Since the property (W) is preserved in the passage to the limit, and since the functional $\|x\|$ does not have the property (W) in any ball, we infer that $\|x\|$ is not a limit of polynomials in any ball.

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Some properties of the norm in F-spaces

by

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We deal in this paper with the properties of the norm in F-spaces[1]). In section 2 we give a construction of a norm equivalent to the basic norm and having very desirable properties. In sections 3 and 4 we give the characterizations of the spaces having some peculiar properties.

1. Let X be an F^*-space and let $\|x\|_1$ and $\|x\|_2$ be two norms defined on X.

DEFINITION 1. The norms $\|x\|_1$ and $\|x\|_2$ are said to be *equivalent* (in symbols $\|x\|_1 \sim \|x\|_2$) if for every sequence $(x_n) \subset X$ the condition

$$\lim_n \|x_n\|_1 = 0$$

is equivalent to

$$\lim_n \|x_n\|_2 = 0.$$

DEFINITION 2. The norm $\|x\|$ is called *monotone (strictly monotone), concave, of class C_k, of class C_∞, or analytic* if for every $x \in X$ the function $f_x(t) = \|tx\|$ is for $t > 0$ monotone (strictly monotone), concave, k times differentiable, infinitely differentiable or analytic respectively. A norm having all these properties except analycity will be said *to have the property* W_1.

DEFINITION 3. The norm $\|x\|$ is called *unbounded* (= *has the property* W_2) if the set of values of the functional $\|x\|$ is unbounded for $x \in X$.

Let (ϑ_n) be a sequence of positive numbers such that

$$\lim_n \vartheta_n = \infty.$$

DEFINITION 4. The norm $\|x\|$ is said *to have the rate of growth* (ϑ_n) if

$$\limsup_n \vartheta_n \left\| \frac{x}{n} \right\| < \infty \quad \text{for every} \quad x \in X.$$

[1]) Concerning the definition and basic properties of the F^*-spaces see [1] and [6].

DEFINITION 5. The norm is said *to have the property* W_3 if it has a rate of growth.

2. In this section we shall prove the following

THEOREM 1. *In every F^*-space with the norm $\|x\|$ there exists an equivalent norm having the property* W_1 [2]).

LEMMA 1. *In every F^*-space there exists a concave norm.*

Proof. From the continuity of multiplication by scalars it follows that the functional

$$\|x\|^* = \sup_{t\leqslant 1} \|tx\| \tag{1}$$

is a monotone norm equivalent to the norm $\|x\|$.

Let us write

$$\|x\|^{**} = \sup_n \sup_{\substack{a_1\geqslant 0, a_2\geqslant 0, \ldots, a_n \geqslant 0 \\ a_1+a_2+\ldots+a_n=n}} \frac{1}{n}\sum_{i=1}^{n} \|a_i x\|^*. \tag{2}$$

The formula (2) defines well the functional $\|x\|^{**}$. Indeed, by the triangle-inequality it follows that $\|kx\|^* \leqslant k\|x\|^*$ for $k = 1, 2, \ldots,$ whence by the monotonity of the norm $\|x\|^*$ it follows for $a \geqslant 0$ that

$$\|ax\|^* \leqslant (a+1)\|x\|^*,$$

which implies for $a_1+a_2+\ldots+a_n = n$, $a_i \geqslant 0$, $i = 1, 2, \ldots, n$,

$$\frac{1}{n}\sum_{i=1}^{n} \|a_i x_i\|^* \leqslant 2\|x\|^*.$$

Consequently, by (2) we get

$$\|x\|^* \leqslant \|x\|^{**} \leqslant 2\|x\|^*. \tag{3}$$

By (2) the functional $\|x\|^{**}$ satisfies the triangle inequality and by (3) we infer that $\|x\|^{**} = 0$ implies $x = \Theta$ [3]). Hence the functional $\|x\|^{**}$ is a norm; from (3) it immediately follows that $\|x\|^* \sim \|x\|^{**}$. It remains to prove that the norm $\|x\|^{**}$ is concave, *i. e.* that $\lambda \geqslant 0$, $\mu \geqslant 0$ implies

$$\|\lambda x\|^{**} + \|\mu x\|^{**} \leqslant 2\left\|\frac{\lambda+\mu}{2}x\right\|^{**}. \tag{4}$$

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[2]) Eidelheit and Mazur [2] have proved that in every F-space there exists an equivalent strictly monotone norm.

