d'ordre p a exactement p solutions linéairement indépendantes. L'ensemble des solutions est donc, dans notre construction algébrique, plus riche que dans le calcul opérationnel. Par exemple l'équation

$$x''(\lambda)+s^2x(\lambda) = 0 \tag{41}$$

n'a pas de solutions, sauf 0, parmi les fonctions opérationnelles, tandis qu'elle a deux solutions linéairement indépendantes dans l'espace $\mathcal{F}$.

En partant de la clôture algébrique du corps des opérateurs l'espace $\mathcal{F}$ devient un espace des fonctions exponentielles. La fonction $e^{w\lambda}$ existe pour tout opérateur w. En particulier les fonctions $e^{is\lambda}$ et $e^{-is\lambda}$ sont des solutions linéairement indépendantes de l'équation (41). La théorie des équations différentielles est, dans cette interprétation, tout à fait analogue à la théorie classique des équations différentielles ordinaires, mais l'ensemble des fonctions exponentielles est beaucoup plus riche, même plus riche que dans le calcul opérationnel. Par l'adjonction d'un élément transcendant on obtient une construction analogue au calcul opérationnel, mais sur un niveau plus haut, une sorte de „calcul opérationnel du calcul opérationnel".

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[5] — *Sur les solutions linéairement indépendantes des équations différentielles à coefficients constants*, ibidem 16(1957).

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Reçu par la Rédaction le 12. 9. 1956

A property of multilinear operations

by

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1. In this paper I prove a special property of multilinear operations (see [3]) defined in B^*-spaces[1]). It appears that if the space has some additional properties (*e. g.* weak completeness), these operations are continuous with respect to a sequential topology weaker than the topology induced by the norm (it follows hence in particular that in the space C_0 the polynomials are weakly continuous). For a class of spaces the functional $\|x\|$ is not continuous with respect to this topology. From this fact it follows that the functional $\|x\|$ cannot be uniformly approximated in these spaces by polynomials. This result partly coincides with some results of J. Kurzweil [2]. In the sequel I give an example of a non-separable space, in which the functional $\|x\|$ is not representable in any ball as the pointwise limit of polynomials.

I am indebted to Professor S. Mazur for calling my attention to this problem and the aid which he has given me in solving it.

2. Let E be a B^*-space, s — a real number from the interval $\langle 0, 1)$, (ε_i) — a sequence composed of $+1$'s or -1's. I shall define in E a sequential topology τ_s.

Definition 1. The sequence $(x_n)\subset E$ is *τ_s-convergent to Θ*[2]) if there is a constant C such that for $k = 1, 2, \ldots$, for arbitrary different indices $n_1, n_2, \ldots, n_k$ and for every sequence (ε_i) the inequality

$$\|\varepsilon_1 x_{n_1}+\varepsilon_2 x_{n_2}+\ldots+\varepsilon_k x_{n_k}\| \leqslant Ck^s \tag{1}$$

is satisfied.

The sequence (x_n) is τ_s-convergent to the element x_0 if the sequence (x_n-x_0) is τ_s-convergent to Θ[3]). This fact will be denoted in symbols: $x_n \underset{\tau_s}{\to} x_0$.

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[1]) *i. e.*, in (not necessarily complete) linear subsets of B-spaces.

[2]) Θ denotes the null element of the space E.

[3]) It follows in particular from (1) that $x_n \xrightarrow[\tau_s]{} x_0$ implies $x_n \xrightarrow[\text{weakly}]{} x_0$; hence follows the unicity of the τ_s-limit.

