Some remarks on Saks spaces

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1. In the present paper we shall denote by X a linear space with a homogeneous norm (B-norm) $\|\ \|$. This norm will be called the *fundamental norm in* X. Let $\|\ \|^*$ be another norm defined on X. The norm $\|\ \|^*$ may be a B-norm or, more generally, an F-norm ([2], p. 237). On the set

$$X_s = \underset{x}{E}[\|x\| \leqslant 1]$$

we define a metric by the formula

$$d(x_1, x_2) = \|x_1 - x_2\|^*, \quad \text{where} \quad x_1, x_2 \in X_s.$$

To every starred norm on X there corresponds a notion of convergence in X_s. A sequence $x_n \in X_s$ will be called *ω-convergent to* $x_0 \in X$ if $d(x_n, x_0) \to 0$, in symbols

$$x_n \overset{\omega}{\to} x_0 \quad \text{or} \quad \omega\text{-}\lim_{n\to\infty} x_n = x_0.$$

We shall write $X_s(\omega)$ to denote that a fixed starred norm in X_s is considered. By $X(\omega)$ we shall denote the space X with the norm $\|\ \|^*$.

If $X_s(\omega)$ is a complete space, we shall call it a *Saks space* (corresponding to the norm $\|\ \|^*$, see [2], p. 240). The same set X_s may constitute also other Saks spaces corresponding to other starred norms, non-equivalent to $\|\ \|^*$.

In the sequel we assume that $\|x\| \geqslant \|x\|^*$ for every $x \in X$. Under this hypothesis it is possible to show that, if $X_s(\omega)$ is a Saks space, the space X is a complete space under the norm $\|\ \|$ (see [3]).

We shall say that the norm $\|\ \|_0^*$ is *not weaker* than $\|\ \|^*$ in X_s if $\|x_n - x_0\|_0^* \to 0$ implies $\|x_n - x_0\|^* \to 0$ for $x_n \in X_s, x_0 \in X_s$. The norms $\|\ \|^*, \|\ \|_0^*$ will be called *equivalent on* X_s if the first norm is not weaker than the second and conversely. Note that two norms equivalent in X_s need not be equivalent on the whole space X.

We shall denote by Y the space conjugate to the Banach space X with the norm $\|\ \|$. We shall denote by $Y(\omega)$ the space of all functionals,

distributive and continuous with the norm $\|\ \|^*$, *e. g.* linear over the space $X, \|\ \|^*$. We shall denote by $Y_s(\omega)$ the set of all distributive functionals y defined on X such that $x_n \in X_s(\omega)$, $x_0 \in X_s(\omega)$ and

$$\omega\text{-}\lim_{n\to\infty} x_n = x_0$$

implies $y(x_n) \to y(x_0)$. Clearly $Y_s(\omega)$ is a linear subspace of Y.

1.1. *The space* $Y_s(\omega)$ *is closed in* Y.

Let y belong to the closure of $Y_s(\omega)$. Suppose that $\|x_n\| \leqslant 1$, $\|x_0\| \leqslant 1$,

$$\lim_{n\to\infty} \|x_n - x_0\|^* = 0.$$

Let ε be an arbitrary positive number. There exists a functional $y_0 \in Y_s(\omega)$ such that $\|y - y_0\| < \varepsilon/3$. Since y_0 is ω-continuous on $\|x\| \leqslant 1$, we have

$$\lim_{n\to\infty} \big(y_0(x_n) - y_0(x_0)\big) = 0.$$

There exists a natural number n_0 such that $|y_0(x_n) - y_0(x_0)| < \varepsilon/3$ for $n > n_0$. We have then, for $n > n_0$,

$$|y(x_n) - y(x_0)| \leqslant |y_0(x_n) - y_0(x_0)| + 2\|y - y_0\| < \varepsilon,$$

so that y is continuous on $\|x\| \leqslant 1$ as well. It follows that $y \in Y_s(\omega)$, which concludes the proof.

It is thus natural to ask whether, given a Banach space X and a closed subspace Y_0 of the conjugate space Y, it is not possible to define another norm $\|\ \|^*$ on X in such a way that the corresponding $Y_s(\omega)$ be equal to Y_0. The answer is negative, which may be seen from the following example.

Let X be the space of all continuous functions defined on the set T of all real numbers $0 \leqslant t \leqslant 1$. In X let us define the usual norm

$$\|x\| = \max_{\langle 0,1\rangle} |x(t)|.$$

The conjugate space Y consists of all functions of finite variation y defined on T, continuous from the left on $(0,1)$ and such that $y(0) = 0$.

Let us denote by Y_1 the set of all singular functions, *i. e.* the set of all $y \in Y$ which fulfil the following condition:

For every $\varepsilon > 0$, there exists a finite number of disjoint segments $k_i \in T$ such that $\sum |k_i| > 1 - \varepsilon$ and $\sum W(y; k_i) < \varepsilon$ [1]).

