On approximation in real Banach spaces by analytic operations

by

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Analytic operations defined in real Banach spaces were introduced by Alexiewicz and Orlicz in their paper [1]. Let $F(x)$ be a continuous operation from a real Banach space B to an arbitrary Banach space B_1. In the paper [2] I proved that it is possible to approximate uniformly the operation $F(x)$ by an analytic operation, if the space B is separable and fulfils the following condition:

(A) there exists such a real polynomial $q^*(x)$ that

$$q^*(\theta) = 0^{1)}, \quad \inf_{x \in B, \|x\|=1} q^*(x) > 0.$$

The condition (A) must not be omitted. In the paper quoted above I proved that, for example, the functional $\|x\|$ in the space $C\langle 0,1\rangle$ is not the uniform limit of a sequence of analytic functionals [2]).

The aim of this paper is to prove that the condition (A) is necessarily fulfilled if the space B is uniformly convex and if it is possible to approximate uniformly every continuous operation $F(x)$ by an analytic operation.

The main result of this paper is the following

THEOREM 1. *Let the real Banach space B be uniformly convex and let us suppose that every real polynomial $q(x)$ which is defined in B fulfils the condition*

$$\inf_{x \in B, \|x\|=1} |q(x) - q(\theta)| = 0. \tag{1}$$

Let $f(x)$ be a real analytic functional which is defined for $x \in B$, $\|x\| < R$ $(R > 0)$. If r and ε are two given positive numbers, $\varepsilon < r$, $r + \varepsilon < R$, then there exists a point $x \in B$ satisfying the inequalities $r \leqslant \|x\| < r + \varepsilon$, $|f(x) - f(\theta)| < \varepsilon$.

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[1]) θ is the zero element of B.

[2]) See [2], Theorem 3; let us recall that an analytic operation is regularly differentiable.

In order to prove Theorem 1 the following lemmas will be useful:

LEMMA 1. *Let the Banach space B be uniformly convex. Then there is such a positive non-decreasing function $\chi(\varepsilon)$ $(\varepsilon > 0)$ that the following condition holds:*

if λ is a linear functional, $\|\lambda\| = 1$, $x \in B$, $h \in B$, $\|x\| = 1$, $\|h\| > \varepsilon$, $\lambda(x) = \lambda(x+h) = 1$, then $\|x+h\| > 1 + \chi(\varepsilon)$.

Proof. As the space B is uniformly convex, there is such a positive non-decreasing function $\eta(\varepsilon)$ $(\varepsilon > 0)$ that $\|x_1\| = \|x_2\| = 1$, $\|x_1 - x_2\| < \varepsilon$ implies that $\|x_1 + x_2\| < 2(1 - \eta(\varepsilon))$. If the lemma is false, then there exist a number $\varepsilon > 0$ and sequences x_n, h_n, λ_n such that $\|x_n\| = 1$, $\|h_n\| > \varepsilon$, $\|\lambda_n\| = 1$, $\lambda_n(x_n) = \lambda_n(x_n + h_n) = 1$, $\|x_n + h_n\| \to 1$. It follows that $(x_n + h_n)\|x_n + h_n\|^{-1} = x_n + h_n + k_n$, where $k_n \to 0$. As B is uniformly convex, $\|x_n + \frac{1}{2}(h_n + k_n)\| < 1 - \eta(\frac{1}{2}\varepsilon)$ and $\|x_n + \frac{1}{2}h_n\| < 1 - \frac{1}{2}\eta(\frac{1}{2}\varepsilon)$ for great n. From $\|\lambda_n\| = 1$, $\lambda_n(x_n + \frac{1}{2}h_n) = 1$ we get $\|x_n + \frac{1}{2}h_n\| \geqslant 1$ and the proof is complete.

Let us note that the converse of lemma 1 is true as well.

LEMMA 2. *Let us suppose that every real polynomial $q(x)$ defined in B fulfils the following condition:*

$$\inf_{x \in B, \|x\|=1} |q(x) - q(\theta)| = 0.$$

Then we have

$$\inf_{x \in B, \|x\|=r} |q(x) - q(\theta)| = 0$$

for every real polynomial $q(x)$ defined in B and for every positive r.

The proof is obvious: we write $q(x) = q(\theta) + q_1(x) + \ldots + q_k(x)$ where $q_i(x)$ are homogeneous polynomials of degree i and consider the polynomial $\bar{q}(x) = q_1^2(x) + \ldots + q_k^2(x)$.

LEMMA 3. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be linear functionals on B. Let every polynomial $q(x)$ on B fulfil the condition*

$$\inf_{x \in B, \|x\|=1} |q(x) - q(\theta)| = 0,$$

and let $y \in B$.

Then every polynomial $q(x)$ fulfils the condition

$$\inf_{x} |q(x) - q(y)| = 0, \tag{2}$$

where x satisfies the relations $x \in B$, $\|x - y\| = 1$,

$$\varphi_1(x) = \varphi_1(y), \quad \varphi_2(x) = \varphi_2(y), \quad \ldots, \quad \varphi_n(x) = \varphi_n(y).$$

Proof. Let us suppose that $n = 1$ and that condition (2) is not satisfied by every polynomial. In this case there exist such a $y \epsilon B$ and such a polynomial $q(x)$ that

$$\inf\big(q(x)-q(y)\big) > 0,$$

where

$$x \epsilon B, \quad \|x-y\| = 1, \quad \varphi_1(x) = \varphi_1(y).$$

We put $\tilde{q}(x) = q(x+y)+\varphi_1^2(x)$. $\tilde{q}(x)$ is a polynomial and we verify that

$$\inf_{x\epsilon B, \|x\|=1} \big(\tilde{q}(x)-\tilde{q}(\theta)\big) > 0.$$

This means that lemma 3 holds for $n = 1$. The case $n > 1$ may be treatep by induction (or analogously).

We proceed to the proof of Theorem 1. Let $f(x)$ be a real-valued analytic function defined for $x \epsilon B$, $\|x\| < R$. Let us denote by $\beta(x_0)$ the radius of convergence of the power-series $p_0(x-x_0)+p_1(x-x_2)+ +p_2(x-x_0)+\ldots$ which converges to $f(x)$. Analogously to the case of a scalar independent variable $\beta(x_0)$ depends continuously on x_0. We put $\gamma(x_0) = \min\big(\beta(x_0), \varepsilon\big)$. Let us define a sequence $x_0, x_1, x_2, \ldots, x_0 = \theta$.

We choose the point x_1 satisfying the conditions

$$|f(x_1)-f(\theta_0)| < \frac{\varepsilon}{2r}\|x_1\|, \quad \tfrac{1}{2}\gamma(\theta) = \|x_1\|.$$

The point x_1 exists, as $f(x)$ may be developed in a power-series, which converges uniformly for $\|x\| < \frac{1}{2}\gamma(\theta)$ and as every polynomial satisfies (1). Let us suppose that the points $x_0, x_1, x_2, \ldots, x_n$ and linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ are chosen in such a way that

$$(3)\quad \begin{cases} \|\varphi_1\| = \|\varphi_2\| = \ldots = \|\varphi_{n-1}\| = 1, \\ \varphi_1(x_1) = \|x_1\|, \quad \varphi_2(x_2) = \|x_2\|, \quad \ldots, \quad \varphi_{n-1}(x_{n-1}) = \|x_{n-1}\|, \\ \varphi_1(x_n) = \varphi_1(x_{n-1}) = \ldots = \varphi_1(x_2) = \varphi_1(x_1), \\ \varphi_2(x_n) = \varphi_2(x_{n-1}) = \ldots = \varphi_2(x_2), \\ \cdots\cdots\cdots\cdots\cdots \\ \varphi_{n-1}(x_n) = \varphi_{n-1}(x_{n-1}), \\ \tfrac{1}{2}\gamma(x_i) = \|x_{i+1}-x_i\|, \quad |f(x_{i+1})-f(x_0)| < \dfrac{\varepsilon}{2r}\|x_{i+1}\| \\ \qquad\qquad (i = 0, 1, 2, \ldots, n-1). \end{cases}$$

Let us choose the linear functional φ_n, $\|\varphi_n\| = 1$, $\varphi_n(x_n) = \|x_n\|$ and let the point x_{n+1} satisfy the conditions $\varphi_i(x_{n+1}) = \varphi_i(x_n)$, $i = 1, 2, \ldots, n$, $\|x_{n+1}-x_n\| = \frac{1}{2}\gamma(x_n)$. Let us prove that the point x_{n+1} exists if $\|x_n\| < r$.

We write

$$f(x) = f(x_n)+p_{1,n}(x-x_n)+p_{2,n}(x-x_n)+\ldots$$

and the power-series converges uniformly for $\|x-x_n\| \leqslant \frac{1}{2}\gamma(x_n)$. We find such an integer k that

$$\Big|\sum_{i=k+1}^{\infty} p_{i,n}(x-x_n)\Big| < \tfrac{1}{2}\Big(\frac{\varepsilon}{2r}\|x_n\| - |f(x_n)-f(\theta)|\Big),$$

if $\|x-x_n\| \leqslant \frac{1}{2}\gamma(x_n)$.

