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STUDIA MATHEMATICA, T. XVII. (1958)

A generalization of results of R. C. James concerning absolute bases in Banach spaces

by

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The main results obtained by R. C. James in his paper [14][1]) can be presented as follows:

Let X be a Banach space with an absolute basis.

(a) X is weakly complete if and only if no subspace of X is isomorphic to c_0.

(b) Every bounded set in X is weakly conditionally compact[2]) if and only if no subspace of X is isomorphic to l.

The purpose of this paper is to show that the propositions (a) and (b) hold true also in the case where the space X can be imbedded in a space with an absolute basis.

The last part of this paper contains several questions which are connected with the present paper and with the one entitled *On bases and unconditional convergence of series in Banach spaces* (this fasc., p. 151-164) which is subsequently denoted by [0].

Terminology and notation used in this paper are the same as in [0].

1. In the following we shall need several lemmas.

1.1. *In order that an absolute basic sequence (x_n) with $\sup_n \|x_n\| < +\infty$ be equivalent to the unit-vector-basis of the space l it is necessary and sufficient that there exist a linear functional f such that $\inf_n |f(x_n)| > 0$.*

Sufficiency. 1° If $\sum_{n=1}^{\infty} |t_n| < +\infty$, then $\sum_{n=1}^{\infty} t_n x_n$ is convergent.

[1]) The numbers in brackets [ ] refer to the "References" of the paper *On bases and unconditional convergence of series in Banach spaces*; in this fasc., p. 163.

[2]) A set $Z \subset Y$ is said to be *weakly conditionally compact* if every bounded sequence composed of elements of Z contains a weakly convergent subsequence.

2° If the series $\sum_{i=1}^{\infty} t_i x_i$ is convergent, then, according to the criterion 1.4 of [0], the series $\sum_{i=1}^{\infty} \operatorname{sgn} f(t_i x_i)\cdot t_i x_i$ is convergent too. Thus

$$\sum_{i=1}^{\infty} |t_i|\,|f(x_i)| < +\infty .$$

Since $\inf |f(x_n)| > 0$, this is true only in the case $\sum_{i=1}^{\infty} |t_i| < +\infty$.

Necessity is trivial.

1.2. *Let Y be a subspace of a Banach space X with an absolute basis (x_n). If a sequence $(z_n) \subset Y$ converges weakly and $f_i(z_n) \to 0$ $(i = 1, 2, \ldots)$*[3]*, then $z_n \underset{w}{\to} 0$*[4].

1.21. *If no subspace of Y is isomorphic to l, then the conditions $\sup_n \|z_n\| < +\infty$, $f_i(z_n) \to 0$ $(i = 1, 2, \ldots)$ imply $z_n \underset{w}{\to} 0$.*

Suppose that (z_n) does not converge weakly to 0. According to Theorem 3 of [0] and 1.1 a subsequence (z_{p_n}) is a basic sequence equivalent to the unit-vector-basis of l. Thus: according to 1.2 of [0] we obtain 1.21; according to the fact that the sequence of unit vectors of l does not converge weakly, we obtain 1.2.

In a similar way we can obtain the following theorem ([2], IX, § 2):

1.3. *In the space l every weakly convergent sequence is convergent.*

Let $(x_n) \subset l$ be a weakly convergent sequence. It is enough to prove that for every two increasing sequences of indices (p_n) and (q_n) the sequence $(z_n) = (x_{p_n} - x_{q_n})$ converges to 0.

Suppose *a contrario* that $\|z_n\| \not\to 0$. Since $z_n \underset{w}{\to} 0$, according to Theorem 3 of [0] there is a subsequence (z_n') which is a basic sequence equivalent to a block basis with respect to the unit-vector-basis of l. Since (z_n') is bounded it must be equivalent to the unit-vector-basis of l. This gives a contradiction, because the sequence of unit vectors of l is not weakly convergent.

The proposition 1.3 can be formulated in an equivalent form:

1.31 (Theorem of Schur). *In order that every bounded numerical sequence be summable by the method corresponding to a matrix (a_{nm}) it is necessary and sufficient that:*

[3]) By (f_n) are denoted the linear functionals which are biorthogonal with respect to the basis (x_n) (see [0], 1.1).

[4]) The symbol $x_n \underset{w}{\to} 0$ denotes that the sequence (x_n) converges weakly to 0.

