, t)).$$

Let $I_n$ denote the interval $n \leqslant t < n+1$ $(n = 0, \pm 1, \ldots)$. By $\mathcal{B}_n$ we shall denote the σ-field of Lebesgue measurable subset of $I_n$, and by $\mathcal{B}$ the σ-field of Lebesgue measurable subset of the line. Since $\mathcal{E}|F(\omega, t)|$ is integrable over $I_n$ $(n = 0, \pm 1, \ldots)$, $|F(\omega, t)|$ is integrable over $\Omega \times I_n$ $(n = 0, \pm 1, \ldots)$. Consequently, according to the Radon-Nikodym theorem, there is a function $a_n(\omega, t)$ measurable with respect to $\mathcal{F}\times\mathcal{B}_n$ such that for each $A\in\mathcal{F}\times\mathcal{B}_n$

(8) $$\int_A a_n(\omega, u)\,d\omega\,du = \int_A F(\omega, u)\,d\omega\,du\,.$$

Moreover,

$$\int_{\Omega\times I_n} |a_n(\omega, u)|\,d\omega\,du \leqslant \int_{I_n} \mathcal{E}|F(\omega, u)|\,du\,.$$

Since, according to Fubini's theorem, for almost all $\omega$, $a_n(\omega, t)$ is Lebesgue measurable, the last inequality implies that, for almost all $\omega$, $a_n(\omega, t)$ is Lebesgue integrable over $I_n$. Put

(9) $$a(\omega, t) = a_n(\omega, t) \quad \text{if} \quad t\in I_n \quad (n = 0, \pm 1, \ldots)\,.$$

Then, for almost all $\omega$, $a(\omega, t)$ is Lebesgue integrable over every finite interval. Moreover, $a(\omega, t)$ is measurable with respect to $\mathcal{F}\times\mathcal{B}$. Consequently, the function

$$\beta(\omega, t) = \int_0^t a(\omega, u)\,du$$

is measurable with respect to $\mathcal{F}\times\mathcal{B}$. Hence, in view of Fubini's theorem, for almost all $t$ the $\omega$-function $\beta(\omega, t)$ is measurable with respect to $\mathcal{F}$. Taking into account the continuity of the process $\beta(\omega, t)$, we infer that for all $t$ the $\omega$-function $\beta(\omega, t)$ is measurable with respect to $\mathcal{F}$. Further, from (8) and (9) it follows for every $A\in\mathcal{F}$ that

$$\int_A \beta(\omega, t)\,d\omega = \int_A \int_0^t F(\omega, u)\,du\,d\omega\,.$$

Consequently,

$$\beta(\omega, t) = \mathcal{E}\Big(\int_0^t F(\omega,u)\,du|\mathcal{F}\Big)\,.$$

Since the expectation $\mathcal{E}|\int_0^t F(\omega, u)\,du|$ is integrable over every finite interval and, according to (7)

$$\frac{d^{k+1}}{dt^{k+1}}\int_0^t F(\omega, u)\,du = \Phi(\omega, t)\,,$$

the conditional expectation of $\Phi(\omega, t)$ relative to $\mathcal{F}$ exists.

The theorem is thus proved.



Examples. 1. Let $\nu(\omega)$ be a random variable and let $P(\nu(\omega) < t|\mathcal{F})$ be a version, measurable with respect to $(\omega, t)$, of conditional probability distribution of $\nu(\omega)$ relative to $\mathcal{F}$. Put

$$F(\omega, t) = \max\big(0, t-\nu(\omega)\big)+\min\big(0, \nu(\omega)\big)\,.$$

Obviously, $F(\omega, t)$ is a continuous process, $d^2F(\omega, t)/dt^2 = \delta\big(t-\nu(\omega)\big)$ and $|F(\omega, t)| \leqslant |t|$. From the last inequality we infer that $\mathcal{E}|F(\omega, t)|$ is integrable over every finite interval. Moreover, it is easy to verify that

$$\mathcal{E}\big(F(\omega, t)|\mathcal{F}\big) = \int_0^t P\big(\nu(\omega) < u|\mathcal{F}\big)\,du\,.$$

Consequently,

$$E\Big(\delta\big(t-\nu(\omega)\big)|\mathcal{F}\Big) = \frac{d}{dt}P\big(\nu(\omega) < t|\mathcal{F}\big)\,.$$

2. Let $\xi(\omega)$ be a random variable with a continuous and positive density function $g(x)$. Put

$$H(\omega, t) = \frac{\cos\xi(\omega)t}{2\pi g\big(\xi(\omega)\big)}\,, \qquad F(\omega, t) = \frac{1-\cos\xi(\omega)t}{2\pi\xi^2(\omega)g\big(\xi(\omega)\big)}\,.$$

Obviously, $H(\omega, t)$ and $F(\omega, t)$ are continuous processes and $d^2F(\omega, t)/dt^2 = H(\omega, t)$. Moreover, $\mathcal{E}|F(\omega, t)| = \frac{1}{2}|t|$.

Let $A = \{\omega\colon \xi(\omega) > 0\}$ and let $\mathcal{F}$ be the smallest $\sigma$-field containing $A$. Then it is easy to verify that

$$\mathcal{E}\big(F(\omega, t)|\mathcal{F}\big) = \tfrac{1}{2}|t|\,a(\omega)\,,$$

where

$$a(\omega) = \begin{cases} \dfrac{1}{2P(A)} & \text{if} \quad \omega\in A\,, \\ \dfrac{1}{2P(\Omega-A)} & \text{if} \quad \omega\in\Omega-A\,. \end{cases}$$

Consequently, $E\big(H(\omega, t)|\mathcal{F}\big) = a(\omega)\delta(t)$.

**III. Invariant $\sigma$-fields.** Let $\Phi(\omega, t)$ be a strictly stationary generalized process. Let $F(\omega, t)$ be a continuous process such that

(10) $$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t)\,.$$

Then for any $h$ the process $\Delta_h^{(k)}F(\omega, t)$ is strictly stationary [1]). By $\mathcal{F}_h^{(k)}$ we shall denote the $\sigma$-field of invariant $\omega$ sets induced by the process $\Delta_h^{(k)}F(\omega, t)$ (cf. [1], XI, § 1). It is easy to see that $\mathcal{F}_h^{(k)}$ does not depend upon the chice of a continuous process satisfying equality (10). Let $H(\omega, t)$ be a continuous process such that $d^{k+1}H(\omega, t)/dt^{k+1} = \Phi(\omega, t)$. Since

$$\Delta_h^{(k+1)}H(\omega, t) = \int_t^{t+h} \Delta_h^{(k)}F(\omega, u)\,du,$$

we have

$$\mathcal{F}_h^{(k+1)} \subset \mathcal{F}_h^{(k)}. \tag{11}$$

Put

$$\mathcal{F}_\Phi = \bigcap_{k=k_0}^{\infty} \bigcap_{0<h<1} \mathcal{F}_h^{(k)}, \tag{12}$$

where $k_0$ denotes the first integer $k$ for which there exists a continuous process $F(\omega, t)$ satisfying equality (10). $\mathcal{F}_\Phi$ is called *the invariant $\sigma$-field induced by $\Phi(\omega, t)$*.

A strictly stationary generalized process $\Phi(\omega, t)$ is *indecomposable* if all sets belonging to $\mathcal{F}_\Phi$ have probability 0 or 1.

The following theorem is a version of the zero-one law for generalized processes:

THEOREM 2. *Strictly stationary generalized processes with independent values* [2]) *are indecomposable.*

Proof. Let $\Phi(\omega, t)$ be a strictly stationary generalized process with independent values. There are then a continuous process $F(\omega, t)$ and an integer $k$ such that $d^kF(\omega, t)/dt^k = \Phi(\omega, t)$ and, for any $h$, $\Delta_h^{(k)}F(\omega, t)$ is strictly stationary. Moreover, for any $h > 0$, $\Delta_h^{(k)}F(\omega, t)$ has $kh$-independent values (see [2], II. 4), i. e. for every $t_1, t_2, \ldots, t_m$; $u_1, u_2, \ldots, u_m$ satisfying the inequality $|t_i - u_j| > kh$ $(i, j = 1, 2, \ldots, m)$ the random vectors

$$\langle \Delta_h^{(k)}F(\omega, t_1), \Delta_h^{(k)}F(\omega, t_2), \ldots, \Delta_h^{(k)}F(\omega, t_m)\rangle,$$
$$\langle \Delta_h^{(k)}F(\omega, u_1), \Delta_h^{(k)}F(\omega, u_2), \ldots, \Delta_h^{(k)}F(\omega, u_m)\rangle$$

are mutually independent. To prove our assertion it suffices to show that all sets belonging to $\mathcal{F}_h^{(k)}$ $(0 < h < 1)$ have probability 0 or 1. The

[1]) $\Delta_h^{(1)}f(t) = \Delta_h f(t)$, $\Delta_h^{k+1}f(t) = \Delta_h \Delta_h^{(k)}f(t)$.

