# On Cauchy's condensation theorem

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Let us write $a_n = n$ if $n = 2^k$ $(k = 1, 2, \ldots)$ and $a_n = 0$ elsewhere. Cauchy's well-known condensation theorem (test) states that if $\{\varepsilon_n\}$ is a non-increasing sequence with non-negative terms, then the series

$$(1) \qquad \sum_{n=1}^{\infty} a_n \varepsilon_n, \qquad \sum_{n=1}^{\infty} \varepsilon_n$$

converge or diverge simultaneously. In this paper we determine all the sequences $\{a_n\}$ with non-negative terms for which the above proposition is valid. To be precise, we shall call the sequence $\{a_n\}$ with non-negative terms *effective* for monotone series if, for every sequence $\{\varepsilon_n\}$ with non-increasing non-negative terms, the convergence of either of the series (1) implies the convergence of the other one. We prove that the sequence $\{a_n\}$ is such if and only if

$$0 < \varliminf_{n\to\infty} \frac{a_1+\ldots+a_n}{n} \leqslant \varlimsup_{n\to\infty} \frac{a_1+\ldots+a_n}{n} < \infty.$$

This shows that in the generalized Cauchy's test with

$$\sum_{n=1}^{\infty} a_n \varepsilon_n \equiv \sum_{n=1}^{\infty} (a_{n+1}-a_n)\, \varepsilon_{a_n}$$

the hypothesis

$$\varliminf_{n\to\infty} (a_{n+1}-a_n)(a_n-a_{n-1})^{-1} > 0$$

is indispensable.

We also give the necessary and sufficient conditions for one-side implications of the convergence of the series (1).

**1.** In the sequel $\mathfrak{a} = \{a_n\}$ will denote a sequence of non-negative numbers such that $a_1 \neq 0$. By $\mathfrak{X}(\mathfrak{a})$ we shall denote the set of all sequences $\mathfrak{x} = \{x_n\}$ tending to 0 for which there exists a non-increasing sequence $\{\varepsilon_n\}$ such that $|x_n| \leqslant \varepsilon_n$ and

$$\sum_{n=1}^{\infty} a_n \varepsilon_n < \infty.$$



Under the usual definition of addition and multiplication by scalars, $\mathfrak{X}(\mathfrak{a})$ becomes a linear space. Let us write, for $\mathfrak{x} \in \mathfrak{X}(\mathfrak{a})$,

$$\mu_n(\mathfrak{x}) = \sup\{|x_n|, |x_{n+1}|, \ldots\},$$

then the sequence $\mathfrak{x}$ is in $\mathfrak{X}(\mathfrak{a})$ if and only if

$$(2) \qquad \sum_{n=1}^{\infty} a_n \mu_n(\mathfrak{x}) < \infty.$$

Hence $\mathfrak{X}(\mathfrak{a})$ may be considered as the space of all sequences $\mathfrak{x} = \{x_n\}$ for which the series (2) converges. The sum of the series (2) defines in $\mathfrak{X}(\mathfrak{a})$ a functional $\|\mathfrak{x}\|$ which obviously has all the properties of the norm, moreover, as may easily be verified, the space $\langle \mathfrak{X}(\mathfrak{a}), \|\ \|\rangle$ is complete, whence it is a Banach space. We shall need some properties of this space.

LEMMA 1. *If* $a_n \geqslant 0$, $\varepsilon_n \searrow 0$, *and*

$$\sum_{n=1}^{\infty} a_n \varepsilon_n < \infty,$$

*then*

$$\lim_{n\to\infty} (a_1+\ldots+a_n)\varepsilon_n = 0.$$

This is a trivial generalization of the well-known theorem of Olivier.

Given any element $\mathfrak{x} = \{x_n\}$ of $\mathfrak{X}(\mathfrak{a})$, the element

$$\mathfrak{x}^n = \{x_1, \ldots, x_n, 0, 0, \ldots\}$$

is also in $\mathfrak{X}(\mathfrak{a})$. Since

$$\|\mathfrak{x}-\mathfrak{x}^n\| = (a_1+\ldots+a_n)\mu_n(\mathfrak{x}) + \sum_{i=n+1}^{\infty} a_i \mu_i(\mathfrak{x}),$$

we deduce from the fact that

$$\lim_{n\to\infty} \mu_n(\mathfrak{x}) = 0$$

for every $\mathfrak{x} \in \mathfrak{X}(\mathfrak{a})$ and from the Lemma that

$$\lim_{n\to\infty} \|\mathfrak{x}-\mathfrak{x}^n\| = 0$$

for every $\mathfrak{x} \in \mathfrak{X}(\mathfrak{a})$; this implies that the space $\langle \mathfrak{X}(\mathfrak{a}), \|\ \|\rangle$ is separable.

Now, we wish to determine the general form of linear functionals in $\langle \mathfrak{X}(\mathfrak{a}), \|\ \|\rangle$. Let $\xi$ be such a functional, and let $\mathfrak{x}_n$ denote the $n$-th unit vector. Then

$$\xi(\mathfrak{x}) = \sum_{n=1}^{\infty} b_n x_n \quad \text{where} \quad b_n = \xi(\mathfrak{x}_n).$$

To characterize the sequence $\{b_n\}$ we shall first compute the norm of the functional

$$\xi_k(\mathfrak{x}) = \sum_{n=1}^{k} b_n x_n.$$

Since

$$\|\xi_k\| = \sup_{\|\mathfrak{x}\|\leqslant 1} \left| \sum_{n=1}^{k} b_n x_n \right|,$$

and since $\vartheta_n = \pm 1$, $\mathfrak{x} \in \mathfrak{X}(\mathfrak{a})$ implies $\mathfrak{x}_\vartheta = \{\vartheta_n \mu_n(\mathfrak{x})\} \in \mathfrak{X}(\mathfrak{a})$ and $\|\mathfrak{x}\| = \|\mathfrak{x}_\vartheta\|$, we see that

$$\|\xi_k\| = \sup_{\|\mathfrak{x}\|<1} \sum_{n=1}^{k} |b_n x_n| = \sup_{\|\mathfrak{x}\|<1} \sum_{n=1}^{k} |b_n| \mu_n(\mathfrak{x}),$$

whence $\|\xi_k\|$ is equal to the supremum of

$$\sum_{n=1}^{k} |b_n| \mu_n(\mathfrak{x})$$

under the condition $\|\mathfrak{x}\| = 1$.

We shall prove that

$$\|\xi_k\| = \max\left(\frac{|b_1|}{a_1}, \frac{|b_1|+|b_2|}{a_1+a_2}, \ldots, \frac{|b_1|+\ldots+|b_k|}{a_1+\ldots+a_k}\right).$$

This immediately follows from

LEMMA 2. *Let $b_k \geqslant 0$; then the supremum of the sums*

$$\sum_{n=1}^{k} b_n x_n$$

*under the conditions $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_k \geqslant 0$, $a_1x_1+a_2x_2+\ldots+a_kx_k = 1$ is equal to*

$$\max\left(\frac{b_1}{a_1}, \frac{b_1+b_2}{a_1+a_2}, \ldots, \frac{b_1+\ldots+b_k}{a_1+\ldots+a_k}\right).$$

Proof. Let $\Omega$ be the subset of $R^k$, the space of $k$-dimensional euclidean vectors, composed of those elements $\mathfrak{x} = \{x_1, \ldots, x_k\}$ for which $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_k \geqslant 0$, $a_1x_1+a_2x_2+\ldots+a_kx_k = 1$. This set is evidently closed and convex. Let us now write

$$\mathfrak{x}_n = \{\underbrace{1, \ldots, 1}_{n}, 0, \ldots, 0\}, \qquad c_n = \frac{1}{a_1+\ldots+a_n}, \qquad \mathfrak{z}_n = c_n \mathfrak{x}_n.$$



Then, for every $\mathfrak{x} \in \Omega$,

$$\mathfrak{x} = \frac{x_k}{c_k} c_k \mathfrak{x}_k + \frac{x_{k-1}-x_k}{c_{k-1}} c_{k-1}\mathfrak{x}_{k-1} + \ldots + \frac{x_1-x_2}{c_1} c_1 \mathfrak{x}_1,$$

$$\frac{x_k}{c_k} + \frac{x_{k-1}-x_k}{c_{k-1}} + \ldots + \frac{x_1-x_2}{c_1} = \frac{x_1}{c_1} + x_2\left(\frac{1}{c_2} - \frac{1}{c_1}\right) + \ldots + x_k\left(\frac{1}{c_k} - \frac{1}{c_{k-1}}\right)$$
$$= a_1x_1+a_2x_2+\ldots+a_kx_k = 1,$$

and $\mathfrak{z}_1, \ldots, \mathfrak{z}_k \in \Omega$. Thus every $\mathfrak{x} \in \Omega$ is of the form $\lambda_1\mathfrak{z}_1+\lambda_2\mathfrak{z}_2+\ldots+\lambda_k\mathfrak{z}_k$ with $\lambda_i \geqslant 0$, $\lambda_1+\ldots+\lambda_k = 1$. Since the elements $\mathfrak{z}_n$ are linearly independent (for $c_n \neq 0$), $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_k\}$ is the set of all extreme points of the set $\Omega$. Since the linear functional $\varphi(\mathfrak{x}) = b_1x_1+b_2x_2+\ldots+b_kx_k$ assumes its extrema over $\Omega$ in the extreme points of $\Omega$, we get

$$\sup_{\mathfrak{x}\in\Omega} |\varphi(\mathfrak{x})| = \max[\varphi(\mathfrak{z}_1), \varphi(\mathfrak{z}_2), \ldots, \varphi(\mathfrak{z}_k)],$$

from which the statement of the lemma follows immediately.

