Since 
\[ \lim_{n \to \infty} (2k_n/\delta_n) = 0, \]
the neighbourhood
\[ U = E \{ \|y\| < \epsilon \} \]
satisfies the condition (\(\ast\)), i.e., is bounded.

**Corollary.** From the proof of Theorem 3 it follows directly that if in an \(E^*\)-space a norm has the property \(W_1\), then an equivalent norm has it also.

Remark 1. The above theorem is false in the case of the \(E^*\)-space. An example is provided by the space \(K\) of all the sequences \(x = (x_n)\) almost all elements of which vanish, the norm being
\[ \|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{1 + |x_n|}. \]

It is easily verified that the sequence \(\delta_n = n\) is a rate of growth for the norm \(\|x\|\).

"Since \(K\), being an \(E^*\)-space, is not a \(K^*\)-space (see [6]) there are not any bounded neighbourhoods in \(K\).

**References**


*Reçu par la Rédaction le 13. 9. 1956*

---

Spaces of continuous functions (II)  
(On multiplicative linear functionals over some Hausdorff classes)

by

Z. SEMADENI (Poznań)

8. Mazur [5] has proved that with every bounded sequence \(\{x_n\}\) a real number \(\operatorname{Lim} x_n\) can be associated in such a way that \(\operatorname{Lim} x_n\) is equal to the usual limit of a subsequence of \(\{x_n\}\); consequently

\[
\begin{align*}
(1) & \quad \operatorname{Lim} x_n \leq \operatorname{Lim} x_n \leq \operatorname{Lim} x_n, \\
(2) & \quad \operatorname{Lim}(ax_n + by_n) = a\operatorname{Lim} x_n + b\operatorname{Lim} y_n, \\
(3) & \quad \operatorname{Lim}(x_n y_n) = \operatorname{Lim} x_n \cdot \operatorname{Lim} y_n.
\end{align*}
\]

In this note a construction of generalized limits for some classes of functions is given. This construction is non-effective, just as those of Mazur; it is based on the theorem of Kakutani on the representation of abstract (\(M\))-spaces. It is easily seen that this limit can also be derived from the theorem of Tychonoff, but I think that the way which I have chosen leads to more consequences.

The generalization of the theorem of Mazur to the case of real-valued, bounded functions defined on \(\langle 0, 1 \rangle\) is trivial, e.g., we can put
\[
\lim_{t \to \delta} \operatorname{Limes} x(t) = \operatorname{Lim} x(t),
\]
where \(\operatorname{Lim}\) denotes an arbitrary limit of Mazur and \(t_n \to \delta\). The functional \(\operatorname{"Limes"}\) constructed in the Theorems 1, 1a, 1b and 2 satisfies also some additional conditions. It can be considered as a solution of the following problem: given a space of equivalence classes of functions how to assign in a reasonable way the value to every function at every point.

The second part of this paper contains some applications (the existence of certain multiplicative measures and a negative solution of two questions concerning the extension of linear functionals).
1. The generalized limits. Let $E$ be a topological space and $R$ a $\sigma$-ideal of boundary sets (i.e., $A \in R$ and $B \subseteq A$ imply $B \in R$, $A_n \in R$ imply for $n = 1, 2$
\[ \bigcap_{n=1}^{\infty} A_n \in R, \]
no open non-empty set belongs to $R$). The family $H$ of all sets of form $G \setminus A$ (where $G$ is open and $A \in R$) is multiplicative and $\sigma$-additive.

Denote by $H$ the class of all real-valued functions $x(t)$ on $E$ such that the sets $[a < x(t) < b]$ belong to $H$ for every $a$ and $b$. Hausdorff ([3], p. 235) has established that $H$ is closed with respect to addition, multiplication, supremum and infimum of two elements. Next, note by $\sup_{x(t)}$ the least upper bound of the totality of numbers $a$ such that the set $[x(t) > a]$ belongs to $R$. In particular, $\sup_{x(t)}$ denotes the usual essential supremum and $\sup_{t \in A}$ denotes the essential supremum with respect to the sets of Baire's first category. We introduce also the $R$-essential limit in $t_0$ in the following manner:
\[ \lim_{t \to t_0}^{R} x(t) = \inf_{A \in H} \{ \lim_{t \to t_0} x(t) \}, \quad \lim_{t \to t_0} x(t) = \lim_{t \to t_0} [-x(t)]. \]

**Lemma 1.** For any $\alpha \in H$ the sets
\[ A = [t: \lim_{t \to t_0} x(t) \neq \lim_{t \to t_0} x(t)] \]
and
\[ B = [t: \lim_{t \to t_0} x(t) = \lim_{t \to t_0} x(t)] \]
belong to $R$.