1° *there exist limits* $\lim_n a_{nm}$ $(m = 1, 2, \ldots)$,

2° $\sup_n \sum_{m=k}^{\infty} |a_{nm}| \to 0$ *for* $k \to \infty$.

1.4. *Suppose that the series $\sum_{n=1}^{\infty} t_n x_n$ is w. u. c. If the sequence (u_k) is of the form*

$$u_k = \sum_{i=N_k}^{M_k} t_i' x_i ,$$

where $N_1 < M_1 < N_2 < M_2 < \ldots$, $|t_i'| \leqslant 2|t_i|$ for $N_k \leqslant i \leqslant M_k$ $(k = 1, 2, \ldots)$, then the series $\sum_{k=1}^{\infty} u_k$ is w. u. c.

Indeed,

$$\sum_{k=1}^{\infty} |f(u_k)| \leqslant \sum_{k=1}^{\infty} \Big| f\Big(\sum_{i=N_k}^{M_k} t_i' x_i\Big)\Big| \leqslant C \sum_{k=1}^{\infty} \sum_{i=N_k}^{M_k} |f(t_i x_i)| \leqslant 2\cdot \sum_{i=1}^{\infty} |f(t_i x_i)| < +\infty .$$

1.5. *If the series $\sum_{k=1}^{\infty} u_k$ is not convergent but is w. u. c. and a sequence (y_k) fulfills the condition*

$$\sum_{i=1}^{\infty} \|y_i - u_i\| < +\infty ,$$

then the series $\sum_{k=1}^{\infty} y_k$ is also not convergent and w. u. c.

A trivial proof is omitted.

2. THEOREM 1. *If Y is a subspace of a Banach space X with an absolute basis (x_n), $\|x_n\| = 1$ $(n = 1, 2, \ldots)$, then the following conditions are equivalent:*

(a) *every bounded set in Y is conditionally weakly compact,*

(b) *no subspace of Y is isomorphic to the space l,*

(c) *no subspace of Y^* is isomorphic to the space c_0,*

(d) *every element $g' \in Y^*$ is representable in the form*

$$g' = \sum_{i=1}^{\infty} t_i f_i' ,$$

where $f_i'(x) = f_i(x)$ for $x \in Y$.

Proof. We divide the proof into four parts.

2.1. (a) implies (d). Let $g' \in Y^*$. Let us extend g' to the linear functional $g \in X^*$ in such a way that $\|g\| = \|g'\|$. Since for every $y \in Y$,

$$y = \sum_{i=1}^{\infty} f_i(y) x_i = \sum_{i=1}^{\infty} f_i'(y) x_i ,$$

we obtain

$$g'(y) = g(y) = \sum_{i=1}^{\infty} g(x_i) f'(y) = \sum_{i=1}^{\infty} t_i f'(y), \quad \text{where} \quad t_i = g(x_i).$$

Let $\vartheta = (\vartheta_n)$ be an arbitrary bounded real sequence. According to 1.4 of [0], the series $\sum_{i=1}^{\infty} \vartheta_i f_i(y) x_i$ is convergent. Thus

(1) the series $\sum_{i=1}^{\infty} \vartheta_i t_i f'_i(y)$ is convergent.

The sum of the series (1) defines over the space Y a linear functional $g_\vartheta(y)$.

Suppose that

$$\Big\| \sum_{i=1}^{\infty} t_i f'_i - g \Big\| \to 0.$$

Then there exist an increasing sequence of indices (p_n) and a sequence $(y_n) \subset Y$ with $\|y_n\| = 1$ $(n = 1, 2, \ldots)$ such that

(2) $$\Big\| \sum_{i=p_n}^{\infty} t_i f_i(y_n) \Big\| \geqslant \delta > 0 \quad (n = 1, 2, \ldots).$$

According to (a) it may be additionally supposed that

(3) the sequence (y_n) is weakly convergent.

According to (1) and (3) every bounded numerical sequence is summable by the method corresponding to the matrix $(a_{nm}) = (t_m f'_m(y_n))$. Since the conditions 1.31, 2° and (2) are contradictory, the supposition

$$\Big\| \sum_{i=1}^{M} t_i f'_i - g \Big\| \to 0$$

must be false.

The following is a simple consequence from (d):

(d′) *if Y_1 is a (separable) subspace of Y, then Y_1^* is separable.*

2.2. non (c) implies non (d′). If (c) does not hold, then by [0], Theorem 4, Y contains a subspace Y_1 which is isomorphic to l. Since Y_1^* is isomorphic to m, it is not separable.