[2]) Generalized processes with independent values are defined in [2].



proof of the last statement is similar to that of the stationary processes with independent increments. In fact we can immediately deduce that for every $A \in \mathcal{F}_h^{(k)}$ there is a set $A_0$ such that $P(A - A_0) + P(A_0 - A) = 0$ and for each $T$ the set $A_0$ belongs to the $\sigma$-field spanned by the sets of the form

$$S(t, x) = \{\omega\colon \Delta_h^{(k)}F(\omega, t) < x\} \qquad (t \geqslant T, \ -\infty < x < \infty).$$

Moreover, there are a sequence of sets $B_1, B_2, \ldots$ and a sequence of real numbers $t_1, t_2, \ldots$ such that

$$\lim_{n\to\infty} \big(P(A_0 - B_n) + P(B_n - A_0)\big) = 0$$

and $B_n$ belongs to the $\sigma$-field spanned by sets $S(t_1, x), S(t_2, x), \ldots, S(t_n, x)$ $(-\infty < x < \infty)$. Let $T - t_j > kh$ $(j = 1, 2, \ldots, n)$. Then, according to the $kh$-independence of the values of $\Delta_h^{(k)}F(\omega, t)$, we obtain $P(A_0 \cap B_n) = P(A_0)P(B_n)$. Hence, when $n \to \infty$, $P(A_0) = \big(P(A_0)\big)^2$, which implies $P(A_0) = 0$ or 1. Consequently, $P(A) = 0$ or 1. The theorem is thus proved.

**IV. Lemmas.** In the sequel we shall use the following

LEMMA 2. *Let $f_T(\omega, t)$ be a family of continuous stochastic processes and*

$$\Phi_T(\omega, t) = \frac{d^r}{dt^r} f_T(\omega, t). \tag{13}$$

*If $\Phi_T(\omega, t_1)$ converges in $\Xi(t_1, t_2, \ldots, t_k)$ when $T \to \infty$, then $\Phi_T(\omega, t)$ converges in $\Xi(t)$* [3]).

Proof. From (13) and from the convergence of $\Phi_T(\omega, t_1)$ in $\Xi(t_1, \ldots, t_k)$ it follows that there are continuous processes $F_T(\omega, t)$, $H_T(\omega, t_1, \ldots, t_k)$ and an integer $s$ such that

$$\frac{d^s}{dt^s} F_T(\omega, t) = \Phi_T(\omega, t), \tag{14}$$

$$\frac{\partial^{ks}}{\partial t_1^s \ldots \partial t_k^s} H_T(\omega, t_1, \ldots, t_k) = \Phi_T(\omega, t_1) \tag{15}$$

and $H_T(\omega, t_1, \ldots, t_k)$ converges when $T \to \infty$ for almost all $\omega$ uniformly in every compact. Let $x_1, x_2, \ldots, x_s$ be real numbers for which $x_i \neq x_j$ if $i \neq j$. Put

$$V(t) = \frac{1}{s!}\prod_{j=1}^{s}(t - x_j), \qquad G_T(\omega, t_1, \ldots, t_k) = F_T(\omega, t_1)\prod_{r=2}^{k} V(t_r). \tag{16}$$

[3]) The assumption that $\Phi_T(\omega, t)$ are derivatives of the same order of continuous processes can be omitted.