Proof. According to Alexiewicz ([1], p. 64) a function $x(t)$ belongs to $H$ if and only if the set $D$ of its points of discontinuity belongs to $R$. It follows that $A \in R$ and $B \in R$, because $A \cap B \subseteq D$.

In the class $X_0$ of bounded functions of $H$ we introduce the reflexive, symmetric and transitive relation $x_t \sim y_t$ when $[x_t(t) = y_t(t)] \in R$, and we identify $R$-equivalent functions. Denote by $x, y, \ldots$ the classes of equivalence under the relation $\sim$, corresponding to elements $x_t, y_t, \ldots$. Evidently the space $X = X_0/\sim$ is a Banach space with the norm $\|x\| = \sup_{R} \|x_t(t)\|$.

In $X$ we introduce a partial ordering by the relation: $x \leq y$ if and only if the set $[t: x_t(t) > y_t(t)]$ belongs to $R$ for $x_t \in x$ and $y_t \in y$. If $x \wedge y = 0$ then the set $[t: x_t(t) > y_t(t)] \in R$, and it follows that $\sup_{t \in A}$ implies $\|x + y\| = \|x - y\|$. Similarly $x \geq 0$ and $y \geq 0$ imply $\|x \vee y\| = \|x\| \vee \|y\|$, therefore $X$ is an $(M)$-space with a unit element (see [3]). Denote by $\mathcal{F}(E, R)$ and by $\mathcal{B}$ the set of all linear functionals defined on $X$ which satisfy the conditions

\[ \|x\| = 1, \quad \forall (x) \geq 0 \quad \forall x \geq 0, \quad x \wedge y = 0 \implies \forall x, y = 0. \]

By the theorem of Kakutani $X$ can be linearly, isometrically and isotonically mapped on the space $C(E)$ of continuous functions defined on $E$ (which is compact in a weak topology).

**Theorem 1.** Suppose that $E$ satisfies the first axioms of countability at each point $t \in E$, and suppose that there exists a base $\{U_n\}$ of neighborhoods of $t$ and a sequence of continuous functions $\xi_n$ from $E$ into $U_n$ such that no $\xi_n(E)$ belongs to $R$ ($n = 1, 2, \ldots$). Then to every $x \in \mathcal{F}(E, R)$ corresponds a generalized limit
\[ \lim_{t \to t_0} x(t) = \xi_0(x) \]
such that

\[ \lim_{t \to t_0} x(t) \leq \xi_0(x) \leq \lim_{t \to t_0} x(t), \]

\[ \xi_0(\lambda t) = \lambda \cdot \xi_0(t), \]

\[ \xi_0(t + y) = \xi_0(t) \cdot \xi_0(y), \]

\[ \theta (t \vee y) = \max \{\xi_0(x), \xi_0(y)\}. \]

If this limit exists for any $t \in E$, then the function $u(t) = \xi_0(x)$ is $R$-equivalent to $x$, i.e.,
\[ \{t: \xi_0(t) = \xi_0(t) \in R \text{ for } x \in x \}. \]

Proof. Choose an arbitrary fixed $\xi \in \mathcal{B}$. We put $x_n(t) = \xi_0(\xi_n(t))$ for $t \in E$ and $\xi_n(x) = \xi(\xi_n(t))$. Evidently $\xi_n \in R$. Every limit point $\xi_n$ of the set $\{\xi_n\}$ satisfies (6), (7) and (8); (5) follows from the identity
\[ \lim_{t \to t_0} x(t) = \lim_{n \to \infty} \sup_{R} x(t), \]
and (9) results by lemma 1.

Now, we specialize the space $E$ and the family $R$ to obtain some applications of theorem 1.

