Wird oben beiderseitig das Zeichen des Operators Ω gestrichen, so erhalten wir

$$\frac{\omega(\bar{x})}{2}\cdot\omega(y)=\omega[g(\bar{\bar{x}},y)],\qquad \text{oder}\qquad g(\bar{\bar{x}},y)=\Omega\left(\frac{\omega(\bar{x})}{2}\,\omega(y)\right),$$

was zu beweisen war.

Man beweist nun durch vollständige Induktion die Richtigkeit der Formel (2b) für alle

$$x_n=\Omega\left(\frac{\omega(\bar{x})}{2^n}\right)\qquad (n=1,2,\ldots).$$

Die Zahlen

$$x_{k,n}=\Omega\left(\frac{\omega(k)}{2^n}\right)=\Omega\left(\frac{k}{2^n}\right),$$

wo k eine beliebige ganze Zahl und n eine beliebige natürliche Zahl bedeutet, bilden eine dichte Menge M. Die Gleichung

$$g(\bar{x},y)=\Omega[\omega(\bar{x})\cdot\omega(y)] \tag{22}$$

ist für alle x der Menge M und alle y erfüllt. Daraus folgt wegen der Stetigkeit der Funktionen ω, f, g, daß die Gleichung (22) für alle $\bar{x}$, y erfüllt sein muß.

Auf diese Weise ist auch der zweite Teil unseres Satzes bewiesen worden.

Umgekehrt: ist die überall definierte, stark monotone Funktion $\omega(x)$ gegeben, die die Menge aller Zahlen als Wertbereich hat, und sind mit Hilfe von ω die Funktionen f, g durch die Gleichungen (2) definiert (mit Ω ist die zu ω inverse Funktion bezeichnet), so sind, wie man leicht feststellt, alle Voraussetzungen unseres Satzes (außer etwa II) erfüllt.

Zitatennachweis

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Realizations of some stochastic processes

by

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1. Preliminary remarks and summary. We consider a real stochastic process $\{x_t, t\in I_0\}$, where I_0 is a closed finite interval. The x_t are functions of two arguments and can be explicitly written in the form $x_t(\omega)$, where $t\in I_0$ and $\omega\in\Omega$, Ω being the set of elementary events. The stochastic process is thus a family of random variables. The smallest Borel field $\mathfrak{F}_{I_0}$ of ω sets, with respect to which all the x_t are measurable, is generated by the field of ω sets of the form

$$\{[x_{t_1}(\omega),x_{t_2}(\omega),\ldots,x_{t_n}(\omega)]\in A\}, \tag{*}$$

where A is any right-hand semiclosed interval and $(t_1,t_2,\ldots,t_n)$ is any finite set of values of $t\in I_0$. The probability measure P of elements of $\mathfrak{F}_{I_0}$ is, as we know [4], uniquely determined by the P measure of ω sets of the form (*).

We shall assume that the process $\{x_t, t\in I_0\}$ is separable (see [3], p. 51). This implies that if $\{t_j\}$, $j=1,2,3,\ldots$, is a sequence satisfying the separability conditions and if ω does not belong to an exceptional ω set Λ of P measure 0, then

$$\underset{t\in I}{\text{g.l.b.}}\,x_t=\underset{t_j\in I}{\text{g.l.b.}}\,x_t,\qquad \underset{t\in I}{\text{l.u.b.}}\,x_t=\underset{t_j\in I}{\text{l.u.b.}}\,x_t \tag{1}$$

for every open interval $I\subset I_0$.

We assume further that the process $\{x_t, t\in I_0\}$ has no fixed discontinuity points, *i. e.* that

$$P[\lim_{s\to t}x_s(\omega)=x_t(\omega)]=1\qquad (t\in I_0). \tag{2}$$

It follows (see [3], p. 60) that the process considered is measurable, *i. e.* that $x_t(\omega)$ defines a function measurable in the pair of variables (t,ω) where $t\in I_0$, $\omega\in\Omega$.

The main result of this paper consists in stating that if relation (6) given below is satisfied, the ω set for which $x_t(\omega)$ are step functions has probability 1. In other words almost every realization $x_t(\omega)$ has only finitely many points of discontinuity and is constant in every open interval of continuity points, and at each discontinuity point both left- and right-hand limits exist. It is moreover shown that the mathematical expectation of the number of discontinuities is then finite and an explicit expression for it is found (Theorem 1).

2. We shall introduce the following notation.

By x_I we shall denote the increment of x_t in the interval I, where $I\subset I_0$, and by $|I|$ the length of the interval I. By $\xi(I)$ we shall denote the number of discontinuities of x_t in I, where $\xi(I)=\infty$ if the number of discontinuities is not finite. Further let

$$a(I)=P(x_I\neq 0), \tag{3}$$

$$A(I)=\lim_{n\to\infty}\sum_{k=1}^{n}a(I_{nk}), \quad \text{as} \quad \max_{1\leqslant k\leqslant n}|I_{nk}|\to 0, \tag{4}$$

where $\{I_{nk}\}$ is a partition of I in non-overlapping intervals I_{nk}. $A(I)$ is the Burkill integral of $a(I)$. The upper Burkill integral $\bar{A}(I)$ of $a(I)$ is obtained by replacing in (4) the symbol lim by $\overline{\lim}$. The derivative of $a(I)$ at the point $t\in I$ will be denoted by $Q(t)$, *i. e.*

$$Q(t)=\lim\frac{a(I)}{|I|}, \tag{5}$$

as I contracts to a fixed point $t\in I$.