Obviously, $G_T(\omega, t_1, \ldots, t_k)$ are continuous processes,

$$\frac{\partial^{ks}}{\partial t_1^s \ldots \partial t_k^s} G_T(\omega, t_1, \ldots, t_k) = \Phi_T(\omega, t_1)$$

and

$$(17)\quad G_T(\omega, t_1, \ldots, t_k) = 0 \quad \text{if} \quad t_r = x_j \qquad (r = 2, 3, \ldots, k;\ j = 1, 2, \ldots, s).$$

Hence and from (15) we obtain the equality

$$(18)\quad G_T(\omega, t_1, \ldots, t_k) = H_T(\omega, t_1, \ldots, t_k) + \sum_{r=1}^{k} \sum_{i=0}^{s-1} A_{ri}^T(\omega, t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k) t_r^i.$$

Putting in the last equality $t_n = x_j$ ($j = 1, 2, \ldots, s$; $n = 2, 3, \ldots, k$) and taking into account equality (17) we obtain the linear equations for the functions $A_{ni}^T(\omega, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_k)$ ($i = 0, 1, \ldots, s-1$; $n = 2, 3, \ldots, k$). Hence it follows that the function $A_{ni}^T(\omega, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_k)$ ($i = 0, 1, \ldots, s-1$; $n = 2, 3, \ldots, k$) is a line r combination of the functions $H_T(\omega, t_1, \ldots, t_{n-1}, x_j, t_{n+1}, \ldots, t_k)$ ($j = 1, 2, \ldots, s$), $A_{rm}^T(\omega, t_1, \ldots, t_{n-1}, x_j, t_{n+1}, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k) t_r^m$ ($j = 1, 2, \ldots, s$; $r \neq n$, $r = 1, 2, \ldots, k$; $m = 0, 1, \ldots, s-1$). Consequently, taking into account formula (18) and the convergence of $H_T(\omega, t_1, \ldots, t_k)$, we obtain the following equality:

$$G_T(\omega, t_1, \ldots, t_k) = H_{T1}(\omega, t_1, \ldots, t_k) + \sum_{j=0}^{s-1} D_j^T(\omega, t_2, t_3, \ldots, t_k) t_1^j + \sum_{2 \leqslant r < m \leqslant k} \sum_{i,j=0}^{s-1} C_{rmij}^T(\omega, t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_{m-1}, t_{m+1}, \ldots, t_k) t_r^i t_m^j,$$

where $H_{T1}(\omega, t_1, \ldots, t_k)$ converges when $T \to \infty$ for almost every $\omega$ uniformly in every compact. Putting in the last equality $t_n = x_i$, $t_l = x_j$ ($n, l = 2, 3, \ldots, k$; $i, j = 1, 2, \ldots, s$) we obtain the linear equations for the functions $C_{nlij}^T(\omega, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k)$, which implies that $C_{nlij}^T(\omega, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{l-1}, t_{l+1}, \ldots, t_k)$ is a linear combination of the functions $H_{T1}(\omega, t_1, \ldots, t_{n-1}, x_p, t_{n+1}, \ldots, t_{l-1}, x_q, t_{l+1}, \ldots, t_k)$, $C_{rmij}^T(\omega, t_1, \ldots, t_k) t_r^i t_m^j$, $D_j^T(\omega, t_2, \ldots, t_k) t_1^j$ with $t_n = x_p$, $t_l = x_q$, ($\langle r, m\rangle \neq \langle n, l\rangle$, $p, q = 1, 2, \ldots, s$; $i, j = 0, 1, \ldots, s-1$).

By iterating this procedure we finally obtain the equality

$$(19)\quad G_T(\omega, t_1, \ldots, t_k) = H_{T,k-1}(\omega, t_1, \ldots, t_k) + \sum_{j=0}^{s-1} b_j^T(\omega, t_2, \ldots, t_k) t_1^j + \sum_{0 \leqslant i_1, i_2, \ldots, i_k \leqslant s-1} a_{i_1, i_2, \ldots, i_k}^T(\omega) t_1^{i_1} t_2^{i_2} \ldots t_k^{i_k},$$



where $H_{T,k-1}(\omega, t_1, \ldots, t_k)$ converges when $T \to \infty$ for almost all $\omega$ uniformly in every compact. Let $y_2, y_3, \ldots, y_k$ be a system of real numbers such that $\prod_{r=2}^{k} V(y_r) = 1$. Then, in view of (16), $G_T(\omega, t, y_2, y_3, \ldots, y_k) = F_T(\omega, t)$. Put

$$W_T(\omega, t) = -\sum_{j=0}^{s-1} b_j^T(\omega, y_2, \ldots, y_k) t_j - \sum_{0 \leqslant i_1, \ldots, i_k \leqslant s-1} a_{i_1, \ldots, i_k}^T(\omega) t^{i_1} y_2^{i_2} \ldots y_k^{i_k}.$$

Then

$$(20)\qquad \frac{d^s}{dt^s} W_T(\omega, t) = 0$$

and, in view of (19), $F_T(\omega, t) + W_T(\omega, t)$ converges when $T \to \infty$ for almost all $\omega$ uniformly in every finite interval. Consequently, according to (14) and (20), $\Phi_T(\omega, t) = d^s(F_T(\omega, t) + W_T(\omega, t))/dt^s$ converges in $\Xi(t)$. The lemma is thus proved.