DEFINITION 2. The operation $A(x^1, x^2, \ldots, x^r)$ defined on the space $E^r = \underbrace{E\times E\times \ldots \times E}_{r \text{ times}}$, taking on values from a B^*-space E_1 will be called *τ_s-continuous at the point* $(x_0^1, x_0^2, \ldots, x_0^r)$ if for every sequence $\big((x_n^1, x_n^2, \ldots, x_n^r)\big)$ the relations $x_n^i \underset{\tau_s}{\to} x_0^i$ $(i = 1, 2, \ldots, r)$ imply the convergence of the sequence $\big(A(x_n^1, x_n^2, \ldots, x_n^r)\big)$ to $A(x_0^1, x_0^2, \ldots, x_n^r)$ with respect to the norm. The operation $A(x^1, x^2, \ldots, x^r)$ will be called *τ_s-continuous* if it is τ_s-continuous at every point of the space E^r.

DEFINITION 3. A B-space E_1 will be said *to have the rank s* (*to have a loose rank s*) if for every $0 \leqslant a \leqslant s$ $(a < s)$ the functional $\|x\|$ is a τ_a-continuous operation.

THEOREM 1. *Let $A(x^1, x^2, \ldots, x^r)$ be an r-linear operation from the space E^r to a space E_1 having the (loose) rank s_1. Then A is τ_s-continuous if $rs \leqslant s_1$ $(rs < r_1)$.*

3. The proof of theorem 1 is somewhat complicated and needs some auxiliary concepts and symbols.

If we fix in an r-linear operation $A(x^1, x^2, \ldots, x^r)$ the elements $x_{a_1}, x_{a_2}, \ldots, x_{a_r}$ on the places $a_1, a_2, \ldots, a_p$, $p \leqslant r$, we obtain an $(r-p)$-linear [4]) operation which will be denoted by

$$B^{a_1, a_2, \ldots, a_r}_{x_{a_1}, x_{a_2}, \ldots, x_{a_r}}\ {}^{5})$$

Let $(x_n) \subset E$, $\delta > 0$; we shall write

$$Z_\delta(A) = \mathop{E}_{(n_i)\epsilon N^r} \big(\|A(x_{n_1}, x_{n_2}, \ldots, x_{n_r})\| \geqslant \delta\big). \tag{2}$$

By N^ϱ will be denoted the set of all systems of indices $(n_i) = (n_1, n_2, \ldots, n_\varrho)$ where $n_i = 1, 2, \ldots$ for $i = 1, 2, \ldots, \varrho$.

LEMMA 1. *Let a sequence $(x_n) \epsilon E$ have the following properties:*

(a) *for any $\eta > 0$, the indices $a_1, a_2, \ldots, a_p$ and elements $x_{n_{a_1}}, x_{n_{a_2}}, \ldots, x_{n_{a_r}}$ being fixed, the set*

$$Z_\eta(B^{a_1, a_2, \ldots, a_p}_{x_{n_{a_1}}, x_{n_{a_2}}, \ldots, x_{n_{a_p}}})$$

is finite;

(b) *the set $Z_\delta(A)$ is infinite for some $\delta > 0$.*

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[4]) By 0-*linear operations* we mean the constant operations.

[5]) On principle it would be desirable to make evident in the notation the dependence of $B^{a_1, a_2, \ldots, a_r}_{x_{a_1}, x_{a_2}, \ldots, x_{a_r}}$ on the operation A.

Then there exists a sequence $(n_i^\nu) \supset N^r$, $\nu = 1, 2, \ldots$, of systems of indices such that

$$(\alpha) \qquad (n_i^\nu) \epsilon Z_\delta(A) \quad (\nu = 1, 2, \ldots),$$

$$(\beta) \qquad n_i^\nu > n_i^{\nu-1} \quad (\nu = 2, 3, \ldots;\ i = 1, 2, \ldots, r),$$

(γ) *if the numbers $\nu_1, \nu_2, \ldots, \nu_r$ are not all equal to one another, then*

$$\|A(x_{n_1^{\nu_1}}, x_{n_2^{\nu_2}}, \ldots, x_{n_r^{\nu_r}})\| \leqslant \frac{1}{\max\limits_{i\leqslant r}[(\nu_i)^r \cdot 2^{\nu_i}]} \tag{3}$$