Let y belong to the closure of Y_1. Given an $\varepsilon > 0$, there exists an $y_0 \in Y_1$ such that $\|y - y_0\| < \varepsilon/2$. There exist disjoint segments $k_1, k_2, \ldots, k_n$

[1]) By $W(y; k)$ we denote the variation of the function y on the segment k.

such that $k_i \subset T$, $\sum |k_i| > 1-\varepsilon/2$ and $\sum W(y_0; k_i) < \varepsilon/2$. We have then

$$\sum W(y; k_i) \leqslant \sum W(y_0; k_i) + \sum W(y-y_0; k_i) \leqslant \sum W(y_0; k_i) + \|y-y_0\| < \varepsilon,$$

so that $y \in Y_1$ as well. We have thus shown that Y_1 is closed in Y.

Let us denote by Y_2 the set of all absolutely continuous functions, *i. e.* the set of all $y \in Y$ which fulfil the following condition:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every finite number of disjoint segments $k_i \subset T$ such that $\sum |k_i| < \delta$, we have $\sum W(y; k_i) < \varepsilon$.

Let y belong to the closure of Y_2. Let $\varepsilon > 0$ and suppose that $y_0 \in Y_0$ and $\|y-y_0\| < \varepsilon/2$. There exists a $\delta > 0$ such that $\sum |k_i| < \delta$ implies $\sum W(y_0; k_i) < \varepsilon/2$ for every finite number of disjoint segments $k_i \subset T$. If $k_i \subset T$ are disjoint segmen suchst that $\sum |k_i| < \delta$, we have

$$\sum W(y; k_i) \leqslant \sum W(y_0; k_i) + \|y-y_0\| < \varepsilon.$$

It follows that Y_2 is closed in Y.

Suppose now that there is a norm on X such that, for the corresponding convergence ω, we have $Y_1 \subset Y(\omega)$. Suppose that $x_n \in X_s$ and

$$\omega\text{-}\lim_n x_n = 0.$$

For every $t \in T$, the functional y_t defined by $y_t(x) = x(t)$ belongs to Y_1. Since $Y_1 \subset Y(\omega)$, it follows that $x_n(t) = y_t(x_n) \to 0$. At the same time $\|x_n\| \leqslant 1$, so that x_n is weakly convergent to 0. It follows that $y(x_n) \to 0$ for every $y \in Y$. We thus have $Y = Y_s(\omega)$.

On the other hand, it is easy to see that $Y_2 = Y_s(\omega)$ for the convergence defined by the starred norm

$$\|x\|^* = \int_0^1 |x(t)|\,dt.$$

In this case, however, the space $X_s(\omega)$ is not complete in the metric $\|x_1-x_2\|^*$.

In the sequel we shall examine the following properties of a Saks space $X_s(\omega)$:

(A) Suppose that U is a distributive operation from X to a Banach space Z. Suppose that, for every linear functional η on Z, the functional $\eta\big(U(x)\big)$ belongs to $Y_s(\omega)$[2]. In these conditions, the operation U is continuous on $X_s(\omega)$[3].

[2]) An operation U possessing this property is called *weakly continuous* in $X_s(\omega)$.

[3]) The operation U is said to be *linear* in $X_s(\omega)$ if it is distributive and continuous, *i. e.* $x_n \in X_s(\omega)$, $x_n \overset{\omega}{\to} x_0$ implies $U(x_n) \to U(x_0)$. A distributive operation is continuous in $X_s(\omega)$ if and only if it is continuous in 0.

(B₁) Let $y_n \in Y_s(\omega)$ and suppose that

$$\lim_{n\to\infty} y_n(x) = y(x)$$

exists for every $x \in X_s$. Then y belongs to $Y_s(\omega)$.

(B₂) Let y_n be a sequence of functionals $y_n \in Y_s(\omega)$ and suppose that

$$\lim_{n\to\infty} y_n(x) = y(x)$$

exists for every $x \in X_s$. Then y_n are equicontinuous on $X_s(\omega)$.

(B₃) Let U_n be a sequence of linear operations from $X_s(\omega)$ to a Banach space Z. Suppose that

$$\lim_{n\to\infty} U_n(x) = U(x)$$

exists for each $x \in X_s$. Then the U_n are equicontinuous on $X_s(\omega)$.

The implications $(B_3) \to (B_2) \to (B_1)$ are obvious. We are going to show now that (B_3) *is a consequence of* (A).

Let $X_s(\omega)$ be a Saks space fulfilling the property (A). We begin by showing that (B_1) is fulfilled as well. Suppose that $y_m(x) \to y(x)$ for every $x \in X_s$ and $y_m \in Y_s(\omega)$. Suppose that there exists a sequence x_n and a positive α such that $\|x_n\| \leqslant 1$, $\|x_n\|^* \to 0$ and $|y(x_n)| \geqslant \alpha$.

We are going to construct two sequences, m_i and n_i, of indices. Let $m_1 = 1$ and let n_1 be the least positive integer for which $|y_{m_1}(x_{n_1})| \leqslant \alpha/4$. This is possible since

$$\lim_{n\to\infty} y_{m_1}(x_n) = 0.$$

Let m_2 be the least positive integer for which $|y_{m_2}(x_{n_1})| \geqslant \alpha/2$. Such a number exists since

$$\lim_{m\to\infty} |y_m(x_{n_1})| = |y(x_{n_1})| \geqslant \alpha.$$

Let n_2 be the least positive integer $n_2 \geqslant n_1$ such that $|y_{m_2}(x_{n_2})| \leqslant \alpha/4$. Now let $r > 1$ and suppose we have already defined $m_1, m_2, \ldots, m_r$ and $n_1, n_2, \ldots, n_r$ in such a way that the following conditions are fulfilled:

1° $m_1 < m_2 < \ldots m_r$,

2° $n_1 < n_2 < \ldots n_r$,

3° $|y_{m_i}(x_{n_i})| \leqslant \alpha/4, \quad i = 1, 2, \ldots, r,$

4° $|y_{m_{i+1}}(x_{n_i})| \geqslant \alpha/2, \quad i = 1, 2, \ldots, r-1.$

To obtain m_{r+1}, it is sufficient to note that

$$\lim_{m\to\infty} |y_m(x_{n_r})| \geqslant \alpha.$$

Let us define m_{r+1} as the least positive integer $\geqslant m_r$ such that $|y_{m_{r+1}}(x_{n_r})| \geqslant a/2$. Take n_{r+1} as the least positive integer $\geqslant n_r$ for which $|y_{m_{r+1}}(x_{n_{r+1}})| \leqslant a/4$. It is easy to see that the conditions mentioned above are fulfilled, so that the induction is complete.

Now let $\bar{y}_i = y_{m_{i+1}} - y_{m_i}$, $z_i = x_{n_i}$. We have

$$\lim_{i\to\infty} \bar{y}_i(x) = 0 \quad \text{for every } x,$$

further $\|z_n\| \leqslant 1$ and $\|z_n\|^* \to 0$. At the same time, $|\bar{y}_i(z_i)| \geqslant a/4$ for every i. Since $\bar{y}_i$ are functionals on a Banach space X and

$$\lim_{i\to\infty} \bar{y}_i(x) = 0 \quad \text{for every } x,$$

there is a constant β such that $\|\bar{y}_i\| \leqslant \beta$ for every i. Let a_k be a sequence of real numbers such that $\sum |a_k| < \infty$. Since all $\bar{y}_i$ belong to $Y_s(\omega)$ and are equibounded, the sum $\sum a_k \bar{y}_k$ exists. The space $Y_s(\omega)$ being closed in Y, we have $\sum a_k \bar{y}_k \epsilon Y_s(\omega)$. Let us denote by V the Banach space of all sequences $v = \{t_k\}$ convergent to 0, with the norm

$$\|v\| = \max_k |t_k|.$$

For every $x \epsilon X$, let $U(x)$ be the sequence $\{\bar{y}_k(x)\}$. Every linear functional η on V is of the form $\eta(v) = \sum \lambda_k t_k$, where $\sum |\lambda_k| < \infty$. In view of what has been said above, it follows that the operation U is weakly continuous on $X_s(\omega)$. It follows that U is continuous. Since $\|z_i\| \leqslant 1$ and $\|z_i\|^* \to 0$, we have

$$\lim_{i\to\infty} U(z_i) = 0.$$

This is a contradiction since $\|U(z_i)\| \geqslant |\bar{y}_i(z_i)| \geqslant a/4$ for every i.

Now suppose that we are given a sequence of linear operations U_n from $X_s(\omega)$ to a normed space Z such that $U_n(x) \to U(x)$ for every $x \epsilon X_s$. If η is an arbitrary linear functional on Z, $\eta(U_n(x))$ will be a sequence of linear functionals on $X_s(\omega)$ such that for every $x \epsilon X_s$ we have $\eta(U_n(x)) \to \eta(U(x))$. According to what we have proved above, $\eta(U(x))$ will be a linear functional on $X_s(\omega)$. It follows that $U(x)$ is continuous as well.

Let V_n be defined by $V_n(x) = U_n(x) - U(x)$. Let us denote by W the normed space of all sequences $w = \{w_k\}$, where $w_k \epsilon Z$ and $w_k \to 0$, equipped with the norm

$$\|w\| = \max_k \|w_k\|.$$

For every $x \epsilon X$ let us denote by $W_n(x)$ the element of W defined by the sequence $V_1(x), V_2(x), \ldots, V_n(x), 0, 0, \ldots$

Let us denote by $W(x)$ the sequence $V_1(x), V_2(x), \ldots$ Since

$$\|W_n(x) - W(x)\| = \max_{k>n} \|V_k(x)\|,$$

it follows that $W_n(x) \to W(x)$ for every x. The operations W_n being continuous on $X_s(\omega)$, it follows that W is continuous as well. This, however, already implies the equicontinuity of the operations V_n and, consequently, that of the operations U_n. This concludes the proof.

