d’ordre $p$ a exactement $p$ solutions linéairement indépendantes. L’ensemble des solutions est donc, dans notre construction algébrique, plus riche que dans le calcul opérationnel. Par exemple l’équation

$$x''(t) + x'(t) = 0$$

n’a pas de solutions, sauf 0, parmi les fonctions opérationnelles, tandis qu’elle a deux solutions linéairement indépendantes dans l’espace $F$.

En particulier, les fonctions $x''$ et $x'$ sont des solutions linéairement indépendantes de l’équation (41). La théorie des équations différentielles est, dans cette interprétation, tout à fait analogue à la thèorie classique des équations différentielles ordinaires, mais l’ensemble des fonctions exponentielles est beaucoup plus riche, même plus riche que dans le calcul opérationnel. Par l’adjonction d’un élément transcendant on obtient une construction analogue au calcul opérationnel, mais sur un niveau plus haut, une sorte de “calcul opérationnel du calcul opérationnel”.

**Travaux cités**


A property of multilinear operations

by

A. PELCZYŃSKI (Warszawa)

1. In this paper I prove a special property of multilinear operations (see [3]) defined in $B^*$-spaces. 1 It appears that if the space has some additional properties (e. g. weak completeness), these operations are continuous with respect to a sequential topology weaker than the topology induced by the norm (it follows hence in particular that in the space $C_0$ the polynomials are weakly continuous). For a class of spaces the functional $||x||$ is not continuous with respect to this topology. From this fact it follows that the functional $||x||$ cannot be uniformly approximated in these spaces by polynomials. This result partly coincides with some results of J. Kurzweil [2]. In the sequel I give an example of a non-separable space, in which the functional $||x||$ is not representable in any ball as the pointwise limit of polynomials.

I am indebted to Professor S. Mazur for calling my attention to this problem and the aid which he has given me in solving it.

2. Let $E$ be a $B^*$-space, $s$ — a real number from the interval $(0, 1)$, $(x_k)$ — a sequence composed of $+1$’s or $-1$’s. I shall define in $E$ a sequential topology $\tau_s$.

**Definition** 1. The sequence $(x_n) \subset E$ is $\tau_s$-convergent to $0$ if there is a constant $C$ such that for $k = 1, 2, \ldots$, for arbitrary different indices $n_1, n_2, \ldots, n_k$ and for every sequence $(x_k)$ the inequality

$$||x_{n_1}x_{n_2} + \cdots + x_{n_k}|| \leq C,$$

is satisfied.

The sequence $(x_n)$ is $\tau_s$-convergent to the element $x$ if the sequence $(x_n - x_k)$ is $\tau_s$-convergent to $0$. 2 This fact will be denoted in symbols:

$$x_n \rightarrow x.$$

1) i. e. in (not necessarily complete) linear subsets of $B$-spaces.

2) $0$ denotes the null element of the space $E$.

3. It follows in particular from (1) that $x_n \rightarrow x_k$ implies $x_n \rightarrow x_k$; hence follows the unicity of the $\tau_s$-limit.
Definition 2. The operation $A(x_1, x_2, \ldots, x_r)$ defined on the space $E^r = E \times E \times \cdots \times E$, taking on values from a $B^a$-space $E_b$, will be called $\tau_b$-continuous at the point $(x_1, x_2, \ldots, x_r)$ if for every sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ the relations $x_{i_k} \to x_k$ $(i = 1, 2, \ldots, r)$ imply the convergence of the sequence $(A(x_{i_1}, x_{i_2}, \ldots, x_{i_r}))$ to $A(x_1, x_2, \ldots, x_r)$ with respect to the norm. The operation $A(x_1, x_2, \ldots, x_r)$ will be called $\tau_b$-continuous if it is $\tau_b$-continuous at every point of the space $E^r$.

Definition 3. A $B$-space $E_b$ will be said to have the rank $r$ (to have a loose rank $r$) if for every $0 \leq c < r$ (or $c < r$) the functional $[x_c]$ is a $\tau_b$-continuous operation.