2.3. non (b) implies non (c). This is a consequence of C. 7 of [0].

2.4. (b) implies (a). Let $(y'_n) \subset Y$ be an arbitrary bounded sequence. Since

$$\sup_{i,n} |f_i(y'_n)| \leqslant \sup_i \|f_i\| \cdot \sup_n \|y'_n\| < +\infty,$$

one can choose a subsequence (y_n) of the sequence (y'_n) such that there exist the limits

$$t_i = \lim_n f_i(y_n) \quad (i = 1, 2, \ldots).$$

Let (p_n) and (q_n) be arbitrary increasing sequences of positive integers. We have

$$\sup_n \|y_{p_n} - y_{q_n}\| \leqslant 2 \cdot \sup_n \|y'_n\|$$

and

$$\lim_n f_i(y_{p_n} - y_{q_n}) = t_i - t_i = 0 \quad (i = 1, 2, \ldots).$$

It follows from 1.21 that if the condition (b) is satisfied then $(y_{p_n} - y_{q_n}) \underset{w}{\to} 0$, *i. e.* the sequence (y_n) converges weakly, q. e. d.

3. THEOREM 2. *Let Y be a subspace of a Banach space X with an absolute basis (x_n). Then Y is weakly complete if and only if Y contains no subspace which is isomorphic to the space c_0.*

Proof. Suppose that Y is not weakly complete. We shall prove that Y contains a subspace isomorphic to c_0. (The converse implication is trivial). Let $(y'_n) \subset Y$ be a weakly convergent sequence which weakly converges to no element.

The idea of the proof: We compose a sequence (v_n) of differences of certain convex linear combinations of elements (y'_n) in such a way that the series $\sum_{n=1}^{\infty} v_n$ does not converge but it is w. u. c. Now we apply Theorem 5 of [0].

The proof will be composed of five parts.

Since (y'_n) weakly converges, there exist limits

(4) $$t_i = \lim f_i(y'_n) \quad (i = 1, 2, \ldots).$$

3.1. $\sum_{i=1}^{\infty} t_i x_i$ does not converge but is w. u. c. Suppose that

$$\sum_{i=1}^{\infty} t_i x_i = y.$$

The sequence $(y'_n - y)$ is weakly convergent, because (y_n) converges weakly. According to (4), $f_i(y'_n - y) \to 0$ $(i = 1, 2, \ldots)$. Thus 1.2 implies $(y'_n - y) \underset{w}{\to} 0$. This leads to a contradiction, because (y'_n) converges to no element.

Since (y'_n) is weakly convergent,

(5) $$\sup_n \|y'_n\|.$$

According to (4), (5) and criterion 1.4 of [0] for every $f \in Y^*$ we have

$$\sum_{i=1}^{\infty} |f(t_i x_i)| = \sup_m \sum_{i=1}^{\infty} |f(t_i x_i)| \leqslant K_{ab} \|f\| \sup \|y'_n\| < +\infty.$$

3.2. There exists an increasing sequence (r_n) of indices and a subsequence (y_n) of the sequence (y'_n) which satisfy the following conditions:

(6) $$\inf_n \Big\| \sum_{i=r_n+1}^{r_{n+1}} t_i x_i \Big\| = \delta > 0,$$

(7) $$\sum_{i=1}^{r_n} \| f_i(y_n) x_i - t_i x_i \| \leqslant \frac{1}{n} \quad (n = 1, 2, \ldots).$$

This is an easy consequence of 3.1 and (4).

3.3. *If* $z_n = \sum_{i=r_n+1}^{\infty} f_i(y_n) x_i$, *then* $z_n \underset{w}{\to} 0$.

We have

$$f_k(z_n) = \sum_{i=r_n+1}^{\infty} f_i(y_n) f_k(x_i) = \begin{cases} f_i(y_n) & \text{for} \quad r_n < k, \\ 0 & \text{for} \quad r_n \geqslant k, \end{cases}$$

whence $f_k(z_n) \to 0$ $(k = 1, 2, \ldots)$.