Under the condition of *separability* of the space Z, we may prove the converse implication $(B_2) \to (A)$.

In fact, suppose that $X_s(\omega)$ is a Saks space which fulfils the property (B_2). Let U be a weakly continuous transformation of $X_s(\omega)$ into a separable Banach space Z. Suppose there exists a sequence $x_n \epsilon X$ and a positive a such that $\|x_n\| \leqslant 1$, $\|x_n\|^* \to 0$ and $\|U(x_n)\| \geqslant a$. There exist functionals η_n on Z such that $\|\eta_n\| = 1$ and $\eta_n(U(x_n)) \geqslant a$. The space Z being separable, we may find a subsequence η_{n_i} and a functional η such that $\eta_{n_i}(z) \to \eta(z)$ for every $z \epsilon Z$. It follows that the functionals $\eta_{n_i}(U(x))$ converge to $\eta(U(x))$ for every $x \epsilon X_s$. Since (B_2) is fulfilled, the functionals $\eta_{n_i}(U(x))$ are equicontinuous. This, however, is a contradiction of $\eta_{n_i}(U(x_{n_i})) \geqslant a$ for every i.

1.2. *Suppose that a Saks space $X_s(\omega_0)$ corresponding to the B-norm $\| \|_0^*$, possesses the property* (A). *Suppose that we are given another convergence ω corresponding to the B-norm $\| \|^*$ and such that $Y_s(\omega) \subset Y_s(\omega_0)$. In these conditions the norm $\| \|_0^*$ is not weaker than $\| \|^*$ in X_s.*

Let $x_n \epsilon X_s$ and $\|x_n\|_0^* \to 0$. We are to prove that $\|x_n\|^* \to 0$ as well. Let us denote by U the operation which assigns to an element $x \epsilon X_s(\omega_0)$ the same element considered as an element of the space $X(\omega)$. We have $Y(\omega) \subset Y_s(\omega) \subset Y_s(\omega_0)$, so that U is weakly continuous in $X_s(\omega_0)$. It follows that U is continuous. Since

$$\omega_0\text{-}\lim_{n\to\infty} x_n = 0,$$

we have $\|x_n\|^* = \|U(x_n)\|^* \to 0$ which concludes the proof.

It is important to know the sufficient conditions for a given space $X_s(\omega)$ to possess the property (A). The following condition are given in [2]:

(Σ_1) *Given any $x_0 \epsilon X_s$ and $\varrho > 0$, there exists a positive number δ such that every $x \epsilon X_s$, $\|x\|^* < \delta$ can be written in the form $x = x_1 - x_2$, where $\|x_1 - x_0\|^* < \varrho$, $\|x_2 - x_0\|^* < \varrho$, $x_1, x_2 \epsilon X_s$.*

We are going to give another condition of this kind.

(Σ_{1g}) *For every separable subspace $X_0 \subset X_s(\omega)$ there exists a separable linear subspace X_0' of the normed space $X(\omega)$ such that $X_{0s}'(\omega) = X_s X_0'$ fulfils the condition* (Σ_1).

Let us note, if $X_s(\omega)$ is supposed to be a Banach space, *i. e.* if the equality $\|x\|^* = \|x\|$ holds for all $x \in X$, the condition (Σ_{1g}) is always satisfied. Another example of a space satisfying this condition is to be found in 2.1.

Let us note that in [2] we are given one more condition assuring the fulfilment of property (A) (condition (Σ_2)). The importance of these conditions is seen from the following two results:

1.31. *Suppose that $X_s(\omega)$ is a Saks space and fulfils the condition* (Σ_2). *Then $X_s(\omega)$ possesses the property* (A).

This result has been proved in [2] under the assumption that Z is separable. It is easy to see, however, that this assumption is not necessary in the proof.

1.32. *Suppose that $X_s(\omega)$ is a separable Saks space and fulfils the condition* (Σ_1). *Then $X_s(\omega)$ possesses the property* (A).

The proof has been given in [2].

1.33. *Suppose that a Saks space $X_s(\omega)$ fulfils the condition* (Σ_{1g}). *Then $X_s(\omega)$ possesses the property* (A).

Let U be a distributive operation on $X_s(\omega)$ into a Banach space Z. Suppose that, for every linear functional η on Z, the functional $\eta(U(x))$ belongs to $Y_s(\omega)$. Suppose that there exist an $\alpha > 0$ and a sequence $x_n \in X_s$ such that