Theorem 1. Let $A(x_1, x_2, \ldots, x_r)$ be an $r$-linear operation from the space $E^r$ to a space $E_b$ having the (loose) rank $a_1$. Then $A$ is $\tau_b$-continuous if $r \leq a_1$ ($r < a_1$).

The proof of Theorem 1 is somewhat complicated and needs some auxiliary concepts and symbols.

If we fix in an $r$-linear operation $A(x_1, x_2, \ldots, x_r)$ the elements $x_{a_1}, x_{a_2}, \ldots, x_{a_p}$ on the places $a_1, a_2, \ldots, a_p$, $p \leq r$, we obtain an $(r-p)$-linear operation which will be denoted by

$$B_{a_1, a_2, \ldots, a_p}^{a_1, a_2, \ldots, a_p} \text{ and }$$

Let $(x_n) \subseteq E_b, \delta > 0$; we shall write

$$Z_b(A) = \{ (x_{a_1}, x_{a_2}, \ldots, x_{a_r}) | \| A(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) \| > \delta \}.$$

By $N_G$ will be denoted the set of all systems of indices $(a_i)$ $=$ $(a_1, a_2, \ldots, a_n)$ where $n_1 = 1, 2, \ldots$ for $i = 1, 2, \ldots, n$.

Lemma 1. Let a sequence $(x_n) \subseteq E_b$ have the following properties:

(a) for any $\eta > 0$, the indices $a_1, a_2, \ldots, a_p$ and elements $x_{a_1}, x_{a_2}, \ldots, x_{a_p}$ being fixed, the set

$$Z_b(B_{a_1, a_2, \ldots, a_p}^{a_1, a_2, \ldots, a_p})$$

is finite;

(b) the set $Z_b(A)$ is infinite for some $\delta > 0$.

Theorem 2. Then there exists a sequence $(n_i) \subseteq N$, $i = 1, 2, \ldots$, of systems of indices such that

(a) $n_i^a \notin Z_b(A)$ $(v = 1, 2, \ldots)$,

(b) $n_i^n > n_i^{v-1}$ $(v = 2, 3, \ldots)$; $i = 1, 2, \ldots, r$,

(c) if the numbers $v_1, v_2, \ldots, v_r$ are not all equal to one another, then

$$\| A(x_{n_i}, x_{n_i}, \ldots, x_{n_i}) \| \leq \frac{1}{\max \| (x_{n_i}) \|^2}.$$

Proof. The sequence $(n_i)$ will be defined by induction. As $(n_1)$ we choose an arbitrary system of indices belonging to the set $Z_b(A)$. Suppose we have already defined the systems $(n_i)$ for $i < v, v > 1$. To define the $v$-th system let us set $n = v - 2^{v-1}$; let us consider all the possible operations

$$B_{s_1, s_2, \ldots, s_p}^{s_1, s_2, \ldots, s_p},$$

where $1 \leq s_i < v, i = 1, 2, \ldots, p, p = 1, 2, \ldots, r$. The number of these operations is finite. The sets

$$Z_b(B_{s_1, s_2, \ldots, s_p}^{s_1, s_2, \ldots, s_p})$$

being finite by (a), there exist quantities $M_1, M_2, \ldots, M_r$, such that $n_{p_1} > M_{p_1}, n_{p_2} > M_{p_2}, \ldots, n_{p_{v-2}} > M_{p_{v-2}}$ implies

$$n_{p_{v-2}} > n_{p_{v-2}} > M_{p_{v-2}}$$

where $s_i > \eta_i, s_j > \eta_j, s_k > \eta_k, \ldots, s_l > \eta_l, v = 1, 2, \ldots, r$. Let $N_v = \max(n_i)$ for $i = 1, 2, \ldots, r$. There is only a finite multitude of systems $(n_i) \subseteq Z_b(A)$ such that for a certain $i$ the inequality $n_i < N_v$ holds. Indeed, in the contrary case there would exist an index $v$ such that for infinitely many systems $(n_i) \subseteq Z_b(A)$ the inequality $n_i < N_v$ would be satisfied. Since there are only finitely many positive integers not greater than $N_v$, there must exist an $v < N_v$ such that $n_i = v$.

