On Poisson and composed Poisson stochastic set functions

by

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Introduction

Several investigations has recently been made concerning Poisson and composed Poisson stochastic processes. The *ordinary Poisson process* is conceivable as a sequence of points, distributed at random on the time axis and this idea can be generalized to more than one-dimensional spaces. The latter case occurs in making a blood-count, in counting stars, etc. In [8], [4], [6] and [15] conditions are given ensuring the Poisson character of the distribution of the number of points in a set A of the one, resp. at least one-dimensional Euclidean space. In [8], [14], [1] and [13] similar problems are considered for the one-dimensional Euclidean space and the main purpose is to prove that under some conditions the random variables $\xi_{t_2}-\xi_{t_1}$ $(t_1<t_2)$ have composed Poisson distributions.

We say that a random variable ξ has a *composed Poisson distribution* if its characteristic function $f(u)$ can be written in the form

$$f(u) = \exp \sum_{k=1}^{\infty} C_k(e^{i\lambda_k u}-1), \tag{1}$$

where $C_1, C_2, \ldots$ are non-negative constants,

$$\sum_{k=1}^{\infty} C_k < \infty$$

and $\lambda_1, \lambda_2, \ldots$ is a sequence of real numbers. It is easy to see that if the set $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$ forms a semi-group with respect to addition, and if $P_k = P(\xi = \lambda_k)$, $k = 0, 1, 2, \ldots$, then

$$\sum_{k=0}^{\infty} P_k = 1.$$

This statement follows from (1), if we take into account that ξ can be written (or ξ can be represented in another probability space) as

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k, \tag{2}$$

where $\xi_1, \xi_2, \ldots$ are mutually independent random variables, having Poisson distributions with the parameters $C_1, C_2, \ldots$

In [3] the problem of random point distribution in an abstract space is considered and in [12] the notion of a stochastic set function, and especially the notion of a composed Poisson stochastic set function are introduced.

In the present paper I give the conditions ensuring the Poisson and composed Poisson character of stochastic set functions or, in other words, of an abstract process, and prove some theorems concerning their structure.

Let H be an abstract space and $\mathfrak{R}$ a ring of sets[1]) consisting of some subsets of H. Let us suppose that to every element A of $\mathfrak{R}$ there corresponds a random variable $\xi(A)$ for which the following conditions hold:

I. *If $A_1, A_2, \ldots$ is a sequence of pairwise disjoint sets of $\mathfrak{R}$, then the random variables $\xi(A_1), \xi(A_2), \ldots$ are independent.*

II. *If $A = \sum_{k=1}^{\infty} A_k \in \mathfrak{R}$, then $P\big(\xi(A) = \sum_{k=1}^{\infty} \xi(A_k)\big) = 1$.*

A random variable-valued set function $\xi(A)$, satisfying conditions I-II is called a *completely additive stochastic set function*. For the sake of brevity we often say only that conditions I-II are satisfied.

III. *The random variables $\xi(A)$, $A \in \mathfrak{R}$, can only assume the values of a countable set of real numbers $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$; this set is independent of the special choice of A and with respect to conditions* I-II *we suppose that it is an additive semi-group.*

If for the stochastic set function $\xi(A)$, satisfying conditions I-II, a certain additional condition is fulfilled, for instance, if for every sequence $B_1, B_2, \ldots$ of pairwise disjoint sets of $\mathfrak{R}$ the series

$$\sum_{k=1}^{\infty} \xi(B_k)$$

converges with probability 1, then $\xi(A)$ can be extended to $\mathfrak{S}(\mathfrak{R})$ (see [12], Theorem 3.2)[2]). By the extension of the latter we mean a construction

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[1]) A class of sets $\mathfrak{R}$ is called a *ring of sets* if $A+B \in \mathfrak{R}$, $A-B \in \mathfrak{R}$, whenever $A \in \mathfrak{R}$, $B \in \mathfrak{R}$.