LEMMA 3. *Let $\Phi(\omega, t)$ be a strictly stationary generalized process for which $E(\Phi(\omega, t))$ exists. There is then an integer $k_\Phi$ such that if a continuous process $F(\omega, t)$ satisfies the equality*

$$(21)\qquad \frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t) \qquad (k \geqslant k_\Phi),$$

*then for every $t_0$ the continuous process*

$$(22)\qquad F_{t_0}^*(\omega, t) = \max_{\substack{|h_i| \leqslant t_0 \\ i=1,2,\ldots,k}} |\Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_k} F(\omega, t)|$$

*is strictly stationary and $\mathcal{E} F_{t_0}^*(\omega, t) < \infty$.*

Proof. Let $\Phi(\omega, t) = [f_n(\omega, t)]$, where processes $f_n(\omega, t)$ ($n = 1, 2, \ldots$) are strictly stationary. Moreover, there are continuous processes $H(\omega, t), H_1(\omega, t), \ldots$ and an integer $k_0$ such that

$$(23)\qquad \frac{d^{k_0}}{dt^{k_0}} H_n(\omega, t) = f_n(\omega, t) \qquad (n = 1, 2, \ldots),$$

$$(24)\qquad \frac{d^{k_0}}{dt^{k_0}} H(\omega, t) = \Phi(\omega, t)$$

and $H_1(\omega, t), H_2(\omega, t), \ldots$ converges to $H(\omega, t)$ for almost all $\omega$ uniformly in every finite interval:

$$(25)\qquad H_n(\omega, t) \rightrightarrows H(\omega, t).$$

Assume that equality (21) is satisfied and $k > k_0$. Put

$$F_n(\omega, t) = \frac{1}{(k-k_0-1)!}\int_0^t (t-u)^{k-k_0-1} H_n(\omega, u)\,du \qquad (n = 1, 2, \ldots),$$

$$F_\infty(\omega, t) = \frac{1}{(k-k_0-1)!}\int_0^t (t-u)^{k-k_0-1} H(\omega, u)\,du .$$

Then, according to (23), (24) and (25),

(26) $$\frac{d^k}{dt^k} F_n(\omega, t) = f_n(\omega, t) \qquad (n = 1, 2, \ldots),$$

(27) $$\frac{d^k}{dt^k} F_\infty(\omega, t) = \Phi(\omega, t)$$

and

(28) $$F_n(\omega, t) \rightrightarrows F_\infty(\omega, t) .$$

Further, according to (26), we obtain

$$\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k} F_n(\omega, t) = \int_t^{t+h_k}\int_{x_{k-1}}^{x_{k-1}+h_{k-1}}\ldots\int_{x_1}^{x_1+h_1} f_n(\omega, u)\,du\,dx_1\ldots dx_{k-1} .$$

From this equality it immediately follows that the process

$$F^*_{n,t_0}(\omega, t) = \max_{|h_i|\leqslant t_0} |\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k} F_n(\omega, t)| \qquad (i = 1, 2, \ldots, k)$$

is strictly stationary. Since, according to (21), (22), (27) and (28),

$$F^*_{n,t_0}(\omega, t) \rightrightarrows F^*_{t_0}(\omega, t) \quad \text{when} \quad n\to\infty ,$$

the process $F^*_{t_0}(\omega, t)$ is also strictly stationary.