1) By $O$-linear operations we mean the constant operations.

2) On principle it would be desirable to make evident in the notation the dependence of $B_{x_{a_1}, x_{a_2}, \ldots, x_{a_p}}^{x_{a_1}, x_{a_2}, \ldots, x_{a_p}}$ on the operation $A$. It is sufficient for this proof to suppose that $A$ is a distributive operation on a linear set of an $B$-space.
is satisfied for infinitely many systems \((u_{n_k})_n \in Z(A)\); this implies, however, the infiniteness of the set \(Z(A)\), which is contrary to (a).

The set \(Z(A)\) being infinite, there exist systems \((u_{n_k})_n \in Z(A)\) of indices such that \(n_k > N\) for \(k = 1, 2, \ldots, r\). We choose as \((u_{n_k})_n\) any of these systems. It is easy to see that all the desired conditions are satisfied.

**Lemma 2.** If the conditions of Theorem 1 are satisfied and the sequence \((x_{n_k})_n\) is \(\tau_r\)-convergent to \(\Theta\), then the set \(Z(A)\) is finite for every \(\delta > 0\).

We prove the lemma by induction with respect to \(r\). Let \(A\) be linear \((r = 1\)). We have by (1)

\[
\left| r_1 A(x_{n_k}) + r_2 A(x_{n_k}) + \cdots + r_k A(x_{n_k}) \right| \leq \left( r_1 ||x_{n_k}|| + r_2 ||x_{n_k}|| + \cdots + r_k ||x_{n_k}|| \right) \leq C ||A|| \cdot L_1,
\]

whence the sequence \((A(x_{n_k}))_n\) is \(\tau_r\)-convergent to \(\Theta_1\). The space \(E_k\), having the rank \(s_1\) (the loose rank \(s_1\)), we infer from the inequality \(\tau^{s_1} = s_1 \leq s_1\) that

\[
\lim_{n \to \infty} ||A(x_{n_k})|| = 0.
\]

Now let us suppose Lemma 2 to be true for \(r < r_1 > 1\) and \(x_{n_k} \in x_{n_k} (x_{n_k} \neq s_1)\). Let the operation \(A\) be \(r\)-linear and let the set \(Z(A)\) be infinite for this operation, for some \(\delta > 0\). We can easily verify that the sequence \((x_{n_k})_n\) satisfies the hypotheses of Lemma 1. Thus, we can choose a sequence of systems of indices \((u_{n_k})_n\) with the properties (a), (b), and (c).

We set for \(k = 1, 2, \ldots\)

\[
y = \sum_{j=1}^{k} x_{n_{k_j}} \quad \text{for} \quad i = 2, 3, \ldots, r,
\]

\[
z = \sum_{j=1}^{k} x_{n_{k_j}} \tau_1
\]

where \(\mu_1 < \mu_2 < \cdots < \mu_r = r\).

Then we write

\[
A = \sum_{i \in \mathbb{N}} E_{\max_1 \mathbb{N}} (a_i \in \mathbb{N}) \cdot \sum_{i \in \mathbb{N}} (a_i < \mathbb{N}) \cdot \text{max}_1 \mathbb{N} = 0,
\]

\[
A = \sum_{i \in \mathbb{N}} (a_i \in \mathbb{N}) \cdot \text{max}_1 \mathbb{N} = 0.
\]

By (2)- (7) we have

\[
||A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r)|| = \sum_{i=1}^{k} \sum_{j=1}^{k} A(x_{n_{k_j}}, x_{n_{k_j}}),
\]

\[
||A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r)|| = \sum_{i=1}^{k} \sum_{j=1}^{k} A(x_{n_{k_j}}, x_{n_{k_j}}) \leq \sum_{i=1}^{k} \sum_{j=1}^{k} ||A(x_{n_{k_j}}, x_{n_{k_j}})|| = 1,
\]

\[
\frac{1}{\sum_{i=1}^{k} \sum_{j=1}^{k} ||A(x_{n_{k_j}}, x_{n_{k_j}})||} = 1.
\]