[2]) A ring $\mathfrak{S}$ is called a *σ-ring* if for every sequence $A_1, A_2, \ldots$ of $\mathfrak{S}$ we have

$$\sum_{k=1}^{\infty} A_k \in \mathfrak{S}.$$

If $\mathfrak{R}$ is a ring, then $\mathfrak{S}(\mathfrak{R})$ denotes the smallest σ-ring containing $\mathfrak{R}$. If $A \in \mathfrak{R}$, then $A\mathfrak{R}$ is the ring containing those sets B for which $B \in \mathfrak{R}$, $B \subset A$.

of a stochastic set function $\xi^*(A)$ defined on the elements of $\mathfrak{S}(\mathfrak{R})$ that satisfies on $\mathfrak{S}(\mathfrak{R})$ conditions I-II, and has the following property:

$$P(\xi^*(A) = \xi(A)) = 1, \quad \text{if} \quad A \in \mathfrak{R}.$$

We shall suppose that in conditions I-III $\mathfrak{R}$ is a σ-ring.

In some theorems we shall use the following conditions:

IV. *There is a positive number ϱ such that $|\lambda_k| \geqslant \varrho$, $k = 1, 2, \ldots$*

V. *There is a sequence of divisions $\mathfrak{Z}_n = \{A_1^{(n)}, A_2^{(n)}, \ldots, A_{k_n}^{(n)}\}$ such that the $(n+1)$-st division is a subdivision of the n-th one, and if $h_1 \in H$, $h_2 \in H$, $h_1 \neq h_2$, then there is an N for which $h_1 \in A_i^{(N)}$, $h_2 \in A_j^{(N)}$, where $i \neq j$.*

In section 1 some definitions and lemmas are formulated.

In section 2 conditions are given under which the random variables $\xi(A)$ have composed Poisson distributions. In section 3 some structural theorems are proved concerning a composed Poisson stochastic set function. In section 4 theorems are proved concerning random point distributions.

1. Preliminary lemmas

DEFINITION 1. Let $\mathfrak{R}$ be a ring of sets and $\alpha(A)$ a real-valued set function on the elements of $\mathfrak{R}$. If for every pair A_1, A_2 of disjoint sets of $\mathfrak{R}$ (for every sequence $A_1, A_2, \ldots$ of disjoint sets of $\mathfrak{R}$, for which

$$A = \sum_{k=1}^{\infty} A_k$$

is an element of $\mathfrak{R}$) the relation

$$\alpha(A) \leqslant \alpha(A_1) + \alpha(A_2) \qquad \Big(\alpha(A) \leqslant \sum_{k=1}^{\infty} \alpha(A_k)\Big) \tag{3}$$

holds, then the set function $\alpha(A)$ will be called *subadditive (completely subadditive)*.

DEFINITION 2. A set function $\alpha(A)$ defined on $\mathfrak{R}$ is said to be *of bounded variation* if there is a number K such that for every finite sequence $A_1, A_2, \ldots, A_r$ of pairwise disjoint sets of $\mathfrak{R}$ the relation

$$\sum_{i=1}^{r} |\alpha(A_i)| \leqslant K \tag{4}$$

holds. If $A_i \subset A \in \mathfrak{R}$, $i = 1, 2, \ldots$, then the smallest K for which relation (4) holds will be denoted by $\mathrm{Var}_\alpha(A)$.

The following two lemmas, the first of which is almost trivial, are proved in [11]:

LEMMA 1. *Let $\alpha(A)$ be a real-valued, non-negative, completely subadditive set function of bounded variation defined on the elements of a ring of sets $\mathfrak{R}$. Then the set function $\mathrm{Var}_\alpha(A)$ is a bounded measure*[3]) *on $\mathfrak{R}$.*

LEMMA 2. *Let $\alpha(A)$ be a real-valued, non-negative and subadditive set function, defined on the elements of a ring of sets $\mathfrak{R}$. If there is a number C such that $\alpha(A) \leqslant C$, for $A \in \mathfrak{R}$ and for every sequence $B_1, B_2, \ldots$ of pairwise disjoint sets of $\mathfrak{R}$ the condition*

$$\sum_{i=1}^{\infty} \alpha(B_i) < \infty$$

is fulfilled, then the set function $\alpha(A)$ is of bounded variation.

Though the following notions and theorems are special cases of known general notions and theorems (see *e. g.* [2], Chapter 8), nevertheless, for the reader's convenience we repeat them separately.