Proof [6]). The sequence (n_i^ν) will be defined by induction. As (n_i^1) we choose an arbitrary system of indices belonging to the set $Z_\delta(A)$. Suppose we have already defined the systems (n_i^μ) for $\mu < \nu$, $\nu > 1$. To define the ν-th system let us set $\eta = \nu^{-r}2^{-\nu}$; let us consider all the possible operations

$$B^{a_1, a_2, \ldots, a_p}_{x_{n_{a_1}^{\nu_1}}, x_{n_{a_2}^{\nu_2}}, \ldots, x_{n_{a_p}^{\nu_p}}}$$

where $1 \leqslant \nu_i < \nu$, $i = 1, 2, \ldots, p$, $p = 1, 2, \ldots, r$. The number of these operations is finite. The sets

$$Z_\eta(B^{a_1, a_2, \ldots, a_p}_{x_{n_{a_1}^{\nu_1}}, x_{n_{a_2}^{\nu_2}}, \ldots, x_{n_{a_p}^{\nu_p}}})$$

being finite by (a), there exist quantities $M_1, M_2, \ldots, M_r$ such that $n_{\beta_1} > M_{\beta_1}$, $n_{\beta_2} > M_{\beta_2}, \ldots, n_{\beta_{r-p}} > M_{\beta_{r-p}}$ implies

$$\|B^{a_1, a_2, \ldots, a_p}_{x_{n_{a_1}^{\nu_1}}, x_{n_{a_2}^{\nu_2}}, \ldots, x_{n_{a_p}^{\nu_p}}}(x_{n_{\beta_1}}, x_{n_{\beta_2}}, \ldots, x_{n_{\beta_{r-p}}})\| < \eta = \frac{1}{\nu^r 2^\nu},$$

where $\beta_j \neq a_1$, $j = 1, 2, \ldots, r-p$, $s = 1, 2, \ldots, p$.

Let $N_i = \max(n_i^{\nu-1}, M_i)$ for $i = 1, 2, \ldots, r$. There is only a finite multitude of systems $(n_i) \epsilon Z_\delta(A)$ such that for a certain i the inequality $n_i \leqslant N_i$ holds. Indeed, in the contrary case there would exist an index i_0 such that for infinitely many systems $(n_i) \epsilon Z_\delta(A)$ the inequality $n_{i_0} \leqslant N_{i_0}$ would be satisfied. Since there are only finitely many positive integers not greater than N_{i_0}, there must exist an $\overline{n} \leqslant N_{i_0}$ such that $n_{i_0} = \overline{n}$

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[6]) It is sufficient for this proof to suppose that A is a distributive operation on a linear set of an F^*-space.

is satisfied for infinitely many systems $(n_{i_0}) \epsilon Z_\delta(A)$; this implies, however, the infiniteness of the set $Z_\delta(B^{i_0}_{x_n^{\frac{i}{n}}})$, which is contrary to (a).

The set $Z_\delta(A)$ being infinite, there exist systems $(n_i) \epsilon Z_\delta(A)$ of indices such that $n_i > N_i$ for $i = 1, 2, \ldots, r$. We choose as (n_i^ν) any of these systems. It is eays to see that all the desired conditions are satisfied.

LEMMA 2. *If the conditions of Theorem 1 are satisfied and the sequence (x_n) is τ_s-convergent to Θ, then the set $Z_\delta(A)$ is finite for every $\delta > 0$.*

