On approximation in real Banach spaces by analytic operations

by

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Analytic operations defined in real Banach spaces were introduced by Alexiewicz and Orlicz in their paper [1]. Let $F(x)$ be a continuous operation from a real Banach space $B$ to an arbitrary Banach space $B_1$. In the paper [2] I proved that it is possible to approximate uniformly the operation $F(x)$ by an analytic operation, if the space $B$ is separable and fulfills the following condition:

(A) there exists such a real polynomial $q(x)$ that

$$q(x)(0) = 0, \quad \inf_{x \in B, \|x\|=1} q(x) > 0.$$

The condition (A) must not be omitted. In the paper quoted above I proved that, for example, the functional $\|x\|$ in the space $C(0,1)$ is not the uniform limit of a sequence of analytic functionals.

I aim to prove that the condition (A) is necessarily fulfilled if the space $B$ is uniformly convex and if it is possible to approximate uniformly every continuous operation $F(x)$ by an analytic operation.

The main result of this paper is the following

THEOREM 1. Let the real Banach space $B$ be uniformly convex and assume that every real polynomial $q(x)$ which is defined in $B$ fulfills the condition

$$\inf_{x \in B, \|x\|=1} |q(x) - q(0)| = 0.$$  \hspace{1cm} (1)

Let $f(x)$ be a real analytic functional which is defined for $x \in B$, $\|x\| < R$ ($R > 0$). If $r$ and $e$ are two given positive numbers, $e < r$, $r+e < R$, then there exists a point $x \in B$ satisfying the inequalities $r \leq \|x\| < r+e$, $|f(x) - f(0)| < e$.

In order to prove Theorem 1 the following lemmas will be useful:

**Lemma 1.** Let the Banach space $B$ be uniformly convex. Then there is such a positive non-decreasing function $\chi(e)$ ($e > 0$) that the following condition holds:

$$\lambda = \lambda(x) = 1, \ k \in B, \ \|x\| = 1, \ \|k\| > \epsilon, \ \lambda(x) = \lambda(x + \chi(e)) = 1, \ \|x + \chi(e)\| > 1 + \chi(e).$$

**Proof.** As the space $B$ is uniformly convex, there is such a positive non-decreasing function $\eta(e)$ ($e > 0$) that $\|x_1\| = \|x_2\| = 1, \ |x_1 - x_2| < e$ implies that $\|x_1 + x_2\| < 2(1 - \eta(e))$. If the lemma is false, then there exist a number $\epsilon > 0$ and sequences $x_n, h_n, k_n$ such that $\|x_n\| = 1, \ |h_n| > \epsilon, \ |\lambda_n - h_n| = 1, \ |x_n + h_n| = 1$. It follows that $\lambda_n(x_n) = \lambda_n(x_n + h_n) = 1$, $\|x_n + h_n\| = 1$, $\|\lambda_n - h_n\| = 1$, $\lambda_n(x_n + h_n) = \lambda_n(x_n + h_n)$, where $h_n \to 0$. As $B$ is uniformly convex, $\|x_n + \frac{1}{2}(h_n + h_n)\| < 1 - \eta(\epsilon)$ and $\|x_n + \frac{1}{2}(h_n + h_n)\| < 1 - \eta(\epsilon)$ for large $n$. From $\|\lambda_n - h_n\| = 1$, $\lambda_n(x_n + h_n) = 1$ we get $\|x_n + \frac{1}{2}(h_n + h_n)\| > 1$ and the proof is complete.

Let us note that the converse of Lemma 1 is true as well.

**Lemma 2.** Let us suppose that every real polynomial $q(x)$ defined in $B$ fulfills the following condition:

$$\inf_{x \in B, \|x\|=1} |q(x) - q(0)| = 0.$$  \hspace{1cm} (2)

Then we have

$$\inf_{x \in B, \|x\|=1} |q(x) - q(0)| = 0$$

for every real polynomial $q(x)$ defined in $B$ and for every positive $r$.

The proof is obvious: we write $q(x) = q(0) + q_1(x) + \ldots + q_r(x)$ where $q_i(x)$ are homogeneous polynomials of degree $i$ and consider the polynomial $q(x) = q_1(x) + \ldots + q_r(x)$.

**Lemma 3.** Let $q_1, q_2, \ldots, q_n$ be linear functionals on $B$. Let every polynomial $q(x)$ on $B$ fulfill the condition

$$\inf_{x \in B, \|x\|=1} |q(x) - q(0)| = 0,$$

and let $y \in B$.

Then every polynomial $q(x)$ fulfills the condition

$$\inf_{x \in B, \|x\|=1} |q(x) - q(y)| = 0,$$

where $x$ satisfies the relations $x \in B$, $\|x\| = 1$,

$q_1(x) = q_1(y), \ q_2(x) = q_2(y), \ldots, \ q_n(x) = q_n(y).$
Proof. Let us suppose that $n = 1$ and that condition (2) is not satisfied by every polynomial. In this case there exist such a $y \in B$ and such a polynomial $q(x)$ that

$$\inf_{x \in B, \|x - y\| = 1} \|q(x) - q(y)\| > 0,$$

where

$$x \in B, \quad \|x - y\| = 1, \quad q(x) = q(y).$$

We put $\tilde{q}(x) = q(x) + p_1(x) = q_1(x)$. $\tilde{q}(x)$ is a polynomial and we verify that

$$\inf_{x \in B, \|x - y\| = 1} \|\tilde{q}(x) - \tilde{q}(y)\| > 0.$$ 

This means that lemma 3 holds for $n = 1$. The case $n > 1$ may be treated by induction (or analogously).

We proceed to the proof of Theorem 1. Let $f(x)$ be a real-valued analytic function defined for $x \in B$, $\|x\| < R$. Let us denote by $\beta(x_0)$ the radius of convergence of the power-series $p_1(x - x_0) + p_2(x - x_0)^2 + \ldots + p_n(x - x_0)^n + \ldots$, which converges to $f(x)$. Analogously to the case of a scalar independent variable, $\beta(x_0)$ depends continuously on $x_0$. We put $\gamma(x_0) = \min(\beta(x_0), \epsilon)$. Let us define a sequence $x_1, x_2, \ldots, x_n = 0$.

We choose the point $x_1$ satisfying the conditions

$$\|f(x_1) - f(\theta_1)\| < \frac{\epsilon}{2R}, \quad \frac{1}{2} \gamma(\theta_1) = \|x_1\|.$$ 

The point $x_1$ exists, as $f(x)$ may be developed in a power-series, which converges uniformly for $\|x\| < \frac{1}{2} \gamma(\theta_1)$ and as every polynomial satisfies (1). Let us suppose that the points $x_1, x_2, \ldots, x_n$ and linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ are chosen in such a way that

$$\begin{aligned}
\|\varphi_1\| = \|\varphi_2\| = \ldots = \|\varphi_{n-1}\| = 1, \\
\varphi_1(x_2) = \|x_1\|, \quad \varphi_2(x_2) = \|x_1\|, \ldots, \varphi_{n-1}(x_{n-1}) = \|x_{n-1}\|, \\
\varphi_1(x_3) = \varphi_2(x_3) = \ldots = \varphi_1(x_3) = \varphi_2(x_3), \\
\varphi_1(x_4) = \varphi_2(x_4) = \ldots = \varphi_1(x_4) = \varphi_2(x_4), \\
\ldots, \\
\varphi_1(x_n) = \varphi_2(x_n), \\
\frac{1}{2} \gamma(\theta_1) = \|x_{n+1} - x_1\|, \quad |f(x_{n+1}) - f(x_1)| = \frac{\epsilon}{2R} \|x_{n+1}\| \\
&\quad (\epsilon = 0, 1, 2, \ldots, n - 1).
\end{aligned}$$

Let us choose the linear functional $\varphi_n$, $\|\varphi_n\| = 1$, $\varphi_n(x_n) = \|x_n\|$ and let the point $x_{n+1}$ satisfy the conditions $\varphi_n(x_{n+1}) = \varphi_n(x_n)$, $i = 1, 2, \ldots, n$, $\|x_{n+1} - x_n\| = \frac{1}{2} \gamma(x_n)$. Let us prove that the point $x_{n+1}$ exists if $\|x_n\| < \epsilon$. 

We write

$$f(x) = f(x_1) + p_1(x - x_1) + p_2(x - x_1) + \ldots$$

and the power-series converges uniformly for $\|x - x_1\| \leq \frac{1}{2} \gamma(x_1)$. We find such an integer $k$ that

$$\left| \sum_{a=1}^{k} p_a(x - x_1) \right| < \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)|,$$

if $\|x - x_1\| \leq \frac{1}{2} \gamma(x_1)$.

Let us put $q(x) = p_1(x - x_1) + p_2(x - x_1) + \ldots + p_n(x - x_1)$. As every polynomial satisfies (1), according to lemma 3 we get

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x)\| = 0,$$

and according to lemma 2 ($q(x + x_0)$ is a polynomial in the variable $y$) we have

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x + x_0)\| = 0,$$

and

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x)\| = 0,$$

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x + x_0)\| = 0.$$

Consequently, there exists such a point $x$ that

$$|q(x)| < \left( \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)| \right),$$

$$\|x - x_1\| = \frac{1}{2} \gamma(x_1), \quad q(x) = q(x_1), \quad i = 1, 2, \ldots, n.$$ 

As $\varphi_n(x) = \varphi_n(x_1)$, $\|x_n\| = 1$ we get $\|\varphi_n\| = \|\varphi_n\|$. We put $x_{n+1} = x$ and get

$$|f(x_{n+1}) - f(x_1)| \leq |q(x_{n+1})| + \sum_{a=1}^{n} p_a(x_{n+1} - x_1)^a \leq \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)|,$$

$$\left| f(x_{n+1}) - f(\theta) \right| < \frac{\epsilon}{2R} \|x_n\| \leq \frac{\epsilon}{2R} \|x_n\|.$$

It follows that relations (3) hold if we replace $n$ by $n+1$. 

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We write

$$f(x) = f(x_1) + p_1(x - x_1) + p_2(x - x_1) + \ldots$$

and the power-series converges uniformly for $\|x - x_1\| \leq \frac{1}{2} \gamma(x_1)$. We find such an integer $k$ that

$$\left| \sum_{a=1}^{k} p_a(x - x_1) \right| < \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)|,$$

if $\|x - x_1\| \leq \frac{1}{2} \gamma(x_1)$.

Let us put $q(x) = p_1(x - x_1) + p_2(x - x_1) + \ldots + p_n(x - x_1)$. As every polynomial satisfies (1), according to lemma 3 we get

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x)\| = 0,$$

(4) 

and according to lemma 2 ($q(x + x_0)$ is a polynomial in the variable $y$) we have

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x + x_0)\| = 0,$$

and

$$\inf_{x \in B, \|x - x_1\| = 1} \|q(x)\| = 0.$$ 

Consequently, there exists such a point $x$ that

$$|q(x)| < \left( \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)| \right),$$

$$\|x - x_1\| = \frac{1}{2} \gamma(x_1), \quad q(x) = q(x_1), \quad i = 1, 2, \ldots, n.$$ 

As $\varphi_n(x) = \varphi_n(x_1)$, $\|x_n\| = 1$ we get $\|\varphi_n\| = \|\varphi_n\|$. We put $x_{n+1} = x$ and get

$$|f(x_{n+1}) - f(x_1)| \leq |q(x_{n+1})| + \sum_{a=1}^{n} p_a(x_{n+1} - x_1)^a \leq \frac{\epsilon}{2R} \|x_n\| - |f(x_1) - f(\theta)|,$$

$$\left| f(x_{n+1}) - f(\theta) \right| < \frac{\epsilon}{2R} \|x_n\| \leq \frac{\epsilon}{2R} \|x_n\|.$$ 

It follows that relations (3) hold if we replace $n$ by $n+1$. 

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Finally we prove that the number of points \( x_0, x_1, x_2, \ldots \) which fulfill \( \|x_i\| < r \) is necessarily finite. Let us suppose that the sequence 
\( x_0, x_1, x_2, \ldots \) is infinite, that \( \|x_i\| < r \) for \( i = 1, 2, 3, \ldots \) and let \( j > i \).

As \( \|x_j\| = 1 \), \( \varphi(x_j) = \varphi(x_j) = \|x_j\| = 1 \), applying Lemma 1 we get

\[
\frac{\|x_j\|}{\|x_i\|} \geq 1 + \frac{\|x_j - x_i\|}{\|x_i\|},
\]

\[
\|x_j - x_i\| \geq \|x_i\| \cdot \frac{\|x_j - x_i\|}{\|x_i\|} \geq \|x_i\| \cdot \frac{\|x_j - x_i\|}{\|x_i\|}.
\]

As \( 0 < \|x_i\| \leq \|x_n\| \leq \ldots < r \) and as \( \varphi(x) \) is positive and non-decreasing, the sequence \( x_n \) is a Cauchy-sequence and

\[
\lim_{n \to \infty} x_n = \tilde{x}
\]

exists. It follows that \( \frac{1}{\gamma(x_n)} = \|x_{n+1} - x_n\| \to 0 \), \( \gamma(\tilde{x}) > 0 \), and we arrive at a contradiction of the fact that the function \( \gamma(x) \) is continuous.

Consequently — as \( \frac{1}{\gamma} \geq \frac{1}{\gamma(x_n)} = \|x_{n+1} - x_n\| \) — there exists a point

\[
x_{n+1}, r \leq \|x_{n+1}\| < r + \epsilon.
\]

The proof of Theorem 1 is complete.

Theorem 1 together with the results of [2] enable us to state the following

**Theorem 2.** Let the space \( B \) be separable and uniformly convex. Then the following three conditions (A), (C) and (C') are equivalent:

(A) There exists a polynomial \( q^*(x) \) in \( B \) such that

\[
\inf_{x \in B, \|x\| = 1} |q^*(x) - q^*(\theta)| > 0;
\]

(C) (C') If \( G \) is an open subset of \( B \) and \( \Phi(x) \) is a continuous operation defined in \( G \) with values in an arbitrary Banach space \( B_1 \) (with values in \( B_1 \)) and if \( \epsilon \) is a positive number, then there exists such an analytic operation \( \Phi(x) \) in \( G \) with values in \( B_1 \) (in \( E_1 \)) that

\[
\|\Phi(x) - H(x)\| < \epsilon \quad \text{for} \quad x \in G.
\]

**Proof.** It follows from [2], Theorem 2, that (A) implies (C). Apparently (C) implies (C'). According to Theorem 1 it is not possible to approximate the functional \( \|x\| \) uniformly by analytic functionals if (A) is not fulfilled. Consequently (C') implies (A).