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Ergodic projections for semi-groups of periodic operators

by

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1. Introduction. In 1938 K. Yosida [11] (cf. also [12]) proved the following mean ergodic theorem:

THEOREM 1. *Let T be a bounded linear operator on a Banach space $\mathfrak{X}$ to itself; and let*

$$S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x, \qquad x \in \mathfrak{X}.$$

If the iterates T^n are equibounded, i. e. $\|T^n\| \leqslant M$ for $n = 1, 2, \ldots$, and all sequences $\{S_n x\}$ are weakly compact, then $S_n x$ converges strongly to Px, where P is a linear bounded operator on $\mathfrak{X}$ to itself with $\|P\| \leqslant M$, and $PT = TP = P$, $P^2 = P$.

In this note we consider the mean ergodic theorem for semi-groups of periodic operators; our main interest being in the representation of the ergodic projection operator and the associated manifold of fixed points of the semi-group. This study was suggested, in part, by our investigation of the Kolmogorov differential equations with periodic coefficients and the associated semi-groups of operators on the Banach space l to itself [1]. In section 4 of this note we discuss the application of the mean ergodic theorem for periodic operators to Markov chains and processes with a denumerable state space.

DEFINITION 1. In the discrete parameter case, a linear operator T on a Banach space to itself is said to be *periodic with period* ω (ω finite) if $T^{\omega} = I$ (*the identity operator*), and ω is the smallest positive integer with this property.

DEFINITION 2. In the continuous parameter case, a linear operator T on a Banach space to itself is said to be *periodic with period* ω if $T(\omega) = I$, and ω is the smallest positive real number with this property.

2. Discrete parameter case. Let $\sigma = \{T^n, n = 0, 1, \ldots\}$ be a semi-group of linear bounded periodic operators on a Banach space $\mathfrak{X}$ to itself.

If the period of σ is ω $(\omega = 1, 2, \ldots)$, we have from Definition 1 and the semi-group property

$$T^{n+\omega}x = T^n T^\omega x = T^n I x = T^n x, \quad x \in \mathfrak{X}.$$

THEOREM 2. *Let $\sigma = \{T^n, n = 0, 1, \ldots\}$ be a semi-group of linear bounded periodic operators on a Banach space to itself such that*

(i) $T^\omega = I$, $T^{n+\omega} = T^n$,

(ii) *the iterates are equibounded,*

(iii) *the sequences $\{S_n x\}$ are weakly compact;*

then $S_n x$ converges strongly to $P_\omega x$, where

$$P_\omega x = \frac{1}{\omega}\sum_{i=0}^{\omega-1} T^i x,$$

with $\|P_\omega\| < M$, and $P_\omega T = TP_\omega = P_\omega$, $P_\omega^2 = P_\omega$.

Proof. The existence of a limiting operator which is a projection operator follows from Theorem 1. To see that the projection operator has the above representation, and depends on ω, we observe that as n increases, for ω fixed, the sequence

$$\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right\}$$

assumes the form

$$\left\{\frac{K}{k\omega+1}\sum_{i=0}^{\omega-1} T^i x\right\};$$

hence

$$\lim_{n\to\infty} S_n x = P_\omega x = \frac{1}{\omega}\sum_{i=0}^{\omega-1} T^i x.$$

That P_ω is a projection operator with $\|P_\omega\| < M$ is easily verified.

It is well-known that an ergodic projection P is a projection of $\mathfrak{X}$ onto the linear manifold M of the fixed points of the operator T, and that the operator $I-P$ projects $\mathfrak{X}$ onto the linear manifold N of those elements of $\mathfrak{X}$ which are annihilated by T. Hence, the operator P_ω is a projection of $\mathfrak{X}$ onto the linear manifold

$$M_\omega = \{x \in \mathfrak{X} | Tx = x,\ T \text{ periodic with period } \omega\},$$

and the operator

$$Q_\omega = I - P_\omega = \frac{1}{\omega}\sum_{i=1}^{\omega-1} T^i$$

is a projection of $\mathfrak{X}$ onto the linear manifold

$$N_\omega = \{x \in \mathfrak{X} | Tx = 0,\ T \text{ periodic with period } \omega\}.$$

For applications it would be of great interest to obtain limit theorems for sequences of projection operators $\{P_\omega\}$ analogous to those obtained in the non-periodic case (cf. [10]). Since if we change the period ω we are dealing with an entirely different operator; it is not possible to establish in the general case the monotonicity properties of the manifolds M_ω, $\omega = 1, 2, \ldots$, and hence establish the monotonicity of the projection operators P_ω. However, it has been pointed out by Professor Herman Rubin that the following result obtains:

Let

$$F_\omega = \{M\colon \text{ for some operator } T \text{ with period } \omega,\ M = \{x \in \mathfrak{X}\colon Tx = x\}\};$$

then $F_\omega \subset F_{k\omega}$, $k = 1, 2, \ldots$

A result which might be of interest in certain applications is the following:

THEOREM 3. *For ω fixed, $M_T = M_\omega$, where M_T and M_ω are the manifolds of fixed points of the operators T and P_ω, respectively.*

Proof. Consider $x \in M_\omega$. Then

$$P_\omega x = \frac{1}{\omega}(I + T + \ldots + T^{\omega-1})x = x.$$

Operating on the above with T, we have

$$TP_\omega x = \frac{1}{\omega}(T + T^a + \ldots + T^\omega)x = \frac{1}{\omega}(I + T + \ldots + T^{\omega-1})x = P_\omega x = Tx.$$

Therefore $P_\omega x = Tx = x$, and $F_T = F_\omega$.

3. Continuous parameter case. Let $\sigma = \{T(t), t > 0\}$ be a semi-group of linear bounded periodic operators on a Banach space $\mathfrak{X}$ to itself. If the period of σ is ω, $\omega > 0$, we have from Definition 2 and the semi-group property

$$T(t+\omega)x = T(t)T(\omega)x = T(t)Ix = T(t)x, \quad x \in \mathfrak{X}.$$

We assume $T(t)$ to be continuous in the strong operator topology, and that the semi-group σ is of class C_0, that is

$$\lim_{t\downarrow 0} \|T(t)x - x\| = 0$$

for each $x \in \mathfrak{X}$.

THEOREM 4. *Let $\sigma = \{T(t), t > 0\}$ be a semi-group of linear bounded periodic operators on $\mathfrak{X}$ to itself, with $T(\omega) = I$, $T(t+\omega) = T(t)$, where $\omega > 0$ is fixed. Let*

$$A(\tau)x = \begin{cases} \frac{1}{\tau}\int_0^\tau T(t)x\,dt, & \tau > 0, \\ x, \quad \tau = 0, & x \in \mathfrak{X}, \end{cases}$$

denote the average on the interval $[0, \tau]$ of the semi-group σ which is assumed to be strongly integrable on every finite interval. Assume also that

(i) $\lim\limits_{t\downarrow 0} \frac{T(t)}{t}x = 0$, $x \in \mathfrak{X}$,

(ii) $|A(\tau)| < M$, $\tau \geqslant 0$,

(iii) *for each $x \in S$, where S is a fundamental set in $\mathfrak{X}$, the set $\{A(\tau)x, \tau > 0\}$ is weakly sequentially compact.*

Then the averages $A(\tau)$ converge strongly to the integral operator

$$P(\omega)x = \frac{1}{\omega}\int_0^\omega T(t)x\,dt,$$

with $|P(\omega)| < M$, and $P(\omega)T(t) = T(t)P(\omega) = P(\omega)$, $P^2(\omega) = P(\omega)$.

Proof. We omit the proof since Theorem 4 can be reduced to the discrete parameter case by applying Theorem 2 to the operator $T(1)$ on the space $\overline{A(1)\mathfrak{X}}$ (cf. [2], [3], [6]).

4. Periodic Markov chains and processes[1]**.** Let $T = (t_{ij})$, $i, j = 0, 1, \ldots$, be the matrix of transition probabilities associated with a temporally homogeneous Markov chain with a denumerable state space. The probability of a transition from the state i to the state j in n steps (or time n) is given by the element, $t_{ij}^{(n)}$, in the i-th row and j-th column of T^n. Hence the specification of T completely determines the system. A problem of great interest in both theoretical investigations and applications is the study of the limiting behaviour of the iterates T^n as $n \to \infty$. It is well-known, for example, that the Cesàro limits

$$p_{ij} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_{ij}^{(k)}$$

[1]) The notation used in this section follows that of the previous sections, and is not the usual notation employed in the theory of Markov chains and processes.

always exist, and satisfy the relations

$$p_{ij} \geqslant 0, \qquad \sum_{j=0}^{\infty} p_{ij} \leqslant 1.$$

Let $P = (p_{ij})$ denote the matrix of limit elements p_{ij}. The problem of interest can now be stated as follows: Determine the limit matrix P when the matrix of transition probabilities T is given.