From the assumption of Lemma it follows that there are a continuous process $G(\omega, t)$ and an integer $k_1$ such that the expectation $\mathcal{E}|G(\omega, t)|$ is bounded in every finite interval and

$$\frac{d^{k_1}}{dt^{k_1}} G(\omega, t) = \Phi(\omega, t), \qquad \frac{d^{k_1}}{dt^{k_1}} \mathcal{E}G(\omega, t) = E\big(\Phi(\omega, t)\big) .$$

Let $k > k_1$. Then the equality

$$\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k} F(\omega, t) = \Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k}\frac{1}{(k-k_1-1)!}\int_0^t (t-u)^{k-k_1-1} G(\omega, u)\,du$$



is true. Consequently, for any $|h_i| \leqslant t_0$ $(i = 1, 2, \ldots, k)$

$$|\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_k} F(\omega, t)| \leqslant 2^k \int_{-|t|-kt_0}^{|t|+kt_0} (|t|+kt_0-u)^{k-k_1-1}|G(\omega, u)|\,du .$$

Hence, according to (22),

$$\mathcal{E}F^*_{t_0}(\omega, t) \leqslant 2^k \int_{-|t|-kt_0}^{|t|+kt_0} (|t|+kt_0-u)^{k-k_1-1}\mathcal{E}|G(\omega, u)|\,du .$$

Putting $k_\Phi = \max(k_0+1, k_1+1)$ we obtain the assertion of the lemma.

**V. Ergodic theorem.** For every generalized process $\Phi(\omega, t)$ and constants $A$, $B$ we define the integral $\int_{t+A}^{t+B} \Phi(\omega,u)\,du$ by the following formula:

$$\int_{t+A}^{t+B} \Phi(\omega, u)\,du = \Psi(\omega, t+B) - \Psi(\omega, t+A) ,$$

where $\Psi(\omega, t)$ is a generalized process satisfying the equality $d\Psi(\omega, \ )/dt = \Phi(\omega, t)$. Obviously, $\int_{t+A}^{t+B}\Phi(\omega, u)\,du$ is also a generalized stochastic process.

THEOREM 3. *Let $\Phi(\omega, t)$ be a strictly stationary generalized stochastic process for which $E\big(\Phi(\omega, t)\big)$ exists. Then*

(29) $$\frac{1}{T}\int_t^{t+T} \Phi(\omega, u)\,du \to E\big(\Phi(\omega, t)|\mathcal{F}_\Phi\big)$$

*when $T\to\infty$. The conditional expectation $E\big(\Phi(\omega, t)|\mathcal{F}_\Phi\big)$ is a random variable independent of $t$.*

*In particular, if the process $\Phi(\omega, t)$ is indecomposable, the right-hand side of (29) can be replaced by the constant $E\big(\Phi(\omega, t)\big)$.*

Proof. First we shall prove that

$$\frac{1}{T}\int_t^{t+T} \Phi(\omega, u)\,du$$

converges when $T\to\infty$. Let $k \geqslant k_\Phi$, where $k_\Phi$ is determined by Lemma 3. There is then a continuous process $F(\omega, t)$ such that

(30) $$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t) .$$

Consequently, for any $t_1, t_2, \ldots, t_k$ the process $\Delta_{t_1}\Delta_{t_2}\ldots\Delta_{t_k}F(\omega, t)$ is strictly stationary and, in view of Lemma 3, $\mathcal{E}|\Delta_{t_1}\Delta_{t_2}\ldots\Delta_{t_k}F(\omega, t)| < \infty$. Put

$$(31)\qquad \Gamma_T(\omega, t_1, t_2, \ldots, t_k) = \frac{1}{T}\int\limits_0^T \Delta_{t_1}\Delta_{t_2}\ldots\Delta_{t_k}F(\omega, u)\,du\,.$$

Obviously, $\Gamma_T(\omega, t_1, \ldots, t_k)$ is a continuous process of variables $t_1, t_2, \ldots, t_k$. Using Birkhoff's ergodic theorem (cf. [1], XI, § 2), we infer that for fixed $t_1, t_2, \ldots, t_k$ the limit

$$(32)\qquad \Gamma(\omega, t_1, t_2, \ldots, t_k) = \lim_{T\to\infty}\Gamma_T(\omega, t_1, t_2, \ldots, t_k)$$

exists almost everywhere. Now we shall prove that $\Gamma_T(\omega, t_1, \ldots, t_k)$ converges to $\Gamma(\omega, t_1, \ldots, t_k)$ in the sense of the convergence in $\Xi(t_1, t_2, \ldots, t_k)$. From equalities (22) and (31) it follows that

$$(33)\qquad \max_{\substack{|t_i|\leqslant t_0\\ i=1,2,\ldots,k}} |\Gamma_T(\omega, t_1, \ldots, t_k)| \leqslant \frac{1}{T}\int\limits_0^T F^*_{t_0}(\omega, u)\,du \qquad (T \geqslant 0).$$