According to 1.2, it is enough to show that the sequence (z_n) converges weakly. Let $f \epsilon Y^*$. By (7),

$$|f(z_q - z_p)| = \Big| f(y_q - y_p) - \sum_{i=1}^{r_q} \big(f_i(y_q) - t_i\big) x_i + \sum_{i=1}^{r_q} \big(f_i(y_p) - t_i\big) x_i - \sum_{i=r_p}^{r_q} t_i x_i \Big|$$

$$\leqslant |f(y_q - y_p)| + \sup \|y_n\| \cdot \|f\| \cdot \left(\frac{1}{p} + \frac{1}{q}\right) + \sum_{i=r_p}^{r_p} |f(t_i x_i)| \qquad (p \leqslant q).$$

Since (y_n) weakly converges and $\sum_{i=1}^{\infty} t_i x_i$ is w. u. c.,

$$\lim_{p,q\to\infty} f(z_q - z_p) = 0,$$

therefore the sequence (z_n) converges weakly.

3.4. *For every* $\varepsilon > 0$ *and every positive integer* N *there exists* $y(N, \varepsilon) \epsilon [y_n]$ *which is of the form*

$$y(N, \varepsilon) = u(N, \varepsilon) + v(N, \varepsilon),$$

where

(8) $$u(N, \varepsilon) = \sum_{i=N}^{M(N,\varepsilon)} t'_i x_i \quad (M(N, \varepsilon) > N),$$

(9) $$\|u(N, \varepsilon)\| \geqslant \delta \cdot K_{ab}^{-1},$$

(10) $$|t'_i| \leqslant 2|t_i| \quad (i = N, N+1, \ldots, M(N, \varepsilon)),$$

(11) $$\|v(N, \varepsilon)\| \leqslant \varepsilon.$$

($y(N, \varepsilon)$ is a difference of certain convex linear combinations of elements (y_n)).

Let $n' > \max(N, 4/\varepsilon)$. According to 3.3 and a theorem of Mazur ([18], 2.541) there exists a convex linear combination

$$w' = W'(z_{n'}, z_{n'+1}, \ldots, z_{n'+p}) = \sum_{i=0}^{p} \lambda_i z_{n'+i}, \qquad \lambda_i \geqslant 0, \qquad \sum_{i=1}^{p} \lambda_i = 1,$$

such that

(12) $$\|w'\| \leqslant \varepsilon/4.$$

Let $n'' = n' + p + 1$. Consider the second convex linear combination

$$w'' = W''(z_{n''}, z_{n''+1}, \ldots, z_{n''+q}) = \sum_{i=0}^{q} \mu_i z_{n''+i},$$

$$\mu_i \geqslant 0, \qquad \sum_{i=0}^{q} \mu_i = 1,$$

such that

(13) $$\|w''\| \leqslant \varepsilon/4.$$

According to (7) the elements (y_n) are of the form

(7') $$y_n = \sum_{i=1}^{r_n} t_i x_i + z_n + w_n, \qquad \text{where} \qquad \|w_n\| \leqslant 1/n.$$

Let us set

$$y(N, \varepsilon) = W''(y_{n''}, y_{n''+1}, \ldots, y_{n''+q}) - W'(y_{n'}, y_{n'+1}, \ldots, y_{n'+p}),$$

$$u(N, \varepsilon) = W''\Big(\sum_{i=1}^{r_{n''}} t_i x_i, \sum_{i=1}^{r_{n''+1}} t_i x_i, \ldots, \sum_{i=1}^{r_{n''+q}} t_i x_i\Big) -$$

$$- W'\Big(\sum_{i=1}^{r_{n'}} t_i x_i, \sum_{i=1}^{r_{n'+1}} t_i x_i, \ldots, \sum_{i=1}^{r_{n'+p}} t_i x_i\Big),$$

$$v(N, \varepsilon) = w'' - w' + W''(w_{n''}, w_{n''+1}, \ldots, w_{n''+q}) - W'(w_{n'}, w_{n'+1}, \ldots, w_{n'+p}).$$

The element $u(N, \varepsilon)$ is of the form

$$u(N, \varepsilon) = \sum_{i=1}^{r_{n''+q}} t'_i x_i,$$

where

(14) $$|t'_i| \leqslant \Big(\sum_{k=0}^{q} |\mu_k| + \sum_{k=0}^{p} |\lambda_k|\Big) |t_i| = 2|t_i|.$$

(15) $$t'_i = \Big(\sum_{k=0}^{q} \mu_k - \sum_{k=0}^{p} \lambda_k\Big) t_i = (1-1) t_i = 0, \quad \text{for} \quad i \leqslant r_n,$$

(16) $$t'_i = \Big(\sum_{k=0}^{q} \mu_k\Big) t_i = 1 \cdot t_i = t_i \quad \text{for} \quad r_{n'+p} < i \leqslant r_{n''}.$$

By (16), (6) and 1.4 of [0], we have

$$\|u(N,\varepsilon)\| \geqslant K_{ab}^{-1}\Big\|\sum_{i=r_{n'}+p+1}^{r_{n''}} t_i' x_i\Big\| = K_{ab}^{-1}\Big\|\sum_{i=r_{n'}+p+1} t_i x_i\Big\| \geqslant \delta K_{ab}^{-1}. \tag{17}$$

By (14), (15) and (17) we obtain (8), (9) and (10). The inequality (11) is an immediate consequence from (12), (13) and (7').