Let us put $q(x) = p_{1,n}(x-x_n)+p_{2,n}(x-x_n)+\ldots+p_{k,n}(x-x_n)$. As every polynomial satisfies (1), according to lemma 3 we get

$$(4)\quad \begin{gathered} \inf|q(x)| = 0, \\ x\epsilon B, \quad \|x-x_n\| = 1, \quad \varphi_i(x) = \varphi_i(x_n), \quad i = 1, 2, \ldots, n. \end{gathered}$$

Let us denote by B_n the intersection of the hyperplanes $\varphi_i(y) = 0$, $i = 1, 2, \ldots, n$. We rewrite (4) in the form

$$\inf_{y\epsilon B_n, \|y\|=1} |q(y+x_n)| = 0,$$

and according to lemma 2 ($q(y+x_n)$ is a polynomial in the variable y) we have

$$\inf|q(y+x_n)| = 0, \quad y\epsilon B_n, \quad \|y\| = \tfrac{1}{2}\gamma(x_n),$$

and

$$\begin{gathered} \inf|q(x)| = 0, \\ x\epsilon B, \quad \|x-x_n\| = \tfrac{1}{2}\gamma(x_n), \quad \varphi_i(x) = \varphi_i(x_n), \quad i = 1, 2, \ldots, n. \end{gathered}$$

Consequently, there exists such a point z that

$$|q(z)| < \tfrac{1}{2}\Big(\frac{\varepsilon}{2r}\|x_n\| - |f(x_n)-f(\theta)|\Big),$$

$$\|z-x_n\| = \tfrac{1}{2}\gamma(x_n), \quad \varphi_i(z) = \varphi_i(x_n), \quad i = 1, 2, \ldots, n.$$

As $\varphi_n(z) = \varphi_n(x_n) = \|x_n\|$, $\|\varphi_n\| = 1$ we get $\|z\| \geqslant \|x_n\|$. We put $x_{n+1} = z$ and get

$$|f(x_{n+1})-f(x_n)| \leqslant |q(x_{n+1})| + \Big|\sum_{i=k+1}^{\infty} p_{i,n}(x_{n+1}-x_n)\Big| < \frac{\varepsilon}{2r}\|x_n\| - |f(x_n)-f(\theta)|,$$

$$|f(x_{n+1})-f(\theta)| < \frac{\varepsilon}{2r}\|x_n\| \leqslant \frac{\varepsilon}{2r}\|x_{n+1}\|.$$

It follows that relations (3) hold if we replace n by $n+1$.

Finally we prove that the number of points $x_0, x_1, x_2, \ldots$ which fulfil $\|x_i\| < r$ is necessarily finite. Let us suppose that the sequence $x_0, x_1, x_2, \ldots$ is infinite, that $\|x_i\| < r$ $(i = 1, 2, 3, \ldots)$ and let $j > i$. As $\|\varphi_i\| = 1$, $\varphi_i(x_j) = \varphi_i(x_i) = \|x_i\|$, applying Lemma 1 we get

$$\left\| \frac{x_j}{\|x_i\|} \right\| \geqslant 1 + \chi\left(\frac{\|x_j - x_i\|}{\|x_i\|}\right),$$

$$\|x_j\| - \|x_i\| \geqslant \|x_i\| \chi\left(\frac{\|x_j - x_i\|}{\|x_i\|}\right) \geqslant \|x_1\| \chi\left(\frac{\|x_j - x_i\|}{\|x_1\|}\right).$$

As $0 < \|x_1\| \leqslant \|x_2\| \leqslant \ldots < r$ and as $\chi(\varepsilon)$ is positive and non-decreasing, the sequence x_n is a Cauchy-sequence and

$$\lim_{n\to\infty} x_n = \tilde{x}$$

exists. It follows that $\frac{1}{2}\gamma(x_n) = \|x_{n+1} - x_n\| \to 0$, $\gamma(\tilde{x}) > 0$, and we arrive at a contradiction of the fact that the function $\gamma(x)$ is continuous. Consequently — as $\frac{1}{2}\varepsilon \geqslant \frac{1}{2}\gamma(x_n) = \|x_{n+1} - x_n\|$ — there exists a point x_{n+1}, $r \leqslant \|x_{n+1}\| < r + \varepsilon$.

The proof of Theorem 1 is complete.

Theorem 1 together with the results of [2] enable us to state the following

THEOREM 2. *Let the space B be separable and uniformly convex. Then the following three conditions* (A), (C) *and* (C′) *are equivalent*:

(A) *There exists such a polynomial $q^*(x)$ in B that*

$$\inf_{x \in B, \|x\|=1} |q^*(x) - q^*(\theta)| > 0;$$

(C) ((C′)) *If G is an open subset of B and $F(x)$ is a continuous operation defined in G with values in an arbitrary Banach space B_1 (with values in E_1) and if ε is a positive number, then there exists such an analytic operation $H(x)$ in G with values in B_1 (in E_1) that*

$$\|F(x) - H(x)\| < \varepsilon \quad \textit{for} \quad x \in G.$$

Proof. It follows from [2], Theorem 2, that (A) implies (C). Apparently (C) implies (C′). According to Theorem 1 it is not possible to approximate the functional $\|x\|$ uniformly by analytic functionals if (A) is not fulfilled. Consequently (C′) implies (A).

References

[1] A. Alexiewicz and W. Orlicz, *Analytic operations in real Banach spaces*, Studia Math. 14 (1953), p. 57-78.

[2] J. Kurzweil, *On approximation in real Banach spaces*, ibidem 14 (1954), p. 214-231.

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