We prove the lemma by induction with respect to r. Let A be linear $(r = 1)$. We have by (1)

$$\|\varepsilon_1 A(x_{n_1}) + \varepsilon_2 A(x_{n_2}) + \ldots + \varepsilon_k A(x_{n_k})\| = \|A(\varepsilon_1 x_{n_1} + \varepsilon_2 x_{n_2} + \ldots + \varepsilon_k x_{n_k})\|$$
$$\leqslant \|A\| \, \|\varepsilon_1 x_{n_1} + \varepsilon_2 x_{n_2} + \ldots + \varepsilon_k x_{n_k}\| \leqslant C\|A\| k^s,$$

whence the sequence $\big(A(x_n)\big)$ is τ_s-convergent to Θ_1[7]. The space E_1 having the rank s_1 (the loose rank s_1), we infer from the inequality $s \cdot 1 = s \leqslant s_1$ $(s \cdot 1 = s < s_1)$ that

$$\lim_{n=\infty} \|A(x_n)\| = 0.$$

Now let us suppose Lemma 2 to be true for $\bar{r} < r$, $r > 1$ and $rs \leqslant s_1$ $(rs < s_1)$. Let the operation A be r-linear and let the set $Z_\delta(A)$ be infinite for this operation, for some $\delta > 0$. We can easily verify that the sequence (x_n) satisfies the hypotheses of Lemma 1. Thus, we can choose a sequence of systems of indices (n_i^ν) with the properties (α), (β), and (γ).

We set for $k = 1, 2, \ldots$

$$y_i^k = \sum_{j=1}^{k} x_{n_i^{\mu_j}} \quad \text{for} \quad i = 2, 3, \ldots, r, \tag{4}$$

$$z^k = \sum_{j=1}^{k} x_{n_1^{\mu_j}} \varepsilon_j \tag{5}$$

where $\mu_1 < \mu_2 < \ldots < \mu_k = \nu$.

Then we write

$$\Lambda_\varrho = \mathop{E}_{(\alpha_r) \epsilon N^i} \big(\mathop{\exists}_{i \leqslant \varrho} (\alpha_i = \varrho) \cdot \mathop{\exists}_{j \leqslant r} (\alpha_j < \varrho) \cdot \max_{i \leqslant r} \alpha_i = \varrho\big), \tag{6}$$

$$\bar{\Lambda}_\varrho = \mathop{E}_{(\alpha_i) \epsilon N^r} (\max_{i \leqslant r} \alpha_i = \varrho). \tag{7}$$

[7]) Θ_1 denotes the null element of the space E_1.

By (2)-(7) we have

$$\|A(z^k, y_1^k, y_3^k, \ldots, y_r^k)\| = \Big\|\sum_{\varrho=1}^{k} \sum_{\Lambda_\varrho} \varepsilon_{\alpha_1} A(x_{n_1^{\mu_{\alpha_1}}}, x_{n_2^{\mu_{\alpha_2}}}, \ldots, x_{n_r^{\mu_{\alpha_r}}})\Big\|$$

$$\geqslant \Big\|\sum_{\varrho=1}^{k} \varepsilon_\varrho A(x_{n_1^{\mu_\varrho}}, x_{n_2^{\mu_\varrho}}, \ldots, x_{n_r^{\mu_\varrho}}) + \sum_{\varrho=2}^{k} \sum_{\Lambda_\varrho} \varepsilon_{\alpha_1} A(x_{n_1^{\mu_{\alpha_1}}}, x_{n_2^{\mu_{\alpha_2}}}, \ldots, x_{n_r^{\mu_{\alpha_r}}})\Big\|$$

$$\geqslant \Big\|\sum_{\varrho=1}^{k} \varepsilon_\varrho A(x_{n_1^{\mu_\varrho}}, x_{n_2^{\mu_\varrho}}, \ldots, x_{n_r^{\mu_\varrho}})\Big\| - \sum_{\varrho=2}^{k} \sum_{\Lambda_\varrho} \|A(x_{n_1^{\mu_{\alpha_1}}}, x_{n_2^{\mu_{\alpha_2}}}, \ldots, x_{n_r^{\mu_{\alpha_r}}})\|$$

$$\geqslant \Big\|\sum_{\varrho=1}^{k} \varepsilon_\varrho A(x_{n_1^{\mu_\varrho}}, x_{n_2^{\mu_\varrho}}, \ldots, x_{n_r^{\mu_\varrho}})\Big\| - \sum_{\varrho=2}^{k} \sum_{\Lambda_\varrho} \frac{1}{2^{\mu_\varrho}(\mu_\varrho)^r}$$

$$\geqslant \Big\|\sum_{\varrho=1}^{k} \varepsilon_\varrho A(x_{n_1^{\mu_\varrho}}, x_{n_2^{\mu_\varrho}}, \ldots, x_{n_r^{\mu_\varrho}})\Big\| - 1.$$

From (1) it follows that

$$\|A(z^k, y_2^k, y_3^k, \ldots, y_r^k)\| \leqslant \|A\| \|z^k\| \|y_2^k\| \|y_3^k\| \ldots \|y_r^k\| \leqslant \|A\| \cdot C^r k^{sr}.$$

The above inequalities imply that the sequence $\big(A(x_{n_1^\nu}, x_{n_2^\nu}, \ldots, x_{n_r^\nu})\big)$ is τ_{sr}-convergent to Θ_1. The space E_1 having the (loose) rank r_1, it follows from $sr \leqslant s_1$ $(sr < s_1)$ that

$$\lim_{\nu} \|A(x_{n_1^\nu}, x_{n_2^\nu}, \ldots, x_{n_r^\nu})\| = 0,$$

which is impossible, for $n_i^\nu \epsilon Z_\varrho(A)$. Thus the hypothesis that the set $Z_\delta(A)$ is infinite leads to a contradiction.

Proof of Theorem 1. Lemma 2 implies directly that under the hypotheses of Theorem 1 every r-linear operation is τ_s-continuous at Θ. To see this, it suffices to notice that the τ_s-convergence of the sequences (x_n^i) to Θ for $i = 1, 2, \ldots, r$ implies the τ_s-convergence of the sequence $x_1^1, x_1^2, \ldots, x_1^r, x_2^1, x_2^2, \ldots, x_2^r, \ldots$ to Θ, and then to apply Lemma 2.

The proof of τ_s-continuity at an arbitrary point will be proved by induction. This fact is trivial for 0-linear (= constant) operations. Suppose that it is true for p-linear operations, $0 \leqslant p < r$. Let $x_n^i \underset{\tau_s}{\to} x_0^i$ for $i = 1, 2, \ldots, r$. We shall use the identity

$$A(x_n^1, x_n^2, \ldots, x_n^r) = A(x_n^1 - x_0^1, x_n^2 - x_0^2, \ldots, x_n^r - x_0^r) - \tag{8}$$
$$- \sum_{\varrho=0}^{r-1} (-1)^{r-\varrho} \sum_{1 \leqslant \alpha_1, \alpha_2, \ldots, \alpha_{r-\varrho}} B^{\alpha_1, \alpha_2, \ldots, \alpha_p}_{x_0^{\alpha_1}, x_0^{\alpha_2}, \ldots, x_0^{\alpha_p}} (x_n^{\beta_1}, \ldots, x_n^{\beta_p})$$

where $\beta_i \neq \alpha_j$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, r-p$.

By the induction hypothesis

$$\lim_{n=\infty} B^{a_1, a_2, \ldots, a_{r-p}}_{x_0^{a_1}, x_0^{a_2}, \ldots, x_0^{a_{r-p}}} (x_n^{\beta_1}, x_n^{\beta_2}, \ldots, x_n^{\beta_p}) = A(x_0^1, x_0^2, \ldots, x_0^r).$$

Hence follows the existence of the limit of the left-hand side of equality (8) as $n \to \infty$, for $\lim A(x_n^1 - x_0^1, x_n^2 - x_0^2, \ldots, x_n^r - x_0^r) = \Theta_1$ by virtue of the τ_s-continuity of A at Θ. A direct computation shows that this limit is equal to $A(x_0^1, x_0^2, \ldots, x_0^r)$, q. e. d.