3.5. *Let* $N_1 < M(N_1, 2^{-1}) < N_2 < M(N_2, 2^{-2}) < N_3 < \ldots$ *The series*

$$\sum_{k=1}^{\infty} y(N_k, 2^{-k}) \tag{18}$$

does not converge but it is w. u. c.

According to 1.5 it is enough to establish that the series $\sum_{k=1}^{\infty} u(N_k, 2^{-k})$ is w. u. c. but is not convergent. But this is an easy consequence of 3.1, 3.3 and 1.4.

Theorem 5 of [0] and 3.5 imply that the space $[y_n]$ contains a subspace which is isomorphic to c_0, q. e. d.

4. Corollaries. From Theorems 1 and 2 and a well-known result of Eberlein [10] we obtain

COROLLARY 1. *Let Y be a subspace of a Banach space X with an absolute basis. In order that Y be reflexive it is necessary and sufficient, that Y contain no subspace which is isomorphic either to the space c_0 or to the space l.*

DEFINITION. The set $A \subset X$ is called an *absolute basis of the power* $\overline{\overline{A}}$ if

1° the set of linear combinations of elements of A is dense in X,

2° every countable sequence of different elements of A is an absolute basic sequence.

Remark 1. Theorems 1 and 2 and Corollary 1 hold true in the case where the space X has an absolute basis of arbitrary power.

R. C. James has shown in his paper [14] an example of a non-reflexive Banach space without an absolute basis, no subspace of which is isomorphic either to the space l or to the space c_0. We shall denote this space by J.

COROLLARY 2. *If the space X contains a subspace which is isomorphic to J, then there is no absolute basis (of any power) of X.*

It implies that no Banach space that is universal (with respect to the isomorphic mappings) for all separable Banach spaces, has an absolute basis (of any power). In particular:

COROLLARY 3. *The spaces C and m are not imbeddable in a Banach space with an absolute basis (of any power).*

COROLLARY 4. *If X^* is a conjugate space (to a space X) and X^* is a subspace of a space with an absolute basis, then X^* is weakly complete.*

This fact is a simple consequence of Theorem 2, Corollary 3 and Theorem 4 of [0].

COROLLARY 5. *If the space Y is complemented in every space in which it can be imbedded, and Y is a subspace of a space with an absolute basis (of any power), then Y is reflexive*[5].

Suppose that that the space Y fulfills the hypotheses of Corollary 5 and Y is separable. Y may be considered as a subspace of the space m. If Y is not reflexive, then, according to Corollary 1, it contains a subspace Y_1 which is isomorphic either to l or to c_0. Thus, by [0] (C. 6 and C. 7), Y_1 must be complemented in Y, therefore it must be complemented in m. This is not true (see [25]). The proof in the case of a non-separable X can be made without any essential changes.

Remark 2. One can generalize the results of this paper to the case of linear metric spaces which are considered in [0], section 5.

5. Unsolved problems. Finally we quote some problems.

5.1. Does every infinitely dimensional Banach space contain an infinitely dimensional subspace with an absolute basis?

Remark 3. Dworetzky and Rogers [9] have shown that

(D-R) *In every Banach space of an infinite dimension there exists a u. c. series* $\sum_{n=1}^{\infty} x_n$ *such that* $\sum_{n=1}^{\infty}\|x_n\| = +\infty$.

Suppose that the answer to question 5.1 is positive. Then the proof of (D-R) can be immediately reduced to the easy case of the space l.

5.11. Give a solution of 5.1 with the additional assumption that the space X is weakly complete (reflexive).

5.12. Suppose that $x_n \underset{w}{\to} 0$, $\inf_n \|x_n\| > 0$. Does there exist a subsequence (x_{p_n}) which is an absolute basic sequence?