$$\omega\text{-}\lim_{n\to\infty} x_n = 0$$

and $\|U(x_n)\| \geqslant \alpha$. Let us denote by X_0 the subset of X_s consisting of all points x_n. There exists a linear separable subspace X_0' of the normed space $X(\omega)$, with the property defined in (Σ_{1g}), which contains all points x_n. Let us denote by X_{0s}'' the closure (in $X_s(\omega)$) of the space X_{0s}'. It follows that X_{0s}'' is a separable Saks space. Now let us show that the least linear subspace Z_0 containing the set $U(X_{0s}'')$ is separable. The set X_{0s}'' contains a countable dense subset v_n. If x is an arbitrary point of X_{0s}'', there exists a subsequence v_n' of v_n such that $\|v_n'-x\|^* \to 0$. For every linear functional η on Z we have $\eta(U(v_n')) \to \eta(U(x))$. It follows that $U(x)$ may be approximated by linear combinations of points of the countable set $U(v_n)$. Since all $\|U(x_n)\| \geqslant \alpha$, there exist linear functionals η_n on Z_0 such that $\|\eta_n\| = 1$ and $\eta_n(U(x_n)) \geqslant \alpha$. Since Z_0 is separable, there exists a subsequence η_{n_i} which converges to a linear functional η on Z_0. Hence follows the existence of a point $x_0 \in X_{0s}''$ and a positive ϱ such that $x \in X_{0s}''$, $\|x-x_0\|^* < \varrho$ implies $|\eta_{n_i}(U(x)) - \eta_{n_i}(U(x_0))| < \alpha/4$ for every i.

There exists a point $\tilde{x} \in X_{0s}'(\omega)$ such that $\|\tilde{x}-x_0\|^* < \varrho/2$. Now if $x \in X_{0s}'(\omega)$, $\|x-\tilde{x}\|^* < \varrho/2$, we have

$$|\eta_{n_i}(U(x)) - \eta_{n_i}(U(\tilde{x}))| < \alpha/2 \quad \text{for every } i.$$

Now there exists a $\delta > 0$ such that every point $x \in X_{0s}'(\omega)$ which fulfils $\|x\|^* < \delta$ may be written in the form $x = x_1 - x_2$, where $\|x_i - \tilde{x}\| < \varrho/2$ for $i = 1, 2$, $x_i \in X_{0s}'(\omega)$. It follows that, for such an x, we have

$$|\eta_{n_i}(U(x))| \leqslant |\eta_{n_i}(U(x_1)) - \eta_{n_i}(U(\tilde{x}))| + |\eta_{n_i}(U(x_2)) - \eta_{n_i}(U(\tilde{x}))| < \alpha,$$

which is a contradiction since $\|x_{n_i}\|^* \to 0$ and $\eta_{n_i}(U(x_{n_i})) \geqslant \alpha$.

In connection with 1.2 these remarks give the following result:

1.4. Let $X_s(\omega_0)$ be a Saks space. Let us consider all B-norms $\| \ \|^*$ such that $X_s(\omega)$ is a Saks space and that we have $Y_s(\omega) = Y_s(\omega_0)$ for the corresponding convergences. Among these norms, there exists at most one norm $\| \ \|^*$ which fulfils (Σ_2) or (Σ_{g1}). Here, of course, „at most one" is taken in the sense of equivalence of norms on X_s. In the case where $X_s(\omega)$ is separable the same holds for (Σ_1).

2. We shall say that a *set* $Y_0 \subset Y$ *possesses the property* (T) if, for every $x_0 \in X$, $x_0 \neq 0$, there exists an $y_0 \in Y_0$ such that $y_0(x_0) \neq 0$.

Let Y_0 be a closed linear subspace of Y possessing the property (T). A set $B \subset Y_0$ consisting of functionals of norm $\leqslant 1$ will be called a *basis of the space* Y_0 if the linear combinations of elements of B are dense in Y_0. If B is a basis of Y_0, we may define a B-norm in X by the formula

(†) $$\|x\|^* = \sup_{y \in B} |y(x)|.$$

It follows that $\|x\| \geqslant \|x\|^*$.

This norm possesses the following properties:

(1) if $C = \overline{\operatorname{conv} B}$, then $\|x\|^* = \sup_{y \in C} |y(x)|$,

(2) $Y_0 \subset Y_s(\omega)$,

(3) if B is compact, then every B-norm $\| \ \|_0^*$ in X such that $Y_0 \subset Y_s(\omega_0)$ is not weaker in X_s than the norm $\| \ \|^*$,

(4) there exists at most one norm (in the sense of equivalence on X_s) defined by means of a compact basis such that $Y_s(\omega) = Y_0$,

(5) the space Y_0 is separable if and only if there exists a compact basis of Y_0,

(6) suppose that the norm $\| \ \|^*$ is defined by means of a compact basis; then it is possible to define an equivalent norm by means of a count-

able compact basis $\bar{y}_1, \bar{y}_2, \ldots$; this norm will be equivalent to a norm of the form

$$\|x\|_0^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |\bar{y}_n(x)|.$$

The properties (1) and (2) are trivial.

To prove (3), let us take a sequence $\|x_n\| \leqslant 1$ and $\|x_n\|_0^* \to 0$. It follows that $y(x_n) \to 0$ for every $y \in B$. This convergence being uniform on B, we have $\|x_n\|^* \to 0$.

The property (4) follows immediately from (3).