These considerations lead, by a familiar procedure, to

THEOREM 1. *The general form of linear functionals in $\langle\mathfrak{X}(\mathfrak{a}), \|\ \|\rangle$ is*

$$\xi(\mathfrak{x}) = \sum_{n=1}^{\infty} b_n x_n,$$

*where*

$$\sup_{n=1,2,\ldots} \frac{|b_1|+\ldots+|b_n|}{a_1+\ldots+a_n} < \infty, \tag{3}$$

*the last number being equal to the norm of $\xi$.*

We infer

THEOREM 2. *In order that the series*

$$\sum_{n=1}^{\infty} b_n x_n$$

*be convergent for every $\mathfrak{x} = \{x_n\} \in \mathfrak{X}(\mathfrak{a})$ it is necessary and sufficient that the inequality (3) be satisfied.*

Let us denote by $\mathfrak{X}^*(\mathfrak{a})$ the space of all bounded sequences $\mathfrak{x} = \{x_n\}$ for which

$$\sum_{n=1}^{\infty} a_n \mu_n(\mathfrak{x}) < \infty.$$

Then $\mathfrak{X}^*(\mathfrak{a})$ is a Banach space under the norm defined by (1). It is obvious that if

$$\overline{\lim_{n\to\infty}}\, a_n > 0,$$

then

$$\lim_{n\to\infty} \mu_n(\mathfrak{x}) = 0 \quad \text{for every} \quad \mathfrak{x}\,\epsilon\,\mathfrak{X}^*(\mathfrak{a}),$$

whence $\mathfrak{X}^*(\mathfrak{a}) = \mathfrak{X}(\mathfrak{a})$.

**2.** Let $\mathfrak{a}_1 = \{1, 1, \ldots\}$. Then $\mathfrak{X}(\mathfrak{a}_1)$ is the space of all the sequences which can be majorized by non-decreasing sequences with convergent sums. Since $\mathfrak{x}$ is in $\mathfrak{X}(\mathfrak{a}_1)$ if and only if $\{\mu_n(\mathfrak{x})\}$ is in $\mathfrak{X}(\mathfrak{a}_1)$, we get, by Theorem 2,

THEOREM 3. *The series*

$$\sum_{n=1}^{\infty} a_n \varepsilon_n$$

*converges for every convergent series* $\sum_{n=1}^{\infty} \varepsilon_n$ *with non-increasing terms if and only if*

$$\sup_{n=1,2,\ldots} \frac{a_1+\ldots+a_n}{n} < \infty.$$

The assumption $a_1 \neq 0$ with regard to the space referred to in the proof is inessential.

Now we ask what the sequence $\{a_n\}$ with non-negative terms must be like in order that

$$\sum_{n=1}^{\infty} a_n \varepsilon_n < \infty$$

imply $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ for every sequence $\{\varepsilon_n\}$ with non-increasing terms. The assumption $a_1 \neq 0$ does not restrict generality.

THEOREM 4. *Let* $a_n \geqslant 0$. *The sequence* $\{a_n\}$ *is such that*

$$\sum_{n=1}^{\infty} a_n \varepsilon_n < \infty$$

*implies*

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty$$

*for every sequence* $\{\varepsilon_n\}$ *with non-negative, non-increasing terms if and only if*

$$\varliminf_{n\to\infty} \frac{a_1+\ldots+a_n}{n} > 0.$$

Proof. Necessity. For every $\mathfrak{x} = \{x_n\}\,\epsilon\,\mathfrak{X}^*(\mathfrak{a})$ the convergence of

$$\sum_{n=1}^{\infty} a_n x_n$$



implies the convergence of $\sum_{n=1}^{\infty} x_n$. *A fortiori* this is satisfied for every $\mathfrak{x}\,\epsilon\,\mathfrak{X}(\mathfrak{a})$. By Theorem 2 we must have

$$\varlimsup_{n\to\infty} n(a_1+\ldots+a_n)^{-1} < \infty.$$

To prove the condition sufficient, note that

$$\varliminf_{n\to\infty} n^{-1}(a_1+\ldots+a_n) > 0$$

implies

$$\varlimsup_{n\to\infty} a_n > 0,$$

whence $\mathfrak{X}^*(\mathfrak{a}) = \mathfrak{X}(\mathfrak{a})$. Then we apply again Theorem 2.

Theorems 3 and 4 yield

THEOREM 5. *The sequence* $\{a_n\}$ *is effective for monotone series if and only if*

$$0 < \varliminf_{n\to\infty} \frac{a_1+\ldots+a_n}{n} \leqslant \varlimsup_{n\to\infty} \frac{a_1+\ldots+a_n}{n} < \infty.$$

The following proposition is an obvious generalization of Theorem 5. Let $a_n \geqslant 0$, $b_n \geqslant 0$,

$$\varlimsup_{n\to\infty} a_n > 0.$$

The sequence $\{b_n\}$ is such that the convergence of either of the series

$$\sum_{n=1}^{\infty} a_n \varepsilon_n, \qquad \sum_{n=1}^{\infty} b_n \varepsilon_n$$

with non-negative non increasing $\varepsilon_n$'s implies the same for the other series if and only if

$$0 < \varliminf_{n\to\infty} \frac{a_1+\ldots+a_n}{b_1+\ldots+b_n} \leqslant \varlimsup_{n\to\infty} \frac{a_1+\ldots+a_n}{b_1+\ldots+b_n} < \infty.$$

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