[3]) Θ denotes the neutral (zero) element of the space X.

By the definition of the norm $\|x\|^{**}$ we can choose, given any $\varepsilon > 0$, a positive integer n and positive numbers $a_1, a_2, \ldots, a_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ so that $\sum a_i = \sum \beta_i = n$ and

$$\|\lambda x\|^* \leqslant \frac{1}{n}\sum_{i=1}^{n} \|a_i x\|^* + \varepsilon, \qquad \|\mu x\|^* \leqslant \frac{1}{n}\sum_{i=1}^{n} \|\beta_i x\|^* + \varepsilon. \tag{5}$$

To see this let us observe that if the positive numbers $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_p$,

$$\sum_{i=1}^{r} \bar{a}_i = r, \qquad \sum_{j=1}^{p} \beta_j = p$$

are chosen so that

$$\|\lambda x\|^{**} \leqslant \frac{1}{r}\sum_{i=1}^{r} \|a_i x\|^* + \varepsilon, \qquad \|\mu x\|^* = \frac{1}{p}\sum_{i=1}^{p} \|\beta_i x\|^* + \varepsilon,$$

then it is sufficient to set $n = p\cdot r$ and $a_{jp+i} = \bar{a}_i$, $\beta_{ir+j} = \bar{\beta}_j$ for $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, p$.

Let us write for $i = 1, 2, \ldots, n$

$$\gamma_i = \frac{2\lambda}{\lambda+\mu} a_i, \qquad \gamma_{n+i} = \frac{2\mu}{\lambda+\mu}\beta_i. \tag{6}$$

We have $\gamma_j \geqslant 0$ for $j = 1, 2, \ldots, n$, and

$$\sum_{j=1}^{2n} \gamma_j = 2n,$$

whence

$$\frac{1}{2n}\sum_{j=1}^{2n} \left\|\gamma_j \frac{\lambda+\mu}{2} x\right\|^* \leqslant \left\|\frac{\lambda+\mu}{2}x\right\|^{**}.$$

On the other hand, by (5) and (6),

$$\frac{1}{2n}\sum_{j=1}^{2n} \left\|\gamma_j \frac{\lambda+\mu}{2} x\right\|^* = \frac{1}{2n}\sum_{i=1}^{n} \|a_i \lambda x\|^* + \frac{1}{2n}\sum_{i=1}^{n} \|\beta_i \mu x\|^*$$
$$\geqslant \frac{1}{2}(\|\lambda x\|^{**} + \|\mu x\|^{**}) - \varepsilon.$$

This implies inequality (4), q. e. d.

Remark. The above construction of a concave norm is equivalent to the following construction:

Let us consider the functions $f_x(t) = \|tx\|$. Let $f_x^{**}(t)$ be the smallest concave function such that $f_x^{**}(t) \geqslant f_x(t)$, then $\|y\|^{**} = f_y^{**}(1)$.

Let $C^k_{\langle 0,1\rangle}$ be the space of the functions $x = x(t)$ having the k-th derivative [4] continuous in the interval $\langle 0,1\rangle$. The spaces $C^k_{\langle 0,1\rangle}$ are B_0-spaces with the pseudonorms

(7) $$\|x\|^i_a = \sup_{t\in\langle a,1\rangle} |x^{(i)}(t)|, \qquad i = 0,1,\ldots,k,\ 0<a<1 \left(x^{(i)}(t) = \frac{d^ix(t)}{dt^i}\right).$$

Let (ε_i) be a sequence of positive numbers such that

(8) $$\sum_{i=1}^{\infty} \varepsilon_i = 1.$$

Let us set for $p = 1,2,\ldots,\ q = p,\ p+1,\ p+2,\ldots,$

(9) $$U_{p,q}(x) = \frac{1}{\varepsilon_p\varepsilon_{p+1}\ldots\varepsilon_q}\int_{1-\varepsilon_p}^{1}\ldots\int_{1-\varepsilon_q}^{1} x(ts_p\ldots s_q)\,ds_q\ldots ds_p.$$

LEMMA 2. *The sequences $\{U_{p,q}\}$ have the following properties:*

(α) *For every $x\in C^k_{\langle 0,1\rangle}$ there exists*

$$\lim_{q=\infty} U_{p,q}(x^{(k)}) = U_{p,\infty}(x^{(k)}) \quad \textit{for} \quad p = 1,2,\ldots,\ k = 0,1,2,\ldots$$

(β) $U_{p,q}(x)\in C^{q-p+1}_{\langle 0,1\rangle}$ $(q = p,\ p+1,\ldots)$.

(γ) *The operation $U_{1,\infty}$ is linear and maps the space $C_{\langle 0,1\rangle}$ into the space $C^{\infty}_{\langle 0,1\rangle}$.*

Proof. By (8) the infinite product

$$\prod_{i=1}^{\infty}(1-\varepsilon_i)$$

is convergent and

$$\prod_{i=1}^{\infty}(1-\varepsilon_i) = \delta < 1.$$

Let $x\in C^k_{\langle 0,1\rangle}$, then by (7), (9) and the inequality

$$\int_a^b f(t)\,dt \leqslant (b-a)\sup_{t\in(a,b)}|f(t)|$$

we have

(10) $$\|U_{p,q}(x)\|^i_a \leqslant \|x\|^i_{a\delta} \quad \text{for} \quad i = 1,2,\ldots,k.$$

[4] By *0-times differentiable function* we mean the continuous functions. We shall write $C_{\langle 0,1\rangle}$ instead of $C^0_{\langle 0,1\rangle}$.

Let $q_2 > q_1$; then

$$\|U_{p,q_1}(x) - U_{p,q_2}(x)\|^i_a = \|U_{p,q_1}(x - U_{q_1+1,q_2}(x))\|^i_a \leqslant \|x - U_{q_1+1,q_2}(x)\|_{a\delta}$$
$$= \sup_{t\in\langle a\delta,1\rangle}\frac{1}{\varepsilon_{q_1+1}\ldots\varepsilon_{q_2}}\int_{1-\varepsilon_{q_1+1}}^{1}\ldots\int_{1-\varepsilon_{q_2}}^{1}|x^{(i)}(t) - x^{(i)}(t\cdot s_{q_1+1}\ldots s_{q_2})|\,ds_{q_2}\ldots ds_{q_1+1};$$

if $q_1, q_2\to\infty$, then in virtue of the relation

$$\lim\prod_{j=q_1+1}^{j=q_2}(1-\varepsilon_j) = 1$$

and the continuity of the function $x^{(i)}(t)$ we infer that

$$|x^{(i)}(t) - x^{(i)}(t\cdot s_{q_1+1}\ldots s_{q_2})|\to 0$$

uniformly for $t\in\langle a\delta,1\rangle$, $1-\varepsilon_j\leqslant s_j\leqslant 1$, $j = q_1+1,\ldots,q_2$, whence from the completeness of the space $C^k_{\langle 0,1\rangle}$ we deduce that the sequences $\{U_{p,q}\}$ have the property (α).

We shall prove the property (β) by induction with respect to the difference $r = p-q$. For $r = 0$ the theorem follows from the identity (11) by the theorems on the differentiation of the integral of a continuous function

(11) $$\int_{1-\varepsilon}^{1} x(t\cdot s)\,ds = \frac{1}{t}\int_{(1-\varepsilon)t}^{t} x(u)\,du.$$

Suppose that the theorem is true for $r = r_0-1$, *i. e.* that

$$\frac{d^{r_0+1}}{dt^{r_0+1}}\frac{1}{\varepsilon_{p+1}\ldots\varepsilon_{p+r_0+1}}\int\ldots\int x(t\cdot s_{p+1}\ldots s_{p+r_0+1})\,ds_{p+1}\ldots ds_{p+r_0+1}$$
$$= \Phi(t)\in C_{\langle 0,1\rangle}.$$

Differentiating under the sign of the integral we get

$$\frac{d^{r_0+1}}{dt^{r_0+1}}U_{p,p+r_0+1}(x) = \frac{1}{\varepsilon_p}\int_{1-\varepsilon_p}^{1}\Phi(t\cdot s_p)\,ds_p\in C^1_{\langle 0,1\rangle},$$

i. e. $U_{p,p+r_0+1}\in C^{r_0+2}_{\langle 0,1\rangle}$, q. e. d.

The property (γ) follows by the identity

$$U_{1,q}(x) = U_{p+1,q}(U_{1,p}(x)),$$

whence by the property (β) it follows that $U_{1,q}(x)\in C^{p+1}_{\langle 0,1\rangle}$ for $q\geqslant p$.

By the property (α) we get for $p = 1, 2, \ldots$, passing to the limit as $q \to \infty$,

$$\lim_q U_{1,q}(x) = U_{1,\infty}(x) = \lim_q U_{p+1,q}\big(U_{1,p}(x)\big) = U_{p+1,\infty}\big(U_{1,p}(x)\big) \in C^{p+1}_{(0,1)},$$

whence $U_{1,\infty}(x) \in C_{(0,1)}$.

The linearity of the operation $U_{1,\infty}(x)$ follows in virtue of (10) from the inequality

$$\|U_{1,\infty}(x)\|^p_a = \|U_{p+1,\infty}\big(U_{1,p}(x)\big)\|^p_a \leqslant \|U_{1,p}(x)\|^p_{a\delta}$$

and from the fact that the operation $U_{1,p}(x)$ maps linearly the space $C_{(0,1)}$ into the space $C^p_{(0,1)}$, $p = 1, 2, \ldots$, which may be proved by induction similarly to the property (β).

Proof of Theorem 1. Let $\|x\|^{**}$ be a concave (and thus monotone) norm defined in Lemma 1. Let us set $|x| = U_{1,\infty}(\|x\|^{**})$.

By Lemma 2 and the definition of the sequence $\{U_{1,q}\}$ we see that the functional $|x|$ is a convex norm of class C_∞. It remains to prove that $|x| \sim \|x\|^{**}$. We have for $q = 1, 2, \ldots$

$$\inf_{t \in \langle\delta,1\rangle} \|tx\|^{**} \leqslant U_{1,q}(\|x\|^{**}) \leqslant \sup_{t \in \langle\delta,1\rangle} \|tx\|^{**}.$$

Passing to the limit we get

$$\inf_{t \in \langle\delta,1\rangle} \|tx\| \leqslant |x| = \sup_{t \in \langle\delta,1\rangle} \|tx\|^{**}.$$

The norm $\|x\|^{**}$ being monotone, we see that

$$\inf_{t \in \langle\delta,1\rangle} \|tx\|^{**} \leqslant \|\delta x\|^{**};$$

on the other hand,

$$\sup_{t \in \langle\delta,1\rangle} \|tx\|^{**} = \|x\|^{**}.$$

Finally,

$$\|\delta x\|^{**} \leqslant |x| \leqslant \|x\|^{**},$$

which implies $|x| \sim \|x\|^{**}$, q. e. d.

Remark. To obtain a strictly monotone norm it is sufficient to set

$$|x|^* = \int_0^1 |tx|\, dt.$$

We omit the easy proof based on the computation of

$$\frac{d}{ds}\int_0^1 |tsx|\, dt.$$

Problem. Does there exist in every F^*-space a norm equivalent to the analytic norm?

3. In this section we characterize these F^*-spaces in which there exists a norm with the property W_2.

THEOREM 2. *The following condition is necessary and sufficient for the existence of an unbounded norm in an F^*-space:*

(*) *there exists a neighbourhood U such that for ~~no~~ positive integer*

$$U^n = \underbrace{U \oplus U \oplus \ldots \oplus U}_{n \text{ times}} = X\ ^{5}).$$

Proof. Necessity. Let us suppose that (*) is not satisfied and $\|x\|$ has the property W_2. Let $U_1, U_2, \ldots$ be a decreasing sequence of neighbourhoods such that

$$\bigcap_{n=1}^{\infty} U_n = \{\Theta\}.$$

We have

$$\sup_{x \in U_i} \|x\| = \infty \quad \text{for} \quad i = 1, 2, \ldots$$

Indeed, in the contrary case the identity $U_i^{n_i} = X$ would imply for some n_i

$$\prod_{x \in X} \|x\| \leqslant n_i \sup_{x \in U_i} \|x\| < \infty$$

which is impossible. Therefore there exist for $i = 1, 2, \ldots$ elements $x_i \in U_i$ such that $\|x_i\| = 1$; this, however, is impossible, for the sequence of neighbourhoods U_n is chosen so that

$$\lim_{i=\infty} x_i = 0,$$

q. e. d.

Sufficiency. The construction of the norm proceeds similarly to the proof of Theorem 4 in [5] (see also [3]) the only difference is that the hypothesis $U(t) = E$, $t = 1$, is to be replaced by $U^{(n)} = U^n$.

Remark 1. From the above construction it follows that if there exists a bounded neighbourhood of Θ, then there exists a bounded norm, *i. e.* the following condition is satisfied:

$$(\text{the set } Z \text{ is bounded}) \equiv \big(\sup_{z \in Z} \|z\| < \infty\big).$$

[5]) $A \oplus B = E\{x = a + b, \text{ where } a \in A,\ b \in B\}$.

Remark 2. In the proof of Theorem 2 only the fact that X is a metric connected Abelian group was used; thus Theorem 2 is true for metric connected Abelian groups (in this case the norm is understood as $\varrho(x,0)$ where $\varrho(x,y)$ is the distance in the group X).

Remark 3. If the space X fulfils the condition (*), then starting in the proof of Theorem 1 from the unbounded norm one can easily show that there exists in X a norm with the properties W_1 and W_2.

Remark 4. If the space X has not the property (*), then no non-trivial linear functional (*i. e.* different from the functional $\varphi(x)\equiv 0$) exist in X.

Indeed, if there is a non-trivial linear functional in X, then the norm $\|x\|^* = \|x\|+|\varphi(x)|$ has the property W_2 and obviously $\|x\|\sim\|x\|^*$.

The example of the space L^p of the functions $x=x(t)$ integrable in $\langle 0,1\rangle$ with the p-th power (with the norm

$$\|x\| = \int_0^1 |x(t)|^p\,dt, \qquad 0<p<1)$$

shows that the converse theorem is not true. Indeed, the norm L^p is unbounded, but no non-trivial linear functionals exist in L^p, $0<p<1$ (see [4]).

It is easy to show that the space S of all measurable functions in $\langle 0,1\rangle$, with the norm

$$\|x\| = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\,dt,$$

has not the property (*), whence in S there are neither non-trivial linear functionals nor unbounded equivalent norms.

4. In this section we shall characterize those F-spaces in which there exists an equivalent norm having the property W_3.

THEOREM 3. *There exists in an F-space X a norm having the property W_3 if and only if there exists a bounded*[6]) *neighbourhood of Θ*[7]).

Proof. Sufficiency. Let U be a bounded neighbourhohd. Let us write

$$r_n = \sup_{x\in U}\left\|\frac{x}{n}\right\|.$$

[6]) The set Z is called *bounded* if $\prod_{\varepsilon>0}\sum_{\delta>0}\prod_{z\in Z}\prod_{\eta\leqslant\delta}\|\eta z\|\leqslant\varepsilon$.

[7]) During the print of this paper it was proved that there exists in the space X a bounded neighbourhood if and only if there exists for some $0<p\leqslant 1$ an equivalent p-homogeneous norm (*i. e.* the norm satisfying the condition $\|tx\|=|t|^p\|x\|$).

The boundedness of the set U implies

$$\lim_n r_n = 0.$$

Obviously $r_n>0$ for $n=1,2,\ldots$ If $\vartheta_n = 1/r_n$, then the sequence (ϑ_n) is a rate of growth of the norm $\|x\|$. Indeed, let $x\in X$ and let n_0 be the smallest positive integer such that $x/n_0\in U$ (such integer exists, for $x/n\to\Theta$ and U is a neighbourhood of Θ). For $n=1,2,\ldots$ we have

$$\vartheta_n\left\|\frac{x}{n}\right\|\leqslant n_0\vartheta_n\left\|\frac{x/n_0}{n}\right\|\leqslant n_0.$$

Necessity. We prove first the

LEMMA. *The set Z is bounded if and only if*

$$(**)\qquad \prod_{\varepsilon>0}\sum_{\delta>0}\prod_{x\in Z}\|\delta x\|<\varepsilon.$$

The necessity of the condition (**) is trivial, only the sufficiency must be proved. Let $\|x\|^*$ be a monotone norm equivalent to the norm $\|x\|$. Boundedness being an invariant property under isomorphisms, it is sufficient to show that the set Z is bounded under the norm $\|x\|^*$. Let $\varepsilon>0$; since $\|x\|\sim\|x\|^*$, there exists an $\varepsilon_1>0$ such that $\|x\|<\varepsilon_1$ implies $\|x\|^*<\varepsilon$. By (**) there is a $\delta>0$ such that $\|\delta x\|<\varepsilon_1$ for $x\in Z$, whence $\|\delta x\|^*<\varepsilon$.

The monotonity of the norm $\|x\|^*$ implies $\|\eta x\|^*<\varepsilon$ for $\eta<\delta$, whence the set Z is bounded, q. e. d.

Let the sequence (ϑ_n) be a rate of growth for the norm $\|x\|$. Let us set for $k=1,2,\ldots$

$$Z_k = \bigcap_{n=1}^{\infty}\mathop{E}_{x}\left\{\vartheta_n\left\|\frac{x}{n}\right\|\leqslant k\right\}.$$

These sets are closed and

$$\bigcup_{k=1}^{\infty} Z_k = X,$$

whence by Baire's theorem there is an index k_0, an $\varepsilon>0$ and a point x_0 such that $\|x-x_0\|<\varepsilon$ implies $\vartheta_n\|x/n\|\leqslant k_0$ for $n=1,2,\ldots$ Thus for every $\|y\|<\varepsilon$ and $n=1,2,\ldots$

$$\vartheta_n\left\|\frac{y}{n}\right\|\leqslant\vartheta_n\left\|\frac{x_0+y}{n}\right\|+\vartheta_n\left\|\frac{x_0}{n}\right\|\leqslant 2k_0,$$

whence $\|y/n\|\leqslant 2k_0/\vartheta_n$.

Since

$$\lim_{n=\infty} (2k_0/\vartheta_n) = 0,$$

the neighbourhood

$$U = \underset{y}{E}\{\|y\| < \varepsilon\}$$

satisfies the condition (**), *i. e.* is bounded.

COROLLARY. *From the proof of Theorem 3 it follows directly that if in an F-space a norm has the property* W_3, *then an equivalent norm has it also.*

Remark 1. The above theorem is false in the case of the F^*-space. An example is provided by the space K of all the sequences $x = (\xi_n)$ almost all elements of which vanish, the norm being

$$\|x\| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k|}{1+|\xi_k|}.$$

It is easily verified that the sequence $\vartheta_n = n$ is a rate of growth for the norm $\|x\|$.

Since K, being a B_0^*-space, is not a B^*-space (see [6]) there are not any bounded neighbourhoods in K.

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Spaces of continuous functions (II)
(On multiplicative linear functionals over some Hausdorff classes)

by

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S. Mazur [5] has proved that with every bounded sequence $\{x_n\}$ a real number $\operatorname{Lim}_n x_n$ can be associated in such a way that $\operatorname{Lim} x_n$ is equal to the usual limit of a subsequence of $\{x_n\}$; consequently

(1) $$\underline{\lim}\, x_n \leqslant \operatorname{Lim} x_n \leqslant \overline{\lim}\, x_n,$$

(2) $$\operatorname{Lim}(ax_n+by_n) = a\operatorname{Lim} x_n + b\operatorname{Lim} y_n,$$

(3) $$\operatorname{Lim}(x_n y_n) = \operatorname{Lim} x_n \cdot \operatorname{Lim} y_n.$$

In this note a construction of generalized limits for some classes of functions is given. This construction is non-effective, just as those of Mazur; it is based on the theorem of Kakutani on the representation of abstract (M)-spaces. It is easily seen that this limit can also be derived from the theorem of Tychonoff, but I think that the way which I have chosen leads to more consequences.

The generalization of the theorem of Mazur to the case of real-valued, bounded functions defined on $\langle 0, 1\rangle$ is trivial, *e. g.*, we can put

$$\operatorname*{Limes}_{t\to t_0} x(t) = \operatorname*{Lim}_n x(t_n)$$

where Lim denotes an arbitrary limit of Mazur and $t_n \to t_0$. The functional "Limes" constructed in the Theorems 1, 1a, 1b and 2 satisfies also some additional conditions. It can be considered as a solution of the following problem: given a space of equivalence classes of functions how to assign in a reasonable way the value to every function at every point.

The second part of this paper contains some applications (the existence of certain multiplicative measures and a negative solution of two questions concerning the extension of linear functionals).