4. We get the following corollaries:

4.1. The space l^p $(p \geqslant 1)$ has the following property ([1], p. 200):

(w_1) *If the sequence $(x_n) \subset l^p$ converges weakly to Θ, then it contains a subsequence $\tau_{1/p}$-convergent to Θ.*

This implies in particular that l^p has a loose rank $1/p$. Hence from Theorem 1 we deduce the following corollary:

Any r-linear operation from the space $(l^p)^r$ to l^p $(p, q \geqslant 1)$ is weakly[8]) *continuous for $r < p/q$.* (We set $s = 1/p$, $s_1 = 1/q$).

4.2. It is easily shown that the space of reals (the finite dimensional space) has a loose rank 1. Hence by the property (w_1) we find that

Any r-linear form (finitely dimensional operation) on the space $(l^p)^r$ $(p \geqslant 1)$ is weakly continuous for $r < p$.

The polynomials of degree $r < p$ defined on l^p are therefore weakly continuous.

4.3. *Any r-linear operation from the space E^r to a weakly complete space E_1 is τ_0-continuous for $r = 0, 1, \ldots$*

This follows from Theorem 1 with $s = 0$, $s_1 = 0$ and the following property:

(w_2) *Any weakly complete B^*-space has the rank 0.*

Proof of the property (w_2). It is easily verified that the τ_0-convergence of the sequence (x_n) to Θ is equivalent to the following condition (0):

(0) there exists a constant C such that for arbitrary different indices $n_1, n_2, \ldots, n_k$ the inequality

$$\|x_{n_1} + x_{n_2} + \ldots + x_{n_k}\| \leqslant C$$

is valid for $k = 1, 2, \ldots$

[8]) The definition of weakly continuous operation is analogous to the definition of τ_0-continuous operation.

A theorem of Orlicz [4] states that if in a weakly complete space the sequence (x_n) satisfies the condition (0), then the series

$$\sum_{n=1}^{\infty} x_n$$

converges unconditionally. It follows that $\|x_n\| \to 0$ in our case, q. e. d.

4.4. The space c_0 has the following property, easy to verify:

(w_3) *Every sequence $(x_n) \subset c_0$ weakly convergent to Θ contains a subsequence τ_0-convergent to Θ.*

Using this property and the result of 4.3 we find that

Any r-linear operation from the space c_0 to a weakly complete space E_1 is weakly continuous for $r = 0, 1, \ldots$

It follows in particular that *multilinear forms (whence also the polynomials) defined on the Cartesian product of the space c_0 are weakly continuous functionals.* The last result was obtained in another way by W. Bogdanowicz[9]).

5. The corollary obtained in 4.3 implies some negative results concerning the approximation of functionals by polynomials.

5.1. I shall need the following

DEFINITION 4. The sequence (x_n) is called *essentially τ_s-convergent to x_0* if it is τ_s-convergent to x_0 but does not converge to x_0 in the norm-topology.

THEOREM 2. *If there exist in the space E essentially τ_0-convergent sequences, then the functional $\|x\|$ cannot be uniformly approximated by polynomials in any ball with centre Θ.*

Proof. It is easily shown that if there exist in E essentially τ_0-convergent sequences, then every ball $K(\Theta, r)$ with centre Θ and radius $r > 0$ contains a sequence (x_n) with the following properties:

1° $x^n \underset{\tau_0}{\to} \Theta$;

2° $\|x_n\| = r/2$ for $n = 1, 2, \ldots$

Indeed, let (y_m) be essentially τ_0-convergent to Θ. Since

$$\limsup_m \|y_m\| > 0,$$

we can choose an increasing sequence of indices (m_n) such that $\|y_{m_n}\| > \delta$ for $n = 1, 2, \ldots$; then we set $x_n = (r/2\|y_{m_n}\|) y_{m_n}$.

[9]) See W. Bogdanowicz, *On the weak continuity of the polynomial functionals defined on the space c_0* (in Russian), Bull. Acad. Pol. Sci. Cl. III, 5(1957). p. 243-246.

Suppose that there exists a sequence $P_m(x)$ of polynomials uniformly convergent to the functional $\|x\|$ in $K(\Theta, r)$. Then there must exist an M such that

$$|P_m(x) - \|x\|| < r/4$$

for every $x \in K(\Theta, r)$ and $m > M$. In particular

$$|P_m(x_n) - r/2| < r/4 \qquad (n = 1, 2, \ldots).$$

It follows that $P_m(x_n) \geqslant r/4$ ($n = 1, 2, \ldots, m > M$). The polynomials are, however, τ_0-continuous, whence $\lim P_m(x_n) = P_m(\Theta)$ for $m = 1, 2, \ldots$, and this implies $|P_m(\Theta)| \geqslant r/4$ for $m > M$ which is contrary to the hypothesis

$$\lim_m P_m(\Theta) = \|\Theta\| = 0.$$

Remark. The space $C(Q)$ of continuous real functions defined on a compact infinite metric space Q contains always essentially τ_0-convergent sequences.

This follows from the fact that the space $C(Q)$ contains a subspace equivalent to the space c_0 of numerical sequences convergent to 0 and in this space the sequence (e_n) where $e_n = (0, \ldots, 0, \underset{n}{1}, 0, \ldots)$ is essentially τ_0-convergent.

5.2. Let $c\langle 0, 1\rangle$ denote the set of all real functions $x = x(t)$ defined for $t \in \langle 0, 1\rangle$, such that for every $\delta > 0$ the set

$$\underset{t \in \langle 0,1\rangle}{E} \big(|x(t)| \geqslant \delta\big)$$

is finite. It is easily seen that, with the usual definition of addition and multiplication by scalars, $c_0\langle 0, 1\rangle$ is a B-space, the norm being

$$\|x\| = \sup_{t \in \langle 0,1\rangle} |x(t)|.$$

THEOREM 3. *It is impossible to represent in the space $c\langle 0, 1\rangle$ the functional $\|x\|$ as a limit of a sequence of functionals in any ball.*

Proof. Let Φ stand for the family of functions e_s, $s \in \langle 0, 1\rangle$ defined as follows:

$$e_s(t) = \begin{cases} 0 & \text{for} \quad s \neq t, \\ 1 & \text{for} \quad s = t. \end{cases}$$

One can easily verify that this family has the following properties:

(α) Every element x of the space $C\langle 0, 1\rangle$ may be represented uniquely in the form

$$x = \sum_{i=1}^{\infty} t_i e_{s_i} \qquad \text{where} \qquad t_i \neq 0$$

and $(s_i) \subset \langle 0, 1\rangle$ is a finite or infinite sequence. The set of finite linear combinations of elements of Φ is therefore dense in $c\langle 0, 1\rangle$.

(β) The smallest linear space spanned upon any denumerable subfamily of Φ is equivalent to the space c_0.

(γ) For every sequence $(s_i) \subset \langle 0, 1\rangle$ ($s_i \neq s_j$ for $i \neq j$), the sequence (e_{s_i}) is τ_0-convergent to Θ.

LEMMA. *For every homogeneous polynomial $P(x)$ there exists a separable set $T \subset \langle 0, 1\rangle$, such that $s \bar{\in} T$ implies $P(x + \lambda e_s) = P(x)$ for every $x \in c_0\langle 0, 1\rangle$ and every real λ.*

Proof of the lemma. Let $A(x^1, x^2, \ldots, x^r)$ be the (r-linear) generating form of $P(x)$. By (γ) and (β) it follows from Lemma 2 (p. 176) that the set $Z_\delta(A)$ is finite for every sequence $(e_{s_i}) \in \Phi$ $\big((s_i) \in \langle 0, 1\rangle$, $s_i \neq s_j$ for $i \neq j\big)$ and every $\delta > 0$. It implies the existence of an at most denumerable set T such that if at least one of the numbers $s_1, s_2, \ldots, s_r$ is not in T, then