Property (6) is an easy consequence of the fact that, if y_n is an arbitrary sequence of elements $\|y_n\| \leqslant 1$ such that linear combinations of these elements are dense in Y_0, and if ε_n is an arbitrary sequence of positive numbers convergent to 0, clearly $\varepsilon_n y_n$ is a compact basis of Y_0.

The space $X_s(\omega)$ defined by means of a basis need not be complete in the general case. We can state the following result:

2.1. *Let B be a compact basis and let the norm $\| \ \|^*$ be defined by (†). The corresponding space $X_s(\omega)$ will be complete if and only if it is compact.*

Suppose that $X_s(\omega)$ is complete. Let $x_n \in X_s$. The space Y_0 being separable, it is possible to construct a subsequence x_{m_n} such that $y(x_{m_n})$ is convergent for every $y \in Y_0$. The convergence of the sequence $y(x_{m_p}) - y(x_{m_q})$ being uniform on B, we have $\|x_{m_p} - x_{m_q}\|^* \to 0$ for $p, q \to \infty$. Since $X_s(\omega)$ is complete, there exists an $x_0 \in X_s$ such that $\|x_{m_n} - x_0\|^* \to 0$ for $n \to \infty$. The other implication being trivial, the proof is complete.

To define norms by means of bases gives a method to obtain different „starred" norms in the case where the space X and the fundamental norm $\| \ \|$ are given and a starred norm with prescribed properties is sought. In the examples given below we define the starred norms by means of bases and explain some properties of the corresponding $X_s(\omega)$.

I. Let us denote by X the normed space consisting of all bounded sequences $x = \{v_i\}$ of elements of a given Banach space V. The fundamental norm in X will be defined by

$$\|x\| = \sup_i \|v_i\|.$$

Suppose that there is defined on V a sequence of norms $\| \ \|_i^*$ such that V_s is a Saks space under each of those norms. Suppose further that in each of these Saks spaces the condition (Σ_1) is fulfilled. It is easy to show that, under these conditions, the space X_s is a Saks space satisfying condition (Σ_1) under the norm

$$\|x\|^* = \sup_n \frac{1}{n} \|v_n\|_n^*.$$

Now suppose that V is non-separable and that $\|v\|_i^* = \|v\|$ for every i. According to what has been said above, the space $X_s(\omega)$ satisfies condition (Σ_1) and is not separable. In this case, however, the condition (Σ_{1g}) is fulfilled as well.

In fact, let X_0 be a separable subset of $X_s(\omega)$. There exists a countable dense subset x_n of X_0. If $x_n = \{v_{ni}\}_i$, let us denote by x_{nk} the element

$$x_{nk} = v_{n1}, v_{n2}, \ldots, v_{nk}, 0, 0, \ldots$$

Let us denote by X_0' the closed linear subspace of $X(\omega)$ generated by the sequence $\{x_{nk}\}_{nk}$. It follows that X_0' is separable and that $X_0 \subset X_0'$. It is easy to show that the space X_{0s}' fulfils the condition (Σ_1).

II. Let X denote the space of all bounded sequences of real numbers $x = \{t_n\}$. The fundamental norm will be defined by

$$\|x\| = \sup_n |t_n|.$$

Let us denote by Y_0 the space of all linear functionals on X of the form

$$y(x) = \sum_{n=1}^{\infty} t_n a_n, \quad \text{where} \quad \sum_{n=1}^{\infty} |a_n| < \infty.$$

In this case $\|y\| = \sum |a_n|$ and Y_0 fulfils the condition (T). We may take a basis consisting of the elements $y_k = \{t_{km}\}$ where $t_{km} = 0$ for $m \neq k$ and $t_{kk} = 1/k$. We may thus define a norm

$$\|x\|^* = \sup_{y \in B} |y(x)| = \sup_n \frac{1}{n} |t_n|.$$

This norm is the weakest starred norm in X_s for which $Y_s(\omega) = Y_0$, $Y_s(\omega)$ fulfils conditions (Σ_1) and (Σ_2) and is a compact space.

III. Let us denote by X the space of all sequences of real numbers $x = \{t_n\}$ such that $\sum |t_n| < \infty$. The fundamental norm will be defined by

$$\|x\| = \sum_{n=1}^{\infty} |t_n|.$$

Let us denote by Y_0 the space of all linear functionals on X of the form

$$y(x) = \sum_{n=1}^{\infty} t_n b_n, \quad \text{where} \quad b_n \to 0.$$

We have

$$\|y\| = \sup_n |b_n|,$$

and Y_0 fulfils the condition (T); we may take the same compact basis as in example II. In the corresponding starred norm $X_s(\omega)$ is separable and, moreover, compact. We have $Y_s(\omega) = Y_0$ and Y_0 is separable.

The space $X_s(\omega)$ *does not possess the property* (B_1). It is sufficient to take the sequence of functionals

$$y_k(x) = \sum_{n=1}^{k} t_n \quad \text{for} \quad k = 1, 2, \ldots$$

We have $y_k \epsilon Y_0$, and the sequence $y_k(x)$ is convergent to the limit $\bar{y}(x) = \sum t_n \bar{\epsilon} Y_0$. It follows from [2], theorem 7 (compare also 1.1, 1.31, 1.32) that there exists no starred norm $\| \ \|_0^*$ such that $X_s(\omega_0)$ will be a Saks space fulfilling the conditions (Σ_1) or (Σ_2) and $Y_s(\omega_0) = Y_0$. We note that the set of the points of continuity of the function $\bar{y}(x)$ on the space $X_s(\omega)$ (this set cannot be empty by the theorem of Baire) is exactly the set of all points $x \epsilon X_s$ which fulfil $\|x\| = 1$.

IV. Let X be the space L^a, $a > 1$. The fundamental norm will be defined by

$$\|x\| = \Big(\int_a^b |x(t)|^a dt\Big)^{1/a}.$$

Let Y_0 denote the space of all linear functionals of the form

$$y(x) = \int_a^b x(t) y(t)\, dt,$$

where $y \epsilon L^\beta$, $1/a + 1/\beta = 1$. Clearly, Y_0 fulfils the condition (T) and $Y_0 = Y$. The basis B_1 will be defined as the set of all functions y_τ where

$$y_\tau(t) = \begin{cases} 1 & \text{for} \quad a \leqslant t \leqslant \tau, \\ 0 & \text{for} \quad \tau < t \leqslant b. \end{cases}$$

The set B_1 is compact and generates the norm

$$\|x\|_1^* = \sup_{\langle a,b \rangle} \Big| \int_a^\tau x(t)\, dt \Big|.$$

In this norm $X_s(\omega_1)$ is a Saks space. Another norm may be obtained in the following manner. Let $\varphi_1, \varphi_2, \ldots$ be a sequence of functions orthonormal in $\langle a, b \rangle$ and such that $|\varphi_n(t)| \leqslant K$, $t \epsilon \langle a, b \rangle$, for every n. Suppose that the sequence φ_n is complete in L^a. For a suitable $c > 0$, the sequence $c\varphi_1, c\varphi_2, \ldots$ is a basis B_2 of the space Y_0 which generates the norm

$$\|x\|_2^* = c \sup_n \Big| \int_a^b x(t) \varphi_n(t)\, dt \Big|.$$

In this norm, $X_s(\omega_2)$ is a Saks space. Evidently B_2 is not compact. The set B_3 consisting of the elements $c\varphi_n/n$ is compact and generates the norm

$$\|x\|_3^* = c \sup_n \frac{1}{n} \Big| \int_a^b x(t) \varphi_n(t)\, dt \Big|.$$

The space $X_s(\omega_3)$ is a Saks space too. Let us denote by B_4 the set of all measurable functions y such that $|y(t)| \leqslant (b-a)^{-1/\beta}$ for $t \epsilon \langle a, b \rangle$. The corresponding norm is

$$\|x\|_4^* = \sup_{y \epsilon B_4} \Big| \int_a^b x(t) y(t)\, dt \Big| = (b-a)^{-1/\beta} \int_a^b |x(t)|\, dt,$$

and $X_s(\omega_4)$ is a Saks space. We note that B_4 is weakly compact but not compact. It is easy to see that the norms $\| \ \|_1^*$ and $\| \ \|_3^*$ are equivalent in X_s and that convergence in $X_s(\omega_1)$, $X_s(\omega_3)$ coincides with the weak convergence in X. Further, the inequalities $K\|x\|_4^* \geqslant \|x\|_2^* \geqslant \|x\|_3^*$ hold, but $\| \ \|_2^*$ and $\| \ \|_3^*$ are not equivalent.

The additive operation $U(x) = x$ on $X_s(\omega_1)$ into L^a is not continuous, but for every $y \epsilon Y_0 = Y$ the functional $y(U(x))$ is continuous in $X_s(\omega_1)$. It follows that the property (A) is not fulfilled in $X_s(\omega_1)$. According to what has been said about the connection between the properties (A) and (B_2), the *property* (B_2) *cannot be fulfilled* since $U(x)$ is mapped into a separable space. On the other hand, *the property* (B_1) *is fulfilled.*

V. Let X be the space of all bounded measurable functions defined in $\langle a, b \rangle$ with the norm

$$\|x\| = \sup_{\langle a,b \rangle}{}^* |x(t)| \text{ [4]}.$$

Let Y_0 be the space of all linear functionals of the form

$$y(x) = \int_a^b x(t) y(t)\, dt \quad \text{where} \quad y \epsilon L^1,$$

so that Y_0 is separable and possesses the property (T). The bases B^i $(i = 1, 2, 3, 4)$ given in IV are bases of Y_0 as well. The same norms may be defined as in IV (in the case of B_2 we assume that the functions φ_n are complete in M; in the definitions of B_4 and $\| \ \|_4^*$ we replace $(b-a)^{-1/\beta}$ by $(b-a)^{-1}$). Just as in IV, the corresponding $X_s(\omega_i)$ are Saks spaces and $Y_s(\omega_i) = Y_0$. This follows from the known result of Fichtenholz and from the relations between the norms $\| \ \|_i^*$, which remain unchanged in the present case.