The structure of the limit matrix P can be described by classifying a state j as *positive* when $p_{jj} > 0$ and *dissipative* when $p_{jj} = 0$. The set of all positive states, if not empty, can be divided into disjoint subsets, with states i and j being in the same class if and only if $p_{ij} > 0$. Let j be a positive state; then we can write $p_j = p_{jj}$, and the limit elements p_{ij} can be expressed in terms of the p_j and a set of numbers $a(i, C)$, defined for each state i and each positive class C, where $0 \leqslant a(i, C) \leqslant 1$. The $a(i, C)$ give the probability that the system starting at state i will ultimately enter the set of states C, and remain in C. The positive states are those states of the chain which are recurrent with finite mean recurrence time μ_j, given by $\mu_j = 1/p_j$; and the dissipative states are those states which are either recurrent with infinite mean recurrence time, or non-recurrent. The properties of the p_{ij}, p_j and $a(i, C)$ can be summarised as follows (cf. [9]):

(i) $p_{ij} = 0$ for all i, if j is dissipative.

(ii) $p_{ij} = p_j a(i, C)$ for all i, if $j \in C$.

(iii) $\sum\limits_{j\in C} p_j = 1$ for each positive class C.

(iv) $$a(i, C) = \begin{cases} 1, & i \in C, \\ 0, & i \notin C, \end{cases}$$

where i is a positive state and C is a positive class.

(v) If i is dissipative and $\{C^\gamma : \gamma = 1, 2, \ldots\}$ are positive classes, then $\sum\limits_{\gamma} a(i, C^\gamma) \leqslant 1$.

(vi) if C is a positive class, then

$$\sum_{k\in C} p_k t_{kj} = \begin{cases} p_j, & j \in C, \\ 0, & j \notin C, \end{cases}$$

and

$$\sum_{k=0}^{\infty} t_{ik}a(k, C) = a(i, C) \qquad \text{for all} \quad i.$$

(vii) $\sum\limits_{k=0}^{\infty} p_{ik}p_{kj} = \sum\limits_{k=0}^{\infty} p_{ik}t_{kj} = \sum\limits_{k=0}^{\infty} t_{ik}p_{kj} = p_{ij}$, for all i and j.

The problem of determining the limit matrix P has been considered by many investigators using different methods (cf. [5], [12]). In a recent paper D. G. Kendall and G. E. H. Reuter [9] have utilised the theory of semi-groups of operators to calculate the ergodic projection operator. In the semi-group theory of Markov chains with a denumerable state space, the Banach space $\mathfrak{X}$ involved is the sequence space l whose elements are sequences $x = (x_0, x_1, \ldots)$ with norm

$$\|x\| = \sum_{k=0}^{\infty} |x_k| < \infty .$$

In this case T and P determine bounded linear operators on l to itself as follows:

$$(Tx)_j = \sum_{k=0}^{\infty} x_k t_{kj}, \qquad (Px)_j = \sum_{k=0}^{\infty} x_k p_{kj}.$$

The operators T and P satisfy the following conditions:

(a) $Tx \geqslant 0$, $\|Tx\| = \|x\|$, $x \geqslant 0$;

(b) $Px \geqslant 0$, $\|Px\| \leqslant \|x\|$, $x \geqslant 0$, $PT = TP = P$, $P^2 = P$.

Hence T is a *transition operator* and P is an *idempotent contraction operator*. We also have that for each positive class C^γ we can define a vector $\psi^\gamma \epsilon\, l$ by

$$(p^\gamma)_j = \begin{cases} p_j, & j \epsilon C, \\ 0, & j \notin C. \end{cases}$$

Hence $p^\gamma \geqslant 0$, $\|p^\gamma\| = 1$, since $\sum_{j \epsilon C} p_j = 1$, and the positive class C^γ is the support of the vector p^γ. From property (vii) we see that $Tp^\gamma = p^\gamma$, so that p^γ is a fixed point of the transition operator T, and is therefore, an element of the linear manifold M.

In order to calculate the ergodic projection operator for periodic Markov chains[2]) it is first necessary to replace Definition 1 for general periodic operators, and consider the following

DEFINITION 3. A Markov chain characterised by a matrix of transition probabilities T is said to be *periodic with period* ω ($\omega > 0$) if $T^{n+\omega} = T^n$, and ω is the smallest positive integer with this property.