DEFINITION 3. Let $\alpha(A)$ be a set function defined on the elements of a ring of sets $\mathfrak{R}$. We say that *the total of the set function $\alpha(A)$ exists in the set $B \in \mathfrak{R}$*, if we can find a number $\beta(B)$ such that for every $\varepsilon > 0$ there exists a division of the set B into pairwise disjoint sets $A_1, A_2, \ldots, A_r$ of $\mathfrak{R}$ for which

$$\Big|\sum_{i=1}^{r} \alpha(A_i) - \beta(B)\Big| < \varepsilon \tag{5}$$

and also

$$\Big|\sum_{i=1}^{r} \sum_{k=1}^{l_i} \alpha(A_{ik}) - \beta(B)\Big| < \varepsilon, \tag{6}$$

where $\mathfrak{Z} = \{A_{ik}, k = 1, 2, \ldots, l_i;\ i = 1, 2, \ldots, r\}$ is an arbitrary subdivision into pairwise disjoint sets of the ring $\mathfrak{R}$ of the division $\mathfrak{Z}' = \{A_i, i = 1, 2, \ldots, r\}$.

The number $\beta(B)$ will be called the *total of $\alpha(A)$ on the set B* and will be denoted in the following manner:

$$\beta(B) = \int_B \alpha(dA).$$

It is easy to see that the total — if it exists — is always uniquely determined.

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[3]) A finite-valued, non-negative set function $m(A)$, defined on a ring of sets $\mathfrak{R}$, is called a *measure* if for every sequence $B_1, B_2, \ldots$ of disjoint sets of $\mathfrak{R}$, for which $B = \sum_{k=1}^{\infty} B_k \in \mathfrak{R}$, the relation $m(B) = \sum_{k=1}^{\infty} m(B_k)$ holds and $m(0) = 0$.

The following lemmas can be proved in a simple manner:

Lemma 3. *If $a(A)$ is a set function defined on the elements of a ring of sets $\mathfrak{R}$ and its totals on $B_1 \in \mathfrak{R}$ and on $B_2 \in \mathfrak{R}$ exist, where $B_1 B_2 = 0$, then its total exists also on $B_1 + B_2$ and*

$$\int_{B_1+B_2} a(dA) = \int_{B_1} a(dA) + \int_{B_2} a(dA).$$

Lemma 4. *If $a_1(A)$ and $a_2(A)$ are two set functions defined on the elements of a ring of sets $\mathfrak{R}$ and the totals of both exist in $B \in \mathfrak{R}$, then the total of $a_1(A)+a_2(A)$ exists also on $B \in \mathfrak{R}$ and*

$$\int_B \big(a_1(dA)+a_2(dA)\big) = \int_B a_1(dA) + \int_B a_2(dA).$$

Lemma 5. *If $a(A)$ is a subadditive set function of bounded variation defined on the elements of a ring of sets $\mathfrak{R}$, then the total of $a(A)$ exists for every $B \in \mathfrak{R}$ and*

$$\int_B a(dA) = \mathrm{Var}_a(B).$$

2. Composed Poisson stochastic set functions

In this section we shall give conditions under which a completely additive stochastic set function will be of composed Poisson type. The method by which the theorems stated below are proved, is based essentially on two facts ensured by our conditions: the set function $1-P_0(A)$ is of bounded variation and $\mathrm{Var}_{1-P_0}(A)$ is a bounded, atomless measure on $\mathfrak{R}$. First we prove a general theorem, and in special cases we shall verify the fulfilment of the conditions introduced here.

Theorem 1. *Let us suppose that the stochastic set function $\xi(A)$, defined on the elements of the σ-ring $\mathfrak{R}$, satisfies conditions* I, II, III. *Suppose furthermore that the following conditions are fulfilled:*

VI. $\mathrm{Var}_{1-P_0}(H) < \infty$.

VII. *If $A \in \mathfrak{R}$ and $1-P_0(A) > 0$, then there exist such sets $A_1 \in A\mathfrak{R}$, $A_2 \in A\mathfrak{R}$, $A_1 A_2 = 0$, that $1-P_0(A_1) > 0$, $1-P_0(A_2) > 0$.*

Under these conditions the logarithm of $f(u,B)$ can be written for every $B \in \mathfrak{R}$ in the form

$$\log f(u,B) = \sum_{k=1}^{\infty} C_k(B)(e^{i\lambda_k u}-1), \tag{7}$$

where

$$C_k(B) = \int_B P_k(dA), \quad k = 1, 2, \ldots; \tag{8}$$

moreover

$$\int_B \big(1-P_0(dA)\big)$$

exists also and

$$\int_B \big(1-P_0(dA)\big) = \sum_{k=1}^{\infty} C_k(B) < \infty. \tag{9}$$

The set functions (8) *and* (9) *are bounded, atomless measures on the σ-ring $\mathfrak{R}$.*

Proof. Let $D_1, D_2, \ldots$ be a sequence of pairwise disjoint sets of $\mathfrak{R}$ and

$$D = \sum_{k=1}^{\infty} D_k.$$

Condition II implies that

$$1-P_0(D) \leqslant \sum_{k=1}^{\infty} \big(1-P_0(D_k)\big).$$

It follows hence and from Condition VI that the conditions in Lemma 1 are fulfilled for $a(A) = 1-P_0(A)$. Thus $\mathrm{Var}_{1-P_0}(A)$ is a bounded measure on $\mathfrak{R}$. The measure $\mathrm{Var}_{1-P_0}(A)$ is also atomless. Let us suppose the contrary and denote by $E \in \mathfrak{R}$ an atom. Then there exists a set $D \in E\mathfrak{R}$ such that $1-P_0(D) > 0$. Clearly D is also an atom and thus for every $D' \in D\mathfrak{R}$ we have either $\mathrm{Var}_{1-P_0}(D') = \mathrm{Var}_{1-P_0}(D) > 0$ or $\mathrm{Var}_{1-P_0}(D') = 0$.

According to Condition VII there exist such sets $D_1 \in D\mathfrak{R}$, $D_2 \in D\mathfrak{R}$, $D_1 D_2 = 0$, that

$$0 < 1-P_0(D_1) \leqslant \mathrm{Var}_{1-P_0}(D_1), \quad 0 < 1-P_0(D_2) \leqslant \mathrm{Var}_{1-P_0}(D_2).$$

Thus if $D' = D_1$, we must have $\mathrm{Var}_{1-P_0}(D_1) = \mathrm{Var}_{1-P_0}(D) > 0$, but this is impossible as $\mathrm{Var}_{1-P_0}(D_2) > 0$ and

$$\mathrm{Var}_{1-P_0}(D_1) + \mathrm{Var}_{1-P_0}(D_2) \leqslant \mathrm{Var}_{1-P_0}(D).$$

Using the intermediate value theorem of atomless completely additive set functions (cf. [7], p. 51, Theorem 5.6.1), we can choose for every ε a decomposition $\mathfrak{Z}' = \{A_1, A_2, \ldots, A_r\}$ of the set B into pairwise disjoint sets of $\mathfrak{R}$, where

$$\mathrm{Var}_{1-P_0}(A_k) \leqslant \varepsilon, \quad k = 1, 2, \ldots, r. \tag{10}$$

Since

$$f(u, B) = \prod_{k=1}^{r} f(u, A_k)$$

and for every $A \epsilon \mathfrak{R}$

$$|1-f(u, A)| \leqslant 2\bigl(1-P_0(A)\bigr),$$

we find that if $\varepsilon \leqslant \frac{1}{4}$, then

$$|f(u, B)| = \prod_{k=1}^{r} |f(u, A_k)| \geqslant \frac{1}{2^r}. \tag{11}$$

Hence $\log f(u, B)$ exists. Taking into account the definition of $f(u, B)$,

$$f(u, B) = \sum_{k=0}^{\infty} P_k(B) e^{i\lambda_k u}, \tag{12}$$

we conclude that $f(u, B)$ and by (11) also $\log f(u, B)$ are almost periodic functions.

Let $C_0(B), C_1(B), \ldots$ denote the Fourier-coefficients of $\log f(u, B)$. Applying the Taylor expansion of the function $\log z$, we find that

$$\begin{aligned} \Bigl|\log f(u, B) - \sum_{i=1}^{r}\bigl(f(u, A_k)-1\bigr)\Bigr| &\leqslant \sum_{i=1}^{r} |f(u, A_i)-1|^2 \leqslant 4\sum_{i=1}^{r}\bigl(1-P_0(A_i)\bigr)^2 \\ &\leqslant 4 \max_{1\leqslant i\leqslant r}\bigl(1-P_0(A_i)\bigr) \sum_{i=1}^{r}\bigl(1-P_0(A_i)\bigr) \leqslant K \max_{1\leqslant i\leqslant r} \mathrm{Var}_{1-P_0}(A_i) \leqslant K\varepsilon. \end{aligned} \tag{13}$$

Multiplying both sides of (13) by $e^{-i\lambda_k u}/2T$, integrating from $-T$ to T, and taking the limit $T\to\infty$, we obtain

$$\begin{aligned} \Bigl|C_k(B) - \sum_{i=1}^{r} P_k(A_i)\Bigr| &\leqslant K\varepsilon, \quad \text{if} \quad k = 1, 2, \ldots, \\ \Bigl|C_0(B) - \sum_{i=1}^{r}\bigl(P_0(A_i)-1\bigr)\Bigr| &\leqslant K\varepsilon, \quad \text{if} \quad k = 0. \end{aligned} \tag{14}$$

It follows from (10) and (13) that (14) is true even if we replace the division $\mathfrak{Z}' = \{A_1, A_2, \ldots, A_r\}$ by any of its subdivisions, whence

$$\begin{aligned} C_k(B) &= \int_B P_k(dA), \quad k = 1, 2, \ldots, \\ C_0(B) &= \int_B \bigl(P_0(dA)-1\bigr). \end{aligned} \tag{15}$$

Relations (10) and (13) imply also

$$\log f(u, B) = \int_B \bigl(f(u, dA)-1\bigr)^{4)}.$$

The almost periodic function $f(u, B)-C_0(B)$ has non-negative Fourier coefficients. Hence ([5], p. 64-65)

$$\sum_{k=1}^{\infty} C_k(B) < \infty.$$

Thus

$$f(u, B)-C_0(B) = \sum_{k=1}^{\infty} C_k(B) e^{i\lambda_k u}.$$

For $u = 0$ we obtain

$$-C_0(B) = \sum_{k=1}^{\infty} C_k(B),$$

whence

$$f(u, B) = \sum_{k=1}^{\infty} C_k(B)(e^{i\lambda_k u}-1).$$

We have proved relations (7), (8) and (9) in Theorem 1. Now we shall prove the remaining assertions relative to the set functions $C_0(A)$ and $C_k(A)$, $k = 1, 2, \ldots$ By Lemma 5, $\mathrm{Var}_{1-P_0}(A) = -C_0(A)$, whence $-C_0(A)$ is an atomless measure on $\mathfrak{R}$. Since, for every $A \epsilon \mathfrak{R}$, $P_k(A) \leqslant 1-P_0(A)$, $k = 1, 2, \ldots$, it follows that

$$C_k(A) \leqslant -C_0(A), \quad A \epsilon \mathfrak{R}. \tag{16}$$

By Lemma 3 the set function $C_k(A)$ is additive. It follows hence and from relation (16) that $C_k(A)$ is also completely additive on $\mathfrak{R}$. Relation (16) implies also that $C_k(A)$ is an atomless measure on $\mathfrak{R}$. Thus Theorem 1 is proved.

In the following theorem we replace Condition VI by another one, which is fulfilled in all the interesting practical cases.

THEOREM 2. *Let us suppose that for the stochastic set function $\xi(A)$ conditions* I, II, III, IV *and* VII *are fulfilled. Then all the assertions in Theorem* 1 *hold.*

Proof. We have only to show that $\mathrm{Var}_{1-P_0}(H) < \infty$. We shall carry out the proof by using Lemma 2. The set function $1-P_0(A)$, $A \epsilon \mathfrak{R}$, is bounded, non-negative and subadditive, since if $A = A_1+A_2$, $A_1 \epsilon \mathfrak{R}$,

4) This means that this equality holds for the real and imaginary parts separately.

$A_2 \epsilon \mathfrak{R}$, $A_1A_2 = 0$, then the event $\xi(A) \neq 0$ implies that at least one of $\xi(A_1) \neq 0$, $\xi(A_2) \neq 0$ holds. Let $B_1, B_2, \ldots$ be a sequence of pairwise disjoint sets of $\mathfrak{R}$. According to condition II ($\mathfrak{R}$ being a σ-ring) the series

$$\sum_{k=1}^{\infty} \xi(B_k)$$

converges with probability 1, whence, by the three series theorem of Kolmogorov (cf. [9], § 5),

$$\sum_{k=1}^{\infty}\big(1-P_0(B_k)\big) = \sum_{k=1}^{\infty} P\big(|\xi(B_k)| \geqslant \varrho\big) < \infty.$$

Thus all the conditions of Lemma 2 are fulfilled, and this completes the proof of Theorem 2.

3. Structural properties of abstract composed Poisson stochastic set functions

In this section we suppose the fulfilment of Conditions I, II, III, V, VI, VII [5]). Moreover, we assume that for fixed $\omega \epsilon \Omega$ [6]) the number-valued set functions [7]) $\xi(\omega, A)$, $A \epsilon \mathfrak{R}$, are completely additive set functions Let $\nu_k(B)$, $B \epsilon \mathfrak{R}$, denote the number of points $h \epsilon H$ to which correspond discontinuities of magnitude λ_k. We are going to prove some theorems concerning the random variables $\nu_k(B)$.

THEOREM 3. *For every $B \epsilon \mathfrak{R}$ the random variables $\nu_k(B)$, $k = 1, 2, \ldots$, have Poisson distributions with the expectations $C_k(B)$, $k = 1, 2, \ldots$*

Proof. As can be seen from the proof of Theorem 1, there exists a sequence of divisions $\mathfrak{Z}_n = \{A_1^{(n)}, A_2^{(n)}, \ldots, A_{l_n}^{(n)}\}$ of the set B into pairwise disjoint sets of $\mathfrak{R}$ such that

$$\mathrm{Var}_{1-P_0}(A_l^{(n)}) \leqslant c/n, \quad l = 1, 2, \ldots, l_n;\ n = 1, 2, \ldots, \tag{17}$$

where $c = \mathrm{Var}_{1-P_0}(B)$ and

$$\lim_{n\to\infty} \sum_{l=1}^{l_n} P_k(A_l^{(n)}) = \int_B P_k(dA) = C_k(B). \tag{18}$$

[5]) We observe that (as it is proved in Theorem 2) conditions I, II, III and IV imply the fulfilment of VI.

[6]) Ω denotes the space of elementary events.

[7]) These set functions will be called *sample functions*.

We may suppose that at the same time $\mathfrak{Z}_n$ has the property described in Condition V. Let us define the random variables

$$\mu_k(A_l^{(n)}) = \begin{cases} 1, & \text{if} \quad \xi(A_l^{(n)}) = \lambda_k, \\ 0, & \text{if} \quad \xi(A_l^{(n)}) \neq \lambda_k. \end{cases}$$

Let $\chi_k(u, A_l^{(n)})$ and $\psi_k(u, B)$ denote the characteristic functions of $\mu_k(A_l^{(n)})$ and $\nu_k(B)$, respectively. Clearly

$$\begin{gathered} \chi_k(u, A_l^{(n)}) = 1+(e^{iu}-1)P_k(A_l^{(n)}), \\ \psi_k(u, B) = \lim_{n\to\infty} \prod_{l=1}^{l_n} \chi_k(u, A_l^{(n)}). \end{gathered} \tag{19}$$

Taking into account (17) we get

$$\Big|\prod_{l=1}^{l_n} [1+(e^{iu}-1)P_k(A_l^{(n)})] - \exp\Big(\sum_{l=1}^{l_n}(e^{iu}-1)P_k(A_l^{(n)})\Big)\Big|$$

$$\leqslant \sum_{l=1}^{l_n} \big|1+(e^{iu}-1)P_k(A_l^{(n)}) - \exp\big((e^{iu}-1)P_k(A_l^{(n)})\big)\big|$$

$$\leqslant \sum_{l=1}^{l_n} |e^{iu}-1|^2 P_k^2(A_l^{(n)}) \leqslant \frac{4c^2}{n}$$

if n is large enough. It follows hence and from relations (18), (19) that $\psi_k(u, B) = \exp\big(C_k(B)(e^{iu}-1)\big)$, q. e. d.

THEOREM 4. *For every $B \epsilon \mathfrak{R}$, the random variables $\nu_1(B), \nu_2(B), \ldots$ are independent.*

Proof [8]). We prove that for every fixed s the variables $\nu_1(B)$, $\nu_2(B), \ldots, \nu_s(B)$ are independent. Let $\mathfrak{Z}_n = \{A_1^{(n)}, A_2^{(n)}, \ldots, A_{l_n}^{(n)}\}$ be a sequence of divisions of the set B, having the property described in Condition V and satisfying the relations

$$\begin{gathered} \lim_{n\to\infty} \sum_{l=1}^{l_n} P_k(A_l^{(n)}) = \int_B P_k(dA) = C_k(B), \quad k = 1, 2, \ldots, s, \\ \mathrm{Var}_{1-P_0}(A_l^{(n)}) \leqslant c/n, \quad l = 1, 2, \ldots, l_n; \quad n = 1, 2, \ldots, \end{gathered} \tag{20}$$

where $c = \mathrm{Var}_{1-P_0}(B)$.

[8]) The idea of this proof was proposed by A. Rényi.

Clearly

(21) $$P\big(\nu_1(B)=j_1,\ \nu_2(B)=j_2,\ \ldots,\ \nu_s(B)=j_s\big)=\lim_{n\to\infty}P^{(n)}_{j_1j_2\ldots j_s}$$

$$=\lim_{n\to\infty}P\Big(\sum_{l=1}^{l_n}\mu_1(A_l^{(n)})=j_1,\ \sum_{l=1}^{l_n}\mu_2(A_l^{(n)})=j_2,\ \ldots,\ \sum_{l=1}^{l_n}\mu_s(A_l^{(n)})=j_s\Big).$$

On the other hand, comparing the coefficients of

$$\exp\big(i(j_1u_1+j_2u_2+\ldots+j_su_s)\big)$$

it is easy to see that the multidimensional characteristic function of the distribution on the right-hand side of (21) has the form

(22) $$\sum_{j_1,j_2,\ldots,j_s}P^{(n)}_{j_1,j_2,\ldots,j_s}e^{i(j_1u_1+j_2u_2+\ldots+j_su_s)}$$

$$=\prod_{l=1}^{l_n}\{P_1(A_l^{(n)})e^{iu_1}+P_2(A_l^{(n)})e^{iu_2}+\ldots+P_s(A_l^{(n)})e^{iu_s}+$$
$$+1-P_1(A_l^{(n)})+1-P_2(A_l^{(n)})+\ldots+1-P_s(A_l^{(n)})\}$$

$$=\prod_{l=1}^{l_n}\{1+P_1(A_l^{(n)})(e^{iu_1}-1)+P_2(A_l^{(n)})(e^{iu_2}-1)+\ldots+P_s(A_l^{(n)})(e^{iu_s}-1)\}.$$

Taking into account (20) and (21), we obtain from (22)

$$\sum_{j_1,j_2,\ldots,j_s}P\big(\nu_1(B)=j_1,\ \nu_2(B)=j_2,\ \ldots,\ \nu_s(B)=j_s\big)e^{i(j_1u_1+j_2u_2+\ldots+j_su_s)}$$

$$=\exp\big(C_1(B)(e^{iu_1}-1)+C_2(B)(e^{iu_2}-1)+\ldots+C_s(B)(^{iu_s}-1)\big).$$

As $\exp\big(C_k(B)(e^{iu}-1)\big)$ is the characteristic function of $\nu_k(B)$ ($k=1,2,\ldots$), our theorem is proved.

Obviously $\nu_k(B)$ is a completely additive stochastic set function on $\mathfrak{R}$, or, in other terms, conditions I-II hold. We have seen that they *side* are of Poisson type. Finally we prove

THEOREM 5. *If $B_1, B_2, \ldots$ is an arbitrary sequence of sets of $\mathfrak{R}$, then the random variables $\nu_1(B_1), \nu_2(B_2), \ldots$ are independent and if $B\in\mathfrak{R}$, then*

(23) $$\xi(B)=\sum_{k=1}^{\infty}\lambda_k\nu_k(B),$$

where the sum of mutually independent random variables on the right-hand side converges with probability 1 regardless of the order of summation.

Proof. If the sets $B_1, B_2, \ldots$ are identical or disjoint, then $\nu_1(B_1)$, $\nu_2(B_2), \ldots$ are independent. In the general case we consider the first s sets and form the disjoint sets $\bar B_{i_1}\ldots\bar B_{i_r}B_{i_{r+1}}\ldots B_{i_s}$. The number of these sets is 2^s. Since the random variables $\nu_k(\bar B_{i_1}\ldots\bar B_{i_r}B_{i_{r+1}}\ldots B_{i_s})$ are independent, where k runs through the set of the positive integers $1,2,\ldots,s$, and $i_1, i_2, \ldots, i_r$ proceeds through all the combinations of r elements of $1,2,\ldots,s$, and furthermore the variables $\nu_1(B_1)$, $\nu_2(B_2)$, $\ldots, \nu_s(B_s)$ can be represented as sums of disjoint sets of the variables mentioned above, cur first assertion holds.

The convergence of the series in (23) is a consequence of formula (7), since

$$\psi_k(\lambda_ku, B)=\exp\big(C_k(B)(e^{i\lambda_ku}-1)\big)$$

is the characteristic function of the random variable $\lambda_k\nu_k(B)$; moreover, the infinite product

$$f(u,B)=\prod_{k=1}^{\infty}\psi_k(\lambda_ku, B)$$

converges absolutely and is also a characteristic function (see for instance [4], p. 115, Theorem 2.7).

Remark. Since the expectation of $\nu_k(B)$ is equal to $C_k(B)$, relation (9) implies that the sample functions have finite numbers of discontinuities with probability 1.

4. Application to random point distributions and the Poisson stochastic set function

In this section we specialize the set $\{\lambda_k\}$. We suppose that $\{\lambda_k\}$ is identical with the set of the non-negative integers and thus the situation can be described as follows: we throw a finite number of points at random on the set H so that the numbers of random points in disjoint sets belonging to $\mathfrak{R}$ are independent. If $\xi(A)$, $A\in\mathfrak{R}$, denotes the number of points in the set A, then conditions I-IV naturally hold. Thus we obtain

THEOREM 6. *If for the set function $\xi(A)$, $A\in\mathfrak{R}$, defined by a random point distribution Condition VII holds, then for every $B\in\mathfrak{R}$*

(24) $$\log f(u,B)=\sum_{k=1}^{\infty}C_k(B)(e^{iku}-1),$$

where the set functions $C_k(B)$ have all the properties described in Theorem 1.

Proof. Our statement immediately follows from Theorem 2.

Hence we can obtain conditions ensuring the Poisson character of a random point distribution. This is expressed in

THEOREM 7. *If $\mathfrak{R}$ is a σ-algebra and for the set function $\xi(A)$, $A \in \mathfrak{R}$, defined by a random point distribution,* condition VII *and one of the following three conditions hold,*

(a) $$\int_H P_1(dA) = \int_H \bigl(1-P_0(dA)\bigr),$$

(b) $$\mathrm{Var}_{P_1}(H) = \mathrm{Var}_{1-P_0}(H),$$

(c) $$\int_H P_k(dA) = 0 \quad \textit{for} \quad k = 2, 3, \ldots,$$

then the random variables $\xi(B)$, $B \in \mathfrak{R}$, have Poisson distributions with the parameters $\int_B P_1(dA)$, $B \in \mathfrak{R}$.

Proof. If for a random point distribution Condition VII holds, then (a), (b) and (c) are equivalent. In fact $P_1(A)$ and $1-P_0(A)$ are subadditive set functions, whence by Lemma 5, (a) and (b) are equivalent. The equivalence of (a) and (c) is ensured by relations (8) and (9). Thus it is sufficient to consider (c). Our statement follows at once from Theorem 3 if we observe that (c) includes

$$C_k(B) = \int_B P_k(dA) = 0 \quad \text{for} \quad k = 2, 3, \ldots, \quad B \in \mathfrak{R}.$$

If in the random point distribution there are only single points, *i. e.* if we have $\nu_k(B) = 0$, $k = 2, 3, \ldots$, for every $B \in \mathfrak{R}$, then we hope to obtain Poisson distributions for the variables $\xi(B)$, $B \in \mathfrak{R}$. However, we need for our proof condition V concerning the space H. The proof of that condition being unnecessary or a counterexample would be desirable. Our result is contained in

THEOREM 8. *If in a random point distribution there are only single points, and furthermore if Conditions* V *and* VII *are fulfilled, then for every $B \in \mathfrak{R}$*

$$P\bigl(\xi(B) = k\bigr) = \frac{\lambda^k(B)}{k!} e^{-\lambda(B)}, \quad k = 0, 1, 2, \ldots, \tag{25}$$

where $\lambda(B) = C_1(B)$ is the average number of points lying in B and

$$\lambda(B) = \int_B P_1(dA). \tag{26}$$

Proof. Since $\nu_k(B) = 0$ for $k = 2, 3, \ldots$, by Theorem 3 we have $C_k(B) = M\bigl(\nu_k(B)\bigr) = 0$ for $k = 2, 3, \ldots$

Applying Theorem 6 and taking into account the result in Theorem 1 concerning the connection of $C_1(B)$ and $P_1(A)$, we obtain the statements of Theorem 8.

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