process considered is homogeneous in time and $a(I)$ is an absolutely continuous function of an interval, the relation $Q(t)=Q=$ constant holds. Relation (18) will then be of the form

$$A(I)=Q|I|. \tag{20}$$

Consequently, for $|I|=1$ the equality $A(I)=Q$ is obtained. Thus Q is the expectation $\mathrm{E}\,\xi(I)$ during a time-unit.

A condition for the existence of a finite $\mathrm{E}\,\xi(I)$ for Markov processes with a finite number of states, similar to (6) (but not identical with it) has been given by Dobrusin ([2], p. 543).

It is easily seen that the converse to theorem 1 is not true. One can give examples of separable stochastic processes without fixed discontinuity points for which $\mathrm{E}\,\xi(I)<\infty$, and yet relation (6) does not hold. The following theorem is in a certain sense a converse to theorem 1:

THEOREM 3. *Let the stochastic process* $\{x_t, t\in I_0\}$ *be separable without fixed discontinuity points and let it be purely jumping. If* $\mathrm{E}\,\xi(I)$ *exists and is finite, relation* (6) *holds and consequently relation* (7) *holds also.*

Proof. Let the assumptions of the theorem hold. Let $\{\tau_j\}$ and $\{\tau_{nk}\}$ be defined as above. The same reasoning as that used in the proof of theorem 1 leads to the assertion of this theorem.

References

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On a theorem of Mazur and Orlicz

by

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In [1], Mazur and Orlicz have proved an interesting theorem concerning the existence of functionals fulfilling certain inequalities. The original proof of the authors has been simplified by Sikorski in [2]. Because of the importance of this result for applications, the reader will excuse our returning to this theorem once more. It is the object of the present remark to reduce the proof of the theorem of Mazur and Orlicz to the classical theorem of Banach. The resulting proof is both simple and short.

Let X be a linear space, let $\omega(x)$ be a real function defined on X. The function ω will be called a *Banach functional* on X, if

1° $\omega(x_1+x_2)\leqslant\omega(x_1)+\omega(x_2)$ for every $x_1,x_2\in X$,

2° $\omega(\lambda x)=\lambda\,\omega(x)$ for every $x\in X$ and every $\lambda\geqslant 0$.

THEOREM. *Let X be a linear space, let T be an abstract set. Let $x(t)$ be a mapping of T into X, let $\beta(t)$ be a real function on T. Let $\omega(x)$ be a Banach functional defined on X.*

A necessary and sufficient condition for the existence of an additive and homogeneous functional $f(x)$ defined on X and fulfilling

3° $f(x)\leqslant\omega(x)$ *for every* $x\in X$,

4° $\beta(t)\leqslant f(x(t))$ *for every* $t\in T$,

is the following one:

for every finite sequence $t_1,\ldots,t_n\in T$ and for arbitrary non-negative numbers $\lambda_1,\ldots,\lambda_n$

5° $\sum\lambda_i\beta(t_i)\leqslant\omega\left(\sum\lambda_i x(t_i)\right)$.

Proof. The necessity being evident, we shall prove the sufficiency only. Suppose that 5° is fulfilled. For every $x\in X$ let us put

$$\tilde{\omega}(x)=\inf\left[\omega\left(x+\sum\lambda_i x(t_i)\right)-\sum\lambda_i\beta(t_i)\right]$$

where $t_1, \ldots, t_n$ is an arbitrary finite subset of T and λ_i arbitrary non-negative numbers. According to 5°, we have

$$\sum \lambda_i \beta(t_i) \leqslant \omega\Big(\sum \lambda_i x(t_i)\Big) \leqslant \omega\Big(x + \sum \lambda_i x(t_i)\Big) + \omega(-x),$$

whence

$$\omega\Big(x + \sum \lambda_i x(t_i)\Big) - \sum \lambda_i \beta(t_i) \geqslant -\omega(-x),$$

so that $\tilde{\omega}(x)$ is well defined. It is easy to see that $\tilde{\omega}(x)$ is a Banach functional on X[1]).

It follows that there exists an additive and homogeneous functional $f(x)$ on X such that $f(x) \leqslant \tilde{\omega}(x)$ for every $x \epsilon X$. Since clearly $\tilde{\omega}(x) \leqslant \omega(x)$, condition 3° is fulfilled. Further we have

$$\tilde{\omega}\big(-x(t)\big) \leqslant \omega\big(-x(t) + x(t)\big) - \beta(t) = -\beta(t),$$

so that $-f\big(x(t)\big) \leqslant -\beta(t)$, which concludes the proof.

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[1]) The subadditivity of $\tilde{\omega}$ follows immediately from the inequalities

$$\tilde{\omega}(x' + x'') \leqslant \omega\big(x' + x'' + \sum \lambda_i' x(t_i') + \sum \lambda_i'' x(t_i'')\big) - \sum \lambda_i' \beta(t_i') - \sum \lambda_i'' \beta(t_i'')$$

$$\leqslant \omega\big(x' + \sum \lambda_i' x(t_i')\big) - \sum \lambda_i' \beta(t_i') + \omega\big(x'' + \sum \lambda_i'' x(t_i'')\big) - \sum \lambda_i'' \beta(t_i'').$$