From Lemma 3, using Birkhoff's ergodic theorem, we infer that for each $t_0$ the limit

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T F^*_{t_0}(\omega, u)\,du$$

exists and is finite almost everywhere. Consequently, there is a random variable $M_{t_0}(\omega)$, such that

$$\sup_{T\geqslant 0}\frac{1}{T}\int\limits_0^T F^*_{t_0}(\omega, u)\,du \leqslant M_{t_0}(\omega) < \infty$$

almost everywhere. Hence and from (32) and (33) we obtain the convergence

$$\lim_{T\to\infty}\int\limits_{-t_0}^{t_0}\int\limits_{-t_0}^{t_0}\ldots\int\limits_{-t_0}^{t_0}|\Gamma_T(\omega, u_1, \ldots, u_k) - \Gamma(\omega, u_1, \ldots, u_k)|\,du_1du_2\ldots du_k = 0$$

almost everywhere. This implies the convergence

$$\lim_{T\to\infty}\int\limits_0^{t_1}\int\limits_0^{t_2}\ldots\int\limits_0^{t_k}\Gamma_T(\omega, u_1, \ldots, u_k)\,du_1du_2\ldots du_k$$
$$= \int\limits_0^{t_1}\int\limits_0^{t_2}\ldots\int\limits_0^{t_k}\Gamma(\omega, u_1, \ldots, u_k)\,du_1du_2\ldots du_k$$



for almost all $\omega$ uniformly in every compact. Hence by differentiation $\partial^k/\partial t_1\partial t_2\ldots\partial t_k$ we obtain the convergence

$$(34)\qquad \Gamma_T(\omega, t_1, t_2, \ldots, t_k) \to \Gamma(\omega, t_1, t_2, \ldots, t_k) \quad \text{in} \quad \Xi(t_1, t_2, \ldots, t_k).$$

Further, in virtue of (30) and (31), we have

$$\frac{\partial^k}{\partial t_1\partial t_2\ldots\partial t_k}\Gamma_T(\omega, t_1, t_2, \ldots, t_k) = \frac{1}{T}\int\limits_{t_1+t_2+\ldots+t_k}^{t_1+t_2+\ldots+t_k+T}\Phi(\omega, u)\,du,$$

which, in view of (34), implies the convergence of

$$\frac{1}{T}\int\limits_{t_1+t_2+\ldots+t_k}^{t_1+t_2+\ldots+t_k+T}\Phi(\omega, u)\,du$$

in $\Xi(t_1, t_2, \ldots, t_k)$ when $T\to\infty$. Hence, according to Lemma 1,

$$\frac{1}{T}\int\limits_{t_1}^{t_1+T}\Phi(\omega, u)\,du \quad \text{converges in } \Xi(t_1, t_2, \ldots, t_k).$$

Since, according to (30),

$$\frac{d^k}{dt^k}\frac{1}{T}\int\limits_t^{t+T}F(\omega, u)\,du = \frac{1}{T}\int\limits_t^{t+T}\Phi(\omega, u)\,du,$$

and the processes $\frac{1}{T}\int\limits_t^{t+T}F(\omega, u)\,du$ are continuous, there exists, in virtue of Lemma 2, a generalized process $\Psi_0(\omega, t)$ such that

$$(35)\qquad \frac{1}{T}\int\limits_t^{t+T}\Phi(\omega, u)\,du \to \Psi_0(\omega, t) \quad (\text{in } \Xi(t))$$

when $T\to\infty$.