From (1) it follows that

\[
||A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r)|| = \frac{1}{\sum_{i=1}^{k} \sum_{j=1}^{k} ||A(x_{n_{k_j}}, x_{n_{k_j}})||} \leq \frac{1}{\sum_{i=1}^{k} \sum_{j=1}^{k} C ||A|| L_1^r}.
\]

The above inequalities imply that the sequence \((A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r))_n\) is \(\tau_{r}\)-convergent to \(\Theta_1\). The space \(E_k\) having the (loose) rank \(r_1\), it follows from \(x_{n_k} \in x_{n_k} (x_{n_k} \neq s_1)\) that

\[
\lim_{n \to \infty} ||A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r)|| = 0,
\]

which is impossible, for \(u_{n_k} \in Z(A)\). Thus the hypothesis that the set \(Z(A)\) is infinite leads to a contradiction.

**Proof of Theorem 1.** Lemma 2 implies directly that under the hypotheses of Theorem 1 every \(r\)-linear operation is \(\tau_r\)-continuous at \(\Theta\). To see this, it suffices to notice that the \(\tau_r\)-convergence of the sequences \((x_{n_k}^i)\) to \(\Theta\) for \(i = 1, 2, \ldots, r\) implies the \(\tau_{r}\)-convergence of the sequence \((x_{n_k}^1, x_{n_k}^2, x_{n_k}^3, x_{n_k}^4, x_{n_k}^5, x_{n_k}^6)\) to \(\Theta\), and then to apply Lemma 2.

The proof of \(\tau_r\)-continuity at an arbitrary point will be proved by induction. This fact is trivial for \(0\)-linear (= constant) operations. Suppose that it is true for \(p\)-linear operations, \(0 \leq p < r\). Let \(x_{n_k}^i \to x_{n_k}^i \) for \(i = 1, 2, \ldots, r\). We shall use the identity

\[
A(x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^r) = \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} \sum_{k=1}^{r} E_{\max_1 \mathbb{N}} (a_i < \mathbb{N}) \cdot a_i = 0,
\]

where \(\beta_i = c_i, \ i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, r-p\).
A property of multilinear operations

A theorem of Orlicz [4] states that if in a weakly complete space the sequence \((x_n)\) satisfies the condition (0), then the series
\[\sum_{n=1}^{\infty} x_n\]
converges unconditionally. It follows that \(\|x_n\| \to 0\) in our case, q. e. d.

4. The space \(\mathcal{C}\) has the following property, easy to verify:
\[(w_1)\] Every sequence \((x_n)\) in \(\mathcal{C}\) weakly convergent to \(\theta\) contains a subsequence \(\tau\)-convergent to \(\theta\).

Using this property and the result of 4.3 we find that
Any \(r\)-linear operation from the space \(\mathcal{C}\) to a weakly complete space \(E_r\)
is weakly continuous for \(r = 0, 1, \ldots\)

It follows in particular that multilinear forms (whence also the polynomials) defined on the Cartesian product of the space \(\mathcal{C}\) are weakly continuous functionals. The last result was obtained in another way by W. Bogdanowicz \(^1\).

5. The corollary obtained in 4.3 implies some negative results concerning the approximation of functionals by polynomials.

5.1. I shall need the following

**Definition 4.** The sequence \((x_n)\) is called essentially \(\tau\)-convergent to \(x_0\) if it is \(\tau\)-convergent to \(x_0\) but does not converge to \(x_0\) in the norm topology.

**Theorem 2.** If there exist in the space \(E\) essentially \(\tau\)-convergent sequences, then the functional \(\|x\|\) cannot be uniformly approximated by polynomials in any ball with centre \(\theta\).