We shall introduce the following

DEFINITION. The process $\{x_t, t\in I_0\}$ is *purely jumping* (shortly PJ) if the ω set Λ for which the $x_t(\omega)$ have the property

For each $t\in I_0$ there exists such a $\vartheta(t,\omega)>0$ that for all t' satisfying the inequality $|t-t'|<\vartheta(t,\omega)$ the equality $x_{t'}(\omega)=x_t(\omega)$ holds unless $x_t(\omega)$ has a discontinuity at t,

satisfies the equality $P(\Lambda)=1$.

THEOREM 1. *Let the stochastic process $\{x_t, t\in I_0\}$ be separable and without fixed discontinuity points. If the relation*

$$\bar{A}(I)<\infty \tag{6}$$

holds, then for every open interval $I\subset I_0$

(i) *the process is purely jumping*;

(ii) *the mathematical expectation $\mathrm{E}\,\xi(I)$ exists, is finite, and satisfies the relation*

$$\mathrm{E}\,\xi(I)=\bar{A}(I)=A(I)<\infty; \tag{7}$$

(iii) *the ω set with the property that at the discontinuities of $x_t(\omega)$, if any, both left- and right-hand limits exist, has probability* 1;

(iv) *the derivative $Q(t)$ exists almost everywhere, and the relation*

$$\int_I Q(t)\,dt\leqslant A(I) \tag{8}$$

holds.

Proof. Let the assumptions of the theorem hold. Let $T=\{\tau_j\}$, $j=1,2,\ldots$, be a denumerable and everywhere dense set of points in I and let the points $\tau_{n1},\tau_{n2},\ldots,\tau_{nn}$ be, for each n, points $\tau_1,\tau_2,\ldots,\tau_n$ arranged in ascending order, $\tau_{n1}<\tau_{n2}<\ldots<\tau_{nn}$. Let $\eta_n(T,I)$ denote the number of pairs of points $(\tau_{nk},\tau_{n(k+1)})$ such that $x_{\tau_{nk}}-x_{\tau_{n(k+1)}}\neq 0$; $\eta_n(T,I)$ is a random variable. Since for a fixed realization the inequality

$$\eta_n(T,I)\leqslant\eta_{n+1}(T,I)$$

holds, we obtain in virtue of Lebesgue's theorem the relation

$$\mathrm{E}\,\eta(T,I)=\lim_{n\to\infty}\mathrm{E}\,\eta_n(T,I), \tag{9}$$

where

$$\eta(T,I)=\lim_{n\to\infty}\eta_n(T,I). \tag{10}$$

On the other hand we have for every n

$$\mathrm{E}\,\eta_n(T,I)=\sum_{k=1}^{n}a(I_{nk}), \tag{11}$$

where $I_{nk}=[\tau_{nk},\tau_{n(k+1)})$. Relations (6), (9) and (11) imply

$$\mathrm{E}\,\eta(T,I)<\infty, \tag{12}$$

and consequently

$$P[\eta(T,I)<\infty]=1. \tag{13}$$

The separability of the process considered being taken into account relation (13) implies that assertion (i) is satisfied.

Indeed, as the process considered has no fixed discontinuity points, the set T satisfies (see [3], p. 54) the separability conditions. For the same reason the relation

$$P[\lim_{s\to\tau}x_s(\omega)=x_\tau(\omega),\tau\in T]=1 \tag{14}$$

holds, and we can thus consider only the realizations which are continuous at all points $\tau\in T$. Let Λ_0 denote the ω set for which the $x_t(\omega)$

satisfy both the relations in the square brackets of (13) and (14). Clearly $P(A_0)=1$. It is now easily seen that if $\omega \epsilon A_0$, $x_t(\omega)$ can take only finitely many values (depending on ω) on $\tau \epsilon T$, and that the interval I is then divided into finitely many subintervals I_k (depending on ω) such that $x_t(\omega)$ is constant at the points $\tau \epsilon TI_k$. Separability, as expressed by (1), implies then that $x_t(\omega)$ is constant in each interval $I_k(\omega)$. Assertion (i) is thus proved.

We introduce now the following

Definition. At the point $t \epsilon I$ the realization $x_t(\omega)$ *has a discontinuity on the set* T if the oscillation of $x_t(\omega)$ on the set T at the point t is positive.

Let us now observe that in proving assertion (i) we have proved at the same time that almost every realization $x_t(\omega)$ is constant in the intervals $I_k(\omega)$ where $I_1(\omega)+I_2(\omega)+\ldots+I_n(\omega)=I$, and where n may depend on ω but is finite. The probability that $\eta(T,I)$ is equal to the number of points $t \epsilon I$ at which $x_t(\omega)$ has a discontinuity on the set T is equal to 1. Clearly the relation

$$\eta(T,I)=\eta(I)=\xi(I) \tag{15}$$

holds with probability 1. Thus, taking into account relation (12), we obtain assertion (ii)[1]. This assertion immediately implies that $P[\xi(I)<\infty]=1$.

Assertion (iii) immediately follows from (i) and (ii).

It remains only to prove assertion (iv). In order to do this we shall show that if the process is PJ, the function $a(I)$ is a continuous function of an interval, *i. e.*

$$\lim_{|I|\to 0} a(I)=0\,. \tag{16}$$

Indeed, let the process $\{x_t, