and write

$$a_{ik} = \int_{\Delta_k} |x_i(t)|\, dt.$$

Then

$$0 \leqslant \sum_{k=1}^{n} a_{ik} \leqslant 1$$

and

$$\|U(x_1,\ldots,x_n)\| = |U(x_1,\ldots,x_n)| \leqslant \sum_{(\pi_1,\ldots,\pi_n)} \prod_{k=1}^{n} \int_{\Delta_k} |x_{\pi_k}(t)|\, dt$$

$$= \sum_{(\pi_1,\ldots,\pi_n)} \prod_{k=1}^{n} a_{\pi_k k} \leqslant \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \leqslant 1.$$

On the other hand choose

$$x_k(t) = \begin{cases} n & \text{for } t \in \Delta_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\|x_k\| = 1$ and $U(x_1,\ldots,x_n) = 1$. Hence

$$\sup_{\|x_1\| \leqslant 1,\ldots,\|x_n\| \leqslant 1} \|U(x_1,\ldots,x_n)\| = 1.$$

By definition of the operation $U(x_1,\ldots,x_n)$ it easily follows that

$$U(x) = U(x,\ldots,x) = n! \prod_{k=1}^{n} \int_{\Delta_k} x(t)\, dt.$$

Let $\|x\| \leqslant 1$; since

$$\|x\| = \sum_{k=1}^{n} \int_{\Delta_k} |x(t)|\, dt,$$

we obtain

$$\|U(x)\| = |U(x,\ldots,x)| \leqslant n! \prod_{k=1}^{n} \int_{\Delta_k} |x(t)|\, dt \leqslant n! \left(\frac{1}{n} \sum_{k=1}^{n} \int_{\Delta_k} |x(t)|\, dt\right)^n \leqslant \frac{n!}{n^n} \|x\|.$$

On the other hand, $x(t) = 1$ implies $U(x,\ldots,x) = n!/n^n$, whence

$$\sup_{\|x\| \leqslant 1} \|U(x)\| = n!/n^n.$$

Thus the equality (*) is true.

Reçu par la Rédaction le 29. 3. 1954

On quotient-fields generated by pseudonormed rings

by

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I. A linear ring R over the field of complex numbers as scalars with unit element is called a *pseudonormed ring* if R is a B_0-space (Mazur and Orlicz [4], p. 185) with respect to a sequence of pseudonorms $\|\ \|_1, \|\ \|_2, \ldots$ *submultiplicative*, *i. e.* satisfying the condition $\|xy\|_k \leqslant \|x\|_k \|y\|_k$ for every $x, y \in R$.

In this paper we shall deal exclusively with commutative pseudonormed rings without divisors of zero.

Let $Q(R)$ be the quotient-field obtained from the pseudonormed ring R (Van der Waerden [7], p. 46-49). J. G.-Mikusiński has introduced by means of the convergence in the ring R a convergence in the field $Q(R)$ (which will be called the *M-convergence*) as follows: the sequence $a_1, a_2, \ldots$ $(a_n \in Q(R))$ converges to $a \in Q(R)$ (which will be denoted by $M\text{-}\lim_{n\to\infty} a_n = a$) if there exists $x \in R$ $(x \neq 0)$ such that $xa_n \in R$ $(n = 1, 2, \ldots)$, $xa \in R$ and the sequence $xa_1, xa_2, \ldots$ converges to xa in the ring R, *i. e.* $\lim_{n\to\infty} \|xa_n - xa\|_k = 0$ $(k = 1, 2, \ldots)$ (Mikusiński [5], p. 62). It is easily seen that $Q(R)$ with the M-convergence is an L-space of Fréchet (Kuratowski [3], p. 84) and that the addition and the ring- and scalar-multiplication in $Q(R)$ are continuous with respect to the M-convergence.

The M-convergence in $Q(R)$ is called *topological* if there exists in $Q(R)$ a topology satisfying the first axiom of countability (Kuratowski [3], p. 33), such that the convergence according to this topology is equivalent to the M-convergence, and $Q(R)$ is a topological field with respect to this topology (we suppose here that the inverse of an element is a continuous function). In connection with the study of the M-convergence in the field of operators of Mikusiński (Urbanik [6]) arose the problem of determining those pseudonormed rings for which the M-conevrgence in $Q(R)$ is topological. In this paper we shall prove that

If the M-convergence in the field $Q(R)$ is topological, then the pseudonormed ring R is isomorphic and homeomorphic with the field of complex numbers (topologized as usual).

It is sufficient to prove that if the M-convergence is topological, then the field $Q(R)$ with the M-convergence is a convex (=locally convex) linear topological space. Indeed, from a theorem of Arens ([1], p. 625) we infer that the field $Q(R)$ is isomorphic and homeomorphic with the field of complex numbers, whence it immediately follows that the ring R also has this property. (I am indebted to Mr Alexiewicz and to Mr Altman for calling my attention to the fact that my first proof may be simplified by applying the theorem of Arens). Since the B_0^*-spaces (Mazur and Orlicz [4], p. 185) are convex linear topological spaces, the convexity of the space $Q(R)$ follows from

LEMMA. *If the M-convergence in $Q(R)$ is topological, then $Q(R)$ provided with the M-convergence is a B_0^*-space.*

Proof. The fact of the M-convergence in $Q(R)$ being topological implies that $Q(R)$ is a topological group with respect to addition and satisfies the first axiom of countability. Hence by a theorem of S. Kakutani [2] there exists a distance-function ϱ in the field $Q(R)$ such that the M-convergence is equivalent to the convergence involved by this distance and such that

$$\varrho(a,b)=\varrho(a+c,b+c)$$

for every $a,b,c\in Q(R)$. Setting $\|a\|=\varrho(a,0)$ we obtain a (non-homogeneous) norm $\|\ \|$ in $Q(R)$, *i. e.* satisfying

1° $\|a\|=0$ if and only if $a=0$,

2° $\|a+b\|\leqslant\|a\|+\|b\|$,

3° $\|a\|=\|-a\|$.

From the equivalence of the convergence implied by the norm $\|\ \|$ to the M-convergence it immediately follows that

4° If $\lambda_1,\lambda_2,\ldots$ is a sequence of complex numbers convergent to 0, then $\lim\limits_{n\to\infty}\|\lambda_n a\|=0$ for every $a\in Q(R)$,

5° If $\lim\limits_{n\to\infty}\|a_n\|=0$ and λ is a complex number, then $\lim\limits_{n\to\infty}\|\lambda a_n\|=0$ $\big(a_n\in Q(R)\big)$.

Hence we see that $Q(R)$ is an F^*-space (Mazur and Orlicz [4], p. 185) with respect to the norm $\|\ \|$ and hence also to the M-convergence.

Now let $\lim\limits_{n\to\infty}\|a_n\|=0$ $\big(a_n\in Q(R)\big)$, *i. e.* $M\text{-}\lim\limits_{n\to\infty}a_n=0$. By the definition of the M-convergence there exists an element $x\in R$ $(x\neq 0)$ such that $xa_n\in R$ and $\lim\limits_{n\to\infty}\|xa_n\|_k=0$ for $k=1,2,\ldots$ ($\|\ \|_k$ denote the pseudonorms in the ring R). Let $\lambda_1,\lambda_2,\ldots$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty}\lambda_n<\infty$. Then $\sum_{n=1}^{\infty}\lambda_n\|xa_n\|_k<\infty$ for $k=1,2,\ldots$, *i. e.* by the homogeneity of the pseudonorms $\|\ \|_1,\|\ \|_2,\ldots$

$$\sum_{n=1}^{\infty}\|x\lambda_n a_n\|_k<\infty \quad \text{for} \quad k=1,2,\ldots$$

Hence the sequence $\sum_{n=1}^{m}x\lambda_n a_n$ $(m=1,2,\ldots)$ converges in R to the element $a=\sum_{n=1}^{\infty}x\lambda_n a_n$, and therefore the sequence $\sum_{n=1}^{m}\lambda_n a_n$ $(m=1,2,\ldots)$ is M-convergent in the field $Q(R)$ to the element $x^{-1}a$, which implies the boundedness of the norms:

$$\Big\|\sum_{n=1}^{m}\lambda_n a_n\Big\|\leqslant K<\infty.$$

Hence, by a theorem of S. Mazur and W. Orlicz ([4], p. 201), $Q(R)$ is a B_0^*-space with respect to the norm $\|\ \|$ and thus also to the M-convergence, which proves the Lemma.

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Reçu par la Rédaction le 7. 4. 1954