**Proof.** It is easily shown that if there exist in \(E\) essentially \(\tau\)-convergent sequences, then every ball \(B(\theta, r)\) with centre \(\theta\) and radius \(r > 0\) contains a sequence \((x_n)\) with the following properties:

1° \(x_n \to \theta\).
2° \(\|x_n\| = r/2\) for \(n = 1, 2, \ldots\)

Indeed, let \((y_n)\) be essentially \(\tau\)-convergent to \(\theta\). Since
\[\limsup_{n \to \infty} \|y_n\| > 0,
\]
we can choose an increasing sequence of indices \((m_n)\) such that \(\|y_{m_n}\| > \delta\) for \(n = 1, 2, \ldots\); then we set \(x_n = (r/2\|y_{m_n}\|)y_{m_n}\).

Suppose that there exists a sequence \( P_n(x) \) of polynomials uniformly convergent to the functional \([x]\) in \( K(\Theta, r)\). Then there must exist an \( M \) such that

\[
|P_n(x) - [x]| < r/4
\]

for every \( x \in K(\Theta, r) \) and \( m > M \). In particular

\[
|P_n(x_n) - [x_n]| < r/4 \quad (n = 1, 2, \ldots).
\]

It follows that \( P_m(x_n) \geq r/4 \quad (n = 1, 2, \ldots, m > M) \). The polynomials are, however, \( \tau_0 \)-continuous, whence \( \lim P_m(x_n) = P_m(\Theta) \) for \( m = 1, 2, \ldots \), and this implies \( |P_m(\Theta)| \geq r/4 \) for \( m > M \) which is contrary to the hypothesis

\[
\lim_m P_m(\Theta) = |[\Theta]| = 0.
\]

Remark. The space \( C(Q) \) of continuous real functions defined on a compact infinite metric space \( Q \) contains always essentially \( \tau_0 \)-convergent sequences.

This follows from the fact that the space \( C(Q) \) contains a subspace equivalent to the space \( e_0 \) of numerical sequences convergent to 0 and in this space the sequence \( (e_n) \) where \( e_n = (0, 0, \ldots, 1, 0, 0, \ldots) \) is essentially \( \tau_0 \)-convergent.

5.2. Let \( c(0, 1) \) denote the set of all real functions \( x = x(t) \) defined for \( t \in (0, 1) \), such that for every \( \delta > 0 \) the set

\[
E\{x(t) > \delta\}
\]

is finite. It is easily seen that, with the usual definition of addition and multiplication by scalars, \( c(0, 1) \) is a B-space, the norm being

\[
|\langle x \rangle| = \sup_{t \in (0, 1)} |x(t)|.
\]

THEOREM 3. It is impossible to represent in the space \( c(0, 1) \) the functional \([x]\) as a limit of a sequence of functionals in any ball.

Proof. Let \( \Phi \) stand for the family of functionals \( e_\alpha, \alpha \in (0, 1) \) defined as follows:

\[
e_\alpha(t) = \begin{cases} 0 & \text{for } s \neq t, \\ 1 & \text{for } s = t. \end{cases}
\]

One can easily verify that this family has the following properties:

(a) Every element \( x \) of the space \( C(0, 1) \) may be represented uniquely in the form

\[
x = \sum_{t \in \mathbb{N}} t_i e_{t_i} \quad \text{where } t_i \neq 0
\]

and \( \langle e_\alpha \rangle \in C(0, 1) \) is a finite or infinite sequence. The set of finite linear combinations of elements of \( \Phi \) is therefore dense in \( c(0, 1) \).

(b) The smallest linear space spanned upon any denumerable subfamily of \( \Phi \) is equivalent to the space \( e_0 \).

(c) For every sequence \( (\alpha_i) \in \langle 0, 1 \rangle \) \( \alpha_i \neq \alpha_j \) for \( i \neq j \), the sequence \( (\alpha_i) \) is \( \tau_0 \)-convergent to \( \Theta \).

LEMMA. For every homogeneous polynomial \( P(x) \) there exists a separable set \( T \subset (0, 1) \), such that \( \not\exists \Gamma \) implies \( P(x + \lambda x) = P(x) \) for every \( x \in c_0(0, 1) \) and every real \( \lambda \).