(a) Let $E$ be the interval $(0, 1)$ and $R$ the family $L$ of sets of Lebesgue's measure zero. Then
Theorem 2. For every bounded function satisfying the condition of Baire in a complete metric space $E$ there exists a function $u(t) = \xi(x)$ such that $(6)$, $(7)$, $(8)$, $(11)$ and $(12)$ hold.

The proof is analogous to the proof of Theorem 1.

The method presented here may be applied to other functional spaces, but the second part of Theorem 1 is not true in all cases.

2. Multiplicative measures. Let us consider the space $X$ of bounded sequences $x = [x_1, x_2, \ldots]$ with the norm

\[ \|x\| = \sup_B |x(t)|, \]

the ordering $x \leq y$ if $x_n \leq y_n$ for $n = 1, 2, \ldots$ and the unit $e = [1, 1, \ldots]$.

By the above mentioned theorem of Kakutani $X$ is strongly equivalent to the space $G(\sigma)$ (where $\sigma$ is given by $(4)$). Every functional

\[ \xi(x) = x_n \]

obviously belongs to $\mathfrak{a}$. Any limit point $\xi$ of the sequence $\{\xi_n\}$ satisfies $(1)$, $(2)$ and $(3)$, whence it follows that $\xi$ is a limit of $\mathfrak{a}$. Conversely, each functional which satisfies $(1)$, $(2)$ and $(3)$ is a limit point of $\{\xi_n\}$ because it belongs to $\mathfrak{a}$ and by $(1)$ it is none of the functional $(13)$.

In other words: the Stone-Cech compactification $\beta(N)$ of the countable isolate set $N$ consists of the functionals $(13)$ and of the limits of $\mathfrak{a}$.

Let $S$ denote a subset of $N$ and $\chi_S$ its characteristic function. Given $\xi \in \mathfrak{a}$, we put $m(S) = \xi(\chi_S)$.

It is easily seen that $(a) m(S) \geq 0$, $(b) m(S \cap \mathcal{S}_0) = m(S \cap \mathcal{S}_0) + m(S \cap \mathcal{S}_0)$ if $S = \varnothing = 0$, $(c) m(S \cap \mathcal{S}_0) = m(S \cap \mathcal{S}_0) + m(S \cap \mathcal{S}_0)$, $(d) m(N) = 1$, $(e)$ if $S$ is finite and $\xi$ is no of $(13)$, then $m(S) = 0$.

Thus $m(S)$ is a finitely-additive and multiplicative set function defined on all subsets of $N$. The condition $(c)$ can be interpreted as a stochastical independence. Conversely, to every measure of such kind there corresponds a multiplicative functional

\[ \xi(x) = \int_S x \, dm. \]

In other words the functional $\xi$ is multiplicative if and only if it is multiplicative on nought-or-one sequences.

This procedure may be generalized. Let $X$ be the space of Baire-functions in $E$ (see Theorem 2). For an arbitrary Baire-set $A \subset E$ and $\xi \in \mathfrak{a}$ we establish that $\mathfrak{m}(A) = \xi(\chi_A)$ is a finitely-additive and multi-
plicative measure vanishing on sets of the first category. The general form of linear functional over \( X \) is the integral

\[
f(x) = \int_{\xi} x(t) \, dm(t)
\]

and \( ||f|| = \text{Var} m \). More generally, we can consider Hausdorff classes \( \xi \) corresponding to arbitrary Boolean algebras.

**Theorem 3.** The following conditions are equivalent for linear functionals over the space \( C(\Omega) \) of continuous functions defined on a 0-dimensional compact space \( \Omega \):

1. \( \xi \in \mathcal{A} \) (i.e., \( \xi(x) = x(t_0) \) for fixed \( t_0 \in \Omega \));
2. \( \xi \) is multiplicative and \( \xi \neq 0 \);
3. the measure \( m_\xi \) does not vanish everywhere and for \( A, B \subset \Omega \) open and \( A \cap B = \emptyset \), we have \( m_\xi(A \cup B) = m_\xi(A) \cdot m_\xi(B) \);
4. \( m_\xi(\Omega) = 1 \), \( m_\xi \geq 0 \), and \( A \cap B = \emptyset \) implies \( m_\xi(A) \cdot m_\xi(B) = 0 \).

