and write
\[ a_n = \int_0^1 |x(t)| \, dt. \]

Then
\[ 0 \leq \sum_{k=1}^n a_k \leq 1 \]

and
\[ \|U(x_1, \ldots, x_n)\| = \|U(x_1, \ldots, x_n)\| \leq \sum_{k=1}^n \left( \int_0^1 |x_k(t)| \, dt \right) \]
\[ = \sum_{k=1}^n \left( \int_0^1 |x_k(t)| \, dt \right) \leq \sum_{k=1}^n \sum_{j=1}^n a_{kj} \leq 1. \]

On the other hand choose
\[ a_k(t) = \begin{cases} n & \text{for } t \in J_k, \\ 0 & \text{elsewhere}. \end{cases} \]

Then \( \|a_k\| = 1 \) and \( U(x_1, \ldots, x_n) = 1. \) Hence
\[ \sup_{|x| \leq 1} \|U(x_1, \ldots, x_n)\| = 1. \]

By definition of the operation \( U(x_1, \ldots, x_n) \) it easily follows that
\[ U(x) = U(x_1, \ldots, x_n) = n! \sum_{k=1}^n \int_0^1 x(t) \, dt. \]

Let \( |x| \leq 1; \) since
\[ |x| = \sum_{k=1}^n \int_0^1 |x(t)| \, dt, \]
we obtain
\[ \|U(x)\| = \|U(x_1, \ldots, x_n)\| = n! \left( \int_0^1 |x(t)| \, dt \right)^n \leq n! \sum_{k=1}^n \left( \int_0^1 |x(t)| \, dt \right)^n \leq n! |x|. \]

On the other hand, \( x(t) = 1 \) implies \( U(x_1, \ldots, x_n) = n!|x|^n, \) whence
\[ \sup_{|x| \leq 1} \|U(x)\| = n! |x|^n. \]

Thus the equality (*) is true.

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On quotient-fields generated by pseudonormed rings

by

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I. A linear ring \( R \) over the field of complex numbers as scalars with unit element is called a pseudonormed ring if \( R \) is a \( \mathcal{B}_1 \)-space (Mazur and Orlicz [4], p. 185) with respect to a sequence of pseudonorms \( ||| \cdot |||_i \), \( \cdot \leq \cdots \) submultiplicative, i.e. satisfying the condition \( |||xy||| \leq |||x||| |||y||| \), for every \( x, y \in R \).

In this paper we shall deal exclusively with commutative pseudonormed rings without divisors of zero.

Let \( Q(R) \) be the quotient-field obtained from the pseudonormed ring \( R \) (Van der Waerden [7], p. 46-49). J. G.-Mikusiński has introduced by means of the convergence in the ring \( R \) a convergence in the field \( Q(R) \) (which will be called the \( M \)-convergence) as follows: the sequence \( a_1, a_2, \ldots, (a_n, Q(R)) \) converges to \( ax \in Q(R) \) (which will be denoted by \( M \lim a_n = a \)) if there exists \( \varepsilon > 0 \) such that \( sa_n \in R \) \( (n=1,2,\ldots) \), \( sa \in R \) and the sequence \( sa_1, sa_2, \ldots \) converges to \( sa \) in the ring \( R \), i.e. \( \lim|sa_n - sa| = 0 \) \( (n=1,2,\ldots) \) (Mikusiński [5], p. 62). It is easily seen that if \( Q(R) \) with the \( M \)-convergence is an \( L \)-space of Fréchet (Kuratowski [3], p. 84) and that the addition and the ring- and scalar-multiplication in \( Q(R) \) are continuous with respect to the \( M \)-convergence.