Proof of the lemma. Let \( A(x^1, x^2, \ldots, x^r) \) be the (r-linear) generating form of \( P(x) \). By \( \gamma \) and \( \theta \) it follows from Lemma 2 (p. 176) that the set \( Z(A) \) is finite for every sequence \( \alpha_n \) of \( (\alpha_n) \in (0, 1) \), \( \alpha_i \neq \alpha_j \) for \( i \neq j \) and every \( \delta > 0 \). It implies the existence of an at most denumerable set \( T \) such that if at least one of the numbers \( e_1, e_2, \ldots, e_r \) is not in \( T \), then

\[
A(e_1, e_2, \ldots, e_r) = 0.
\]

Therefore, for \( x \in T \)

\[
E^x_j (x^1, x^2, \ldots, x^{r-1}) = 0 \quad (j = 1, 2, \ldots, r).
\]

Indeed, it follows from (10) that formula (11) is true if \( x^1, x^2, \ldots, x^{r-1} \) are finite linear combinations of the elements of \( \Phi \). The validity of formula (11) for arbitrary points \( x^1, x^2, \ldots, x^{r-1} \) follows from (a) and from the continuity of the operation \( E^x_j \).

The lemma follows from (11) and the identity

\[
P(x + \lambda x) = A(x + \lambda x, x + \lambda e_1, \ldots, x + \lambda e_r)
\]

\[
= A(x, x, \ldots, x) + \sum_{1 \leq i < j < r} \sum_{x_{i+1} = \ldots = x_{j-1} = 0} \sum_{x_j = \lambda} \nu(x_{i+1}, \ldots, x_{j-1}) \langle x, x, \ldots, x \rangle \lambda^P.
\]

Every polynomial being a sum of a finite number of homogeneous polynomials, and the union of a denumerable family of denumerable sets being denumerable, we infer from the above lemma that every sequence \( \{P_n(x)\} \) of polynomials has the following property:

(W) There exists an \( s \in (0, 1) \) such that

\[
P_n(x + \lambda x) = P_n(x)
\]

for \( n = 1, 2, \ldots, \) every \( x \in c_0(0, 1) \), and every real \( \lambda \).
Since the property (W) is preserved in the passage to the limit, and since the functional \( |x| \) does not have the property (W) in any ball, we infer that \( |x| \) is not a limit of polynomials in any ball.

References


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Some properties of the norm in \( F \)-spaces

by

C. BESSAGA, A. PEŁCZYŃSKI and S. ROLEWICZ (Warszawa)

We deal in this paper with the properties of the norm in \( F \)-spaces. In section 2 we give a construction of a norm equivalent to the basic norm and having very desirable properties. In sections 3 and 4 we give the characterizations of the spaces having some peculiar properties.

1. Let \( X \) be an \( F \)-space and let \( |x|_1 \) and \( |x|_2 \) be two norms defined on \( X \).

DEFINITION 1. The norms \( |x|_1 \) and \( |x|_2 \) are said to be equivalent (in symbols \( |x|_1 \sim |x|_2 \)) if for every sequence \( (a_n) \subseteq X \) the condition

\[ \lim_{n} |a_n|_1 = 0 \]

is equivalent to

\[ \lim_{n} |a_n|_2 = 0. \]

DEFINITION 2. The norm \( |x| \) is called monotone (strictly monotone), concave, of class \( C_1 \), of class \( C_\infty \), or analytic if for every \( x \in X \) the function \( f_t(t) = |tx| \) is for \( t > 0 \) monotone (strictly monotone), concave, \( k \) times differentiable, infinitely differentiable or analytic respectively. A norm having all these properties except analyticity will be said to have the property \( W_1 \).

DEFINITION 3. The norm \( |x| \) is called unbounded (= has the property \( W_1 \)) if the set of values of the functional \( |x| \) is unbounded for \( x \in X \).

Let \( (\theta_n) \) be a sequence of positive numbers such that

\[ \lim_{n} \theta_n = \infty. \]

DEFINITION 4. The norm \( |x| \) is said to have the rate of growth \( (\theta_n) \) if

\[ \limsup_{n} \delta_n \left| \frac{x}{n} \right| < \infty \text{ for every } x \in X. \]

1) Concerning the definition and basic properties of the \( F \)-spaces see [1] and [6].