That Definition 1 is too strong for Markov chains follows from the fact that the global condition $T^\omega = I$ requires that the system be *deterministic*. Definition 3 does not require that $T^\omega = I$, it simply states that the matrix transition operator T is such that its iterates satisfy the product law $T^{n+\omega} = T^n$.

[2]) For some examples of periodic chains we refer to [5], p. 329-331.

In view of the above, the representation of the ergodic projection operator for general periodic operators

$$P_\omega = \frac{1}{\omega} \sum_{i=0}^{\omega-1} T^i$$

must be replaced by

$$P_\omega = \frac{1}{\omega} \sum_{i=1}^{\omega} T^i$$

in the case of Markov chains. Hence we see that for periodic Markov chains the associated ergodic projection operators can be easily obtained from the simple representation above. As in the case of aperiodic chains, the states of the periodic chain can be classified depending on whether the diagonal elements of P_ω are positive or zero; and for positive states j the mean recurrence times μ_j are given by $\mu_j = 1/p_j$. We close this discussion of periodic Markov chains by remarking that the method given by Kendall and Reuter for the calculation of the absorption probabilities $a(i, C)$ when i is a dissipative state and C a positive class can be carried out for periodic chains.

We now consider periodic Markov processes with denumerable state space[3]). Let $T(s) = (t_{ij}(s))$ $(i, j = 0, 1, \ldots)$, $s \geqslant 0$, denote the matrix of transition probabilities defining a temporally homogeneous Markov process with a denumerable state space. If the continuity condition

$$\lim_{s \downarrow 0} t_{ij}(s) = \delta_{ij}$$

holds, then it is well-known that the limits

$$p_{ij} = \lim_{s \to \infty} t_{ij}(s)$$

exist. Let $P = (p_{ij})$ denote the matrix of limit elements p_{ij}. Before proceeding we state a few of the properties of the semi-groups of operators associated with Markov processes with denumerable state spaces.

We define a semi-group of transition operators $\sigma = \{T(s), s \geqslant 0\}$ on the Banach space l to itself by putting

$$\big(T(s)x\big)_j = \sum_{k=0}^{\infty} x_k t_{kj}(s)$$

[3]) A periodic process of the birth-and-death type is discussed in [8].

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for each $x \in l$, where $x = (x_0, x_1, \ldots)$. The semi-group σ has the following properties:

(a) $T(s+t) = T(s)T(t)$, $s, t \geq 0$, $T(0) = I$.

(b) $T(s)x \geq 0$ and $\|T(s)x\| = \|x\|$, $x \geq 0$, $x \in l$.

(c) $\lim\limits_{s \downarrow 0} \|(T(s)-I)x\| = 0$, for each x.

The problem of determining the ergodic projection operator for Markov processes can be stated in the following ways: (1) Determine the limit matrix P when the matrix of transition probabilities $T(s)$ is given, or (2) Determine the limit matrix P when the infinitesimal generator [4]) Ω of the semi-group is given. The second method has been employed by Kendall and Reuter. However, in the case of periodic processes the infinitesimal generator is a periodic function of time, and as pointed out in [1] a periodic infinitesimal generator will not in general generate a semi-group with the same period. Hence in this study we assume that a semi-group of periodic operators has been generated by some operator $\Omega(s)$, and base our calculations on the matrix of transition probabilities.

As in the case of Markov chains, Definition 2 is too strong for Markov processes. Hence we have

DEFINITION 4. A Markov process characterised by a matrix of transition probabilities $T(s)$ is said to be *periodic with period* ω $(\omega > 0)$ if $T(s+\omega) = T(s)$, and ω is the smallest positive real number with this property [5]).

For periodic Markov processes the representation of the ergodic projection operator given in Theorem 4 can be used to obtain the limit matrix of probabilities. Hence, for periodic Markov processes

$$P(\omega) = \frac{1}{\omega}\int_0^\omega T(s)\,ds,$$

the limit matrix depending on the period ω. The classification of states can be obtained as in the case of Markov chains.

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[4]) The infinitesimal generator is defined by $\Omega x = \text{strong} \lim\limits_{s \downarrow 0} ((T(s)-I)x)/s$ whenever this limit exists.

[5]) K. Jacobs [7] has recently given the following definition: $T(s)$ *is periodic with period* ω if $T(s+\omega) = T(s)$, $s > 0$, with $\|T(s, 0)x\| \leq M\|x\|$, $x \in X$, where $T(u, s) = T(u)T(u-1)\ldots T(s+1)$, $u > s$, and $T(s, s) = I$, $M \geq 1$.

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