Now we shall prove the equality

$$(36)\qquad \Psi_0(\omega, t) = E\big(\Phi(\omega, t)|\mathcal{F}_\Phi\big).$$

From formula (35) it follows that there are continuous processes $G_T(\omega, t)$, $G(\omega, t)$ and an integer $s$ such that

$$(37)\qquad G_T(\omega, t) \rightrightarrows G(\omega, t),$$

$$(38)\qquad \frac{d^s}{dt^s}G_T(\omega, t) = \frac{1}{T}\int\limits_t^{t+T}\Phi(\omega, u)\,du,$$

$$(39)\qquad \frac{d^s}{dt^s}G(\omega, t) = \Psi_0(\omega, t).$$

Without loss of generality, we may assume that $s$ is an arbitrary sufficiently great integer and there is a continuous process $F(\omega, t)$, with locally integrable expectation $\mathcal{E}|F(\omega, t)|$, satisfying the equality

$$\frac{d^s}{dt^s}F(\omega, t) = \Phi(\omega, t). \tag{40}$$

Since for each $h$ the process $\Delta_h^{(s)}F(\omega, t)$ is strictly stationary and $\mathcal{E}|\Delta_h^{(s)}F(\omega, t)| < \infty$, therefore, according to Birkhoff's ergodic theorem, for any $t$ and $h$ the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\Delta_h^{(s)}F(\omega, u)\,du = \mathcal{E}(\Delta_h^{(s)}F(\omega, t)|\mathcal{F}_h^{(s)}) \tag{41}$$

exists almost everywhere and is a continuous version of the conditional expectation of $\Delta_h^{(s)}F(\omega, t)$ relative to $\mathcal{F}_h^{(s)}$, being independent of $t$. Further, in view of (38) and (40),

$$\Delta_h^{(s)}G_T(\omega, t) = \frac{1}{T}\int_t^{t+T}\Delta_h^{(s)}F(\omega, u)\,du.$$

Consequently, according to (37) and (41), for each $h$

$$\Delta_h^{(s)}G(\omega, t) = \mathcal{E}(\Delta_h^{(s)}F(\omega, t)|\mathcal{F}_h^{(s)}). \tag{42}$$

Since the right-hand side of the last equality is independent of $t$, we have

$$G(\omega, t) = \frac{a(\omega)}{s!}t^s + \sum_{j=0}^{s-1}a_j(\omega)t^j, \tag{43}$$

where $a(\omega), a_0(\omega), \ldots, a_{s-1}(\omega)$ are random variables. This implies, according to (39),

$$\Psi_0(\omega, t) = a(\omega). \tag{44}$$

From (42) and (43) it follows that

$$a(\omega) = \frac{1}{h^s}\mathcal{E}(\Delta_h^{(s)}F(\omega, t)|\mathcal{F}_h^{(s)}), \tag{45}$$

which implies that $a(\omega)$ is measurable with respect to all the $\sigma$-fields $\mathcal{F}_h^{(s)}$ $(0 < h < 1;\ s \geqslant s_0)$, where $s_0$ denotes the smallest integer for which relations (37), (38) and (39) are true. Consequently, taking into account formulas (11) and (12), we infer that $a(\omega)$ is measurable with respect to $\mathcal{F}_\Phi = \bigcap_{s=s_0}^{\infty}\bigcap_{0<h<1}\mathcal{F}_h^{(s)}$. Hence, according to equality (45) and property (h) of conditional expectations (p. 271), we obtain



$$a(\omega) = \frac{1}{h^s}\mathcal{E}(\Delta_h^{(s)}F(\omega, t)|\mathcal{F}_\Phi). \tag{46}$$

Further, according to theorem 1, we may assume without loss of generality that there is a continuous version of the conditional expectation $\mathcal{E}(F(\omega, t)|\mathcal{F}_\Phi)$ and, according to (40),

$$\frac{d^s}{dt^s}\mathcal{E}(F(\omega, t)|\mathcal{F}_\Phi) = E(\Phi(\omega, t)|\mathcal{F}_\Phi).$$

Consequently,

$$\frac{1}{h^s}\Delta_h^{(s)}\mathcal{E}(F(\omega, t)|\mathcal{F}_\Phi) = \frac{1}{h^s}\mathcal{E}(\Delta_h^{(s)}E(\omega, t)|\mathcal{F}_\Phi) \to E(\Phi(\omega, t)|\mathcal{F}_\Phi)$$

when $h \to 0$ (cf. [2], § I.6). Hence, in view of (46), $a(\omega) = E(\Phi(\omega, t)|\mathcal{F}_\Phi)$, which, according to (44), implies equality (36). Convergence (29) is thus proved.

For indecomposable generalized processes the assertion of the theorem is a direct consequence of property (e) (p. 270) of conditional expectations.

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