The \( M \)-convergence in \( Q(R) \) is called topological if there exists in \( Q(R) \) a topology satisfying the first axiom of countability (Kuratowski [3], p. 33), such that the convergence according to this topology is equivalent to the \( M \)-convergence, and \( Q(R) \) is a topological field with respect to this topology (we suppose here that the inverse of an element is a continuous function). In connection with the study of the \( M \)-convergence in the field of operators of Mikusiński ([Urbanik [6]] arises the problem of determining these pseudonormed rings for which the \( M \)-convergence in \( Q(R) \) is topological. In this paper we shall prove that

If the \( M \)-convergence in the field \( Q(R) \) is topological, then the pseudonormed ring \( R \) is isomorphic and homeomorphic with the field of complex numbers (topologized as usual).
It is sufficient to prove that if the $M$-convergence is topological, then the field $Q(R)$ with the $M$-convergence is a convex (i.e., locally convex) linear topological space. Indeed, from a theorem of Arens ([1], p. 625) we infer that the field $Q(R)$ is isomorphic to the field of complex numbers, whence it immediately follows that the ring $R$ has this property. I am indebted to Mr. Alexiewicz and to Mr. Altman for calling my attention to the fact that my first proof may be simplified by applying the theorem of Arens. Since the $B^*_t$-spaces (Mazur and Orlicz [4], p. 185) are convex linear topological spaces, the convexity of the space $Q(R)$ follows from

**Lemma.** If the $M$-convergence in $Q(R)$ is topological, then $Q(R)$ provided with the $M$-convergence is a $B^*_t$-space.

**Proof.** The fact of the $M$-convergence in $Q(R)$ being topological implies that $Q(R)$ is a topological group with respect to addition and satisfies the first axiom of countability. Hence by a theorem of S. Kakutani [2] there exists a distance-function $d$ in the field $Q(R)$ such that the $M$-convergence is equivalent to the convergence involved by this distance and such that

$$d(a,b) = d(a + c, b + c)$$

for every $a, b, c \in Q(R)$. Setting $|a| = d(a, 0)$ we obtain a (non-homogeneous) norm $|| \cdot ||$ in $Q(R)$, i.e., satisfying

1. $||a|| = 0$ if and only if $a = 0$,

2. $||a + b|| \leq ||a|| + ||b||$,

3. $||a|| = ||-a||$.

From the equivalence of the convergence implied by the norm $|| \cdot ||$ to the $M$-convergence it immediately follows that

4. If $\lambda_1, \lambda_2, \ldots$ is a sequence of complex numbers convergent to $0$, then $\lim_{n \to \infty} ||\lambda_n a|| = 0$ for every $a \in Q(R)$,

5. If $\lim_{n \to \infty} ||a_n|| = 0$ and $\lambda$ is a complex number, then $\lim_{n \to \infty} ||\lambda a_n|| = 0$.

Hence we see that $Q(R)$ is an $F^*$-space (Mazur and Orlicz [4], p. 185) with respect to the norm $|| \cdot ||$ and hence also to the $M$-convergence.

Now let $\lim_{n \to \infty} ||a_n|| = 0$ (i.e., $M$-limit is $0$). By the definition of the $M$-convergence there exists an element $x \in R$ ($x \neq 0$) such that $x a_n \in R$ and $\lim_{n \to \infty} ||a_n|| = 0$ for $k = 1, 2, \ldots$ ($|| \cdot ||$ denote the pseudonorms in the ring $R$). Let $\lambda_1, \lambda_2, \ldots$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then $\sum_{n=1}^{\infty} ||x a_n|| < \infty$ for $k = 1, 2, \ldots$, i.e., by the homogeneity of the pseudonorms $|| \cdot ||$, $\sum_{n=1}^{\infty} ||x a_n|| < \infty$ for $k = 1, 2, \ldots$

Hence the sequence $\sum_{n=1}^{\infty} \lambda_n a_n$ converges in $R$ to the element $a = \sum_{n=1}^{\infty} \lambda_n a_n$, and therefore the sequence $\sum_{n=1}^{\infty} \lambda_n a_n$ is $M$-convergent in the field $Q(R)$ to the element $x^{-1}a$, which implies the boundedness of the norms:

$$\sum_{n=1}^{\infty} ||a_n|| \leq K < \infty.$$

Hence, by a theorem of S. Mazur and W. Orlicz ([4], p. 201), $Q(R)$ is a $B^*_t$-space with respect to the norm $|| \cdot ||$ and thus also to the $M$-convergence, which proves the Lemma.

**References**