5.2. Let X be reflexive. May X be imbedded in a reflexive Banach space with a basis?

5.21. Let X be a reflexive (separable) Banach space. Can X be imbedded in a Banach space with an absolute basis?

5.22. Let X be a reflexive subspace of a Banach space with an absolute basis. Can X be imbedded in a reflexive Banach space with an absolute basis?

5.3. Give an example of a separable Banach space having no absolute basis which is a subspace of a Banach space with an absolute basis.

[5]) This is a generalization of James's result ([15], Theorem 2).

Let J_{2n+1} be a $(2n+1)$-dimensional space, composed of the real sequences $y = (a_1, a_2, \ldots, a_{2n+1})$, under the norm

$$\|y_n\|_n = \sup_{k\leqslant n} \sup_{p_1<p_2<\ldots<p_{2k+1}} \Big(\sum_{i=1}^{k}(a_{p_{2i}}-a_{p_{2i-1}})^2 - a_{p_{2k+1}}^2\Big)^{1/2}.$$

Let $X = (J_3\times J_5\times\ldots)_{l^2}$ be a space of all sequences $x = (y_n)$, $y_n \epsilon J_{2n+1}$ $(n = 1, 2, \ldots)$, such that $\sum_{n=1}^{\infty}\|y_n\|_n^2 < +\infty$, under the norm $\|x\| = \big(\sum_{n=1}^{\infty}\|y_n\|_n^2\big)^{1/2}$.

The space X can be isomorphically imbedded[6]) in the space $Y = (c_0\times c_0\times\ldots)_{l^2}$ having an absolute basis. We conjecture that the space X has no absolute basis.

5.4. Let X be a separable Banach space. Are the following conditions equivalent:

(a) every bounded set in X is conditionally weakly compact,

(b) no subspace of X is isomorphic to l,

(c'') Y^* is separable?

Added in proof. In other way the results given in this paper can be obtained in a stronger form: for instance to the assertion of Theorem can be added the following equivalent condition:

(e) *The space Y^* is weakly complete.*

(See C. Bessaga and A. Pełczyński, *On subspaces of the space with an absolute basis*, Bull. Acad. Pol. Sci. 6 (1958), p. 313-315.)

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[6]) *i. e.* the space X is isomorphic to a subspace of Y.

STUDIA MATHEMATICA, T. XVII. (1958)

Elliptizität und schwache Halbstetigkeit gewisser Funktionale der Variationsrechnung mehrfacher Integrale. Vollstetigkeit Greenscher Transformationen

von

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In dieser Abhandlung wird gezeigt wie man auf elementare Weise (Ehrlingsche Ungleichungen und Rellichscher Auswahlsatz) die Elliptizität einer wichtigen Klasse quadratischer Funktionale beweisen kann (§ 2). Diese Klasse umfaßt u. a. die Funktionale aus der Theorie der elastischen Platten und die von C. B. Morrey [12] in der Theorie der harmonischen Integrale eingeführten Formen.

Mit Hilfe derselben Schlußweise, aber mit Heranziehung der Kondraschewschen Sätze (und eines Kriteriums von E. Rothe) gelingt es die schwache Halbstetigkeit auch für eine gewisse Klasse nichtquadratischer Funktionale der Variationsrechnung mehrfacher Integrale und die Elliptizität ihrer zweiten Differentiale zu beweisen (§ 3).

Es wird weiter eine notwendige und hinreichende Bedingung für die Annahme des absoluten Minimums für quadratische Funktionale, des im § 2 untersuchten Typus angegeben. Diese Funktionale können auch auf Funktionenräumen definiert sein, die gewissen allgemeinen, von Ehrling eingeführten Randbedingungen genügen. Unser Vorgehen ist elementar und umgeht die Theorie der selbstadjungierten Fortsetzungen (§ 4).

Im § 5 bringen wir einen einfachen Beweis für die Vollstetigkeit der Greenschen Transformationen. Unser Satz ist zugleich eine Verschärfung eines Satzes von Ehrling [1] und umfaßt einen vor kurzem publizierten Satz von L. Schwartz [16]. Durch die Anwendung dieses Satzes auf das im § 3 untersuchte zweite Differential eines Funktionals erhalten wir schließlich einen Eigenwertsatz, der eine weitgehende Verallgemeinerung eines auf anderem Wege von E. Rothe gewonnenen Satzes ([15], § 6) darstellt.

Um die Abhandlung leicht zugänglich zu machen sind im § 1 die benutzten Hilfsmittel in ihrem logischen Zusammenhang zusammengestellt.