**Proof.** The implications \( 1 \Rightarrow 2, 2 \Rightarrow 3, 3 \Rightarrow 4 \) are trivial, we shall prove only \( 4 \Rightarrow 1 \). From the obvious \( \xi = \xi(x) \) we have \( ||\xi|| = ||\xi(x)|| = 1 \). Suppose that \( x_1, x_2, x_3 \in X \) and \( x_1 \cdot x_2 = 0 \). For any \( n \) there exist simple functions \( \alpha_n \cdot x_1 \) and \( \beta_n \cdot x_2 \) such that \( |\alpha_n - x_1| < 1/n \), \( |\beta_n - x_2| < 1/n \), and \( x_1 \wedge x_2 = 0 \). Moreover, there exist sets \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_m \) such that \( A_k \cap B_l = \emptyset \) for \( j, k = 1, 2, \ldots, m \),

\[
x_k = \sum_{\xi_k} a_k x_{\xi_k} \quad \text{and} \quad y_k = \sum_{\xi_k} b_k y_{\xi_k}.
\]

Then

\[
\xi(x_k) \cdot \xi(y_k) = \sum_{\xi_k} a_k b_k m_\xi(A_k \cdot B_k) = \sum_{\xi_k} a_k b_k m_\xi(A_k \cap B_k) = 0.
\]

Passing to the limit we obtain \( \xi(x) \cdot \xi(y) = 0 \), whence \( \xi \in \mathcal{A} \).

**Theorem 4.** A compact space \( \Omega \) is 0-dimensional if and only if the condition \( \xi \in \mathcal{A} \) is equivalent to the following: if \( x \in C(\Omega) \) and \( x(t) \) take only 0 and 1 as the values, then either \( \xi(x) = 0 \), or \( \xi(x) = 1 \).

**Proof.** Necessity. Since every positive function \( x \in X \) can be approximated by non-negative simple functions, we have \( \xi \neq 0 \) and \( ||\xi|| = 1 \) (if \( \xi \neq 0 \)). Suppose that \( E_1 \cap \Omega, E_2 \cap \Omega \) and \( E_1 \cap E_2 = 0 \). Then every number \( m_\xi(E_1), m_\xi(E_2), m_\xi(E_1 \cap \Omega - (E_1 \cap E_2)) \) is equal to 0 or 1, whence from \( m_\xi(E_1) + m_\xi(E_2) - m_\xi(E_1 \cap \Omega - (E_1 \cap E_2)) \) there follows \( m_\xi(E_1), m_\xi(E_2) = 0 \).

By theorem 3 (proposition 4) \( \xi \in \mathcal{A} \).

Sufficiency. Suppose that there exists a connected set \( E \subset \Omega \) containing two different points \( t_1 \) and \( t_2 \). Then the functional \( \eta(x) = \frac{x(t_1) + x(t_2)}{2} \) does not belong to \( \mathcal{A} \), and \( \eta(x) = 0 \) or \( \eta(x) = 1 \) holds for any non-negative one continuous function \( x \in X \).

**5. Extension of linear functionals.** Similarly to Theorem 1 we can prove the existence of a generalized left-hand limit, \( \xi_l \), and a right-hand one, \( \eta_r \), in the space \( X \) of Riemann-integrable functions, at any point \( t \in (0, 1) \). Let \( X_t \) denote the subspace of \( X \) of continuous functions on \((0, 1)\). The functionals \( \xi_l \) and \( \eta_r \) are equal on \( x_0 \). There follow two propositions:

1. A norm-preserving extension of linear functional from an \( M \)-subspace \( X_t \) on \( M \)-space \( X \) is not necessarily unique (even if \( X_t \) and \( X \) have the same unit).

However, by a theorem of M. Krein and S. Krein ([4], p. 7) it follows that if \( \Omega_x \) is an open and closed subset of \( \Omega \), then every linear functional has a unique norm-preserving extension from \( X_t = C(\Omega_x) \) to \( X = C(\Omega) \); this is also easily deducible from the integral representation of functionals.

2. If \( X_t = C(\Omega_x) \) is an \( M \)-subspace \( X = C(\Omega) \), then \( \Omega_x \subset \Omega \) is not necessarily satisfied even if every functional \( \xi \in \mathcal{A} \) has an extension \( \xi \in \mathcal{A} \).

**References**


Repri par la redaction le 21, 9, 1956