(10) $$A(e_{s_1}, e_{s_2}, \ldots, e_{s_r}) = 0.$$

Therefore, for $s \bar{\in} T$

(11) $$B^j_{e_s}(x^1, x^2, \ldots, x^{r-1}) \equiv 0 \qquad (j = 1, 2, \ldots, r).$$

Indeed, it follows from (10) that formula (11) is true if $x^1, x^2, \ldots, x^{r-1}$ are finite linear combinations of the elements of Φ. The validity of formula (11) for arbitrary points $x^1, x^2, \ldots, x^{r-1}$ follows from (α) and from the continuity of the operation $B^j_{e_s}$.

The lemma follows from (11) and the identity

$$P(x + \lambda e_s) = A(x + \lambda e_s, x + \lambda e_s, \ldots, x + \lambda e_s)$$
$$= A(x, x, \ldots, x) + \sum_{\varrho=1}^{r} \sum_{\substack{1 \leqslant a_1, a_2, \ldots, a_r \leqslant r \\ a_1 \neq a_j}} B^{a_1, a_2, \ldots, a_r}_{e_s, e_s, \ldots, e_s}(x, x, \ldots, x) \lambda^p.$$

Every polynomial being a sum of a finite number of homogeneous polynomials, and the union of a denumerable family of denumerable sets being denumerable, we infer from the above lemma that every sequence $\big(P_n(x)\big)$ of polynomials has the following property:

(W) There exists an $s \in \langle 0, 1\rangle$ such that

$$P_n(x + \lambda e_s) = P_n(x)$$

for $n = 1, 2, \ldots$, every $x \in c_0\langle 0, 1\rangle$, and every real λ.

Since the property (W) is preserved in the passage to the limit, and since the functional $\|x\|$ does not have the property (W) in any ball, we infer that $\|x\|$ is not a limit of polynomials in any ball.

References

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Reçu par la Rédaction le 13. 9. 1956

Some properties of the norm in F-spaces

by

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We deal in this paper with the properties of the norm in F-spaces[1]). In section 2 we give a construction of a norm equivalent to the basic norm and having very desirable properties. In sections 3 and 4 we give the characterizations of the spaces having some peculiar properties.

1. Let X be an F^*-space and let $\|x\|_1$ and $\|x\|_2$ be two norms defined on X.

DEFINITION 1. The norms $\|x\|_1$ and $\|x\|_2$ are said to be *equivalent* (in symbols $\|x\|_1 \sim \|x\|_2$) if for every sequence $(x_n) \subset X$ the condition

$$\lim_n \|x_n\|_1 = 0$$

is equivalent to

$$\lim_n \|x_n\|_2 = 0.$$

DEFINITION 2. The norm $\|x\|$ is called *monotone (strictly monotone), concave, of class C_k, of class C_∞, or analytic* if for every $x \in X$ the function $f_x(t) = \|tx\|$ is for $t > 0$ monotone (strictly monotone), concave, k times differentiable, infinitely differentiable or analytic respectively. A norm having all these properties except analycity will be said *to have the property* W_1.

DEFINITION 3. The norm $\|x\|$ is called *unbounded* (= *has the property* W_2) if the set of values of the functional $\|x\|$ is unbounded for $x \in X$.

Let (ϑ_n) be a sequence of positive numbers such that

$$\lim_n \vartheta_n = \infty.$$

DEFINITION 4. The norm $\|x\|$ is said *to have the rate of growth* (ϑ_n) if

$$\limsup_n \vartheta_n \left\| \frac{x}{n} \right\| < \infty \quad \text{for every} \quad x \in X.$$

[1]) Concerning the definition and basic properties of the F^*-spaces see [1] and [6].