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[4]) $\sup^*_{\langle a,b \rangle} y(t)$ means the essential upper bound of the function $y(t)$ in $\langle a, b \rangle$.

The situation is different from IV in the following point. The space $X_s(\omega_4)$ possesses the property (A) since (Σ_1) and (Σ_2) are fulfilled (see [2]). The norms $\| \|_1^*$ and $\| \|_3^*$ being non-equivalent in X_s with $\| \|_4^*$, the corresponding Saks spaces do not possess the property (A).

Especially *in* $X_s(\omega_1)$ *we have* (B_1) *but not* (B_2).

VI. Let us denote by X the space of all bounded continuous functions defined in an open interval (a, b). (The end points need not be finite here). As Y_0 we take the set of all linear functionals of the form

$$\int_{a+}^{b-} x(t)\,dy,$$

where y denotes a function of finite variation in (a, b), continuous from the left and equal to zero at the point $(a+b)/2$. It is easy to see that Y_0 is not identical with Y and possesses the property (T). The space Y_0 is non-separable, since

$$\|y\| = \operatorname*{var}_{(a,b)} y(t) \quad \text{for} \quad y \in Y_0.$$

Let a, b be finite and let us denote by B the set of all $y \in Y_0$ such that $y(t) = 0$ for $t \in (a, a+1/n)+(b-1/n, b)$ and $\operatorname*{var}_{\langle a+1/n,\, b-1/n\rangle} y(t) = 1/n$.

Then

$$\|x\|^* = \sup_{y \in B} |y(x)| = \sup_n \sup_{\langle a+1/n,\, b-1/n\rangle} |x(t)|/n.$$

In the case when a, b are infinite we define B and the norm $\| \|^*$ analogically.

It is possible to show that $X_s(\omega)$ is a Saks space fulfilling conditions (Σ_1) and (Σ_2) and that $Y_s(\omega) = Y_0$ (see [1], [2]).

References

[1] J. Musielak and W. Orlicz, *Linear functionals over the space of functions continuous in an open interval*, Studia Math. 15 (1956), p. 216-224.

[2] W. Orlicz, *Linear operations in Saks spaces (I)*, ibidem 11 (1950), p. 237-272.

[3] — *Linear operations in Saks spaces (II)*, ibidem 15 (1955), p. 1-25.

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On the continuity of linear operations in Saks spaces with an application to the theory of summability

by

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1. Let X be a linear space and let a B-norm $\| \|$ (*fundamental norm*) and a B- or F-norm $\| \|^*$ (*starred norm*) be defined in X. If the set

$$X_s = \mathop{E}_{x}\{x \in X,\ \|x\| \leqslant 1\}$$

with the distance defined as $d(x, y) = \|x-y\|^*$ is a complete space, it will be called a *Saks space* (with the norm $\| \|^*$, see [2][1]). The following theorem is a generalization of the result given in [3]:

1.1. *Let* $X_1, X_2, \ldots, X_n, \ldots$ *be linear subspaces of the space* X *and et an* F*-norm* $\| \|_n^*$ *be defined in* X_n *for* $n = 1, 2, \ldots$ *Writing*

$$X_0 = \bigcap_{n=1}^{\infty} X_n,$$

we suppose the following conditions to be satisfied:

(a) $X_1 \supset X_2 \supset \ldots \supset X_n \supset \ldots$;

(b) *there exists a linear subspace* $Y_0 \subset X_0$ *such that the set* $\overline{X}_n = Y_0 \cap X_n \cap \overline{X}_s$ *is dense in* $X_n \cap X_s$, *the distance being induced by* $\| \|_n^*$ *for* $n = 1, 2, \ldots$;

(c) *the set* $X_n \cap X_s$ *is a Saks space under the norm* $\| \|_n^*$, *satisfying the condition* (Σ_1)[2], *for* $n = 1, 2, \ldots$);

(d) *if* $x_i \in X_0$ *and* $\|x_i\|_k^* \to 0$ *for a fixed* k *and* $i \to \infty$ *then* $\|x_i\|_{k'}^* \to 0$ *for every* $k' < k$.

Further suppose that in X_0 *additive operations* U_n *with values in a Fréchet space* Y *are defined, such that*

(α) *for every* $x \in X_0$ *the sequence* $\{U_n(x)\}$ *is convergent*;

(β) *for every fixed positive integer* n, k, $\|x_i\| \leqslant 1$, $x_i \in X_0$ *and* $\|x_i\|_k^* \to 0$ *for* $i \to \infty$ *imply* $U_n(x_i) \to 0$.

[1]) The numbers in square brackets refer to the references at the end of this paper.

[2]) Concerning the definition of the condition (Σ_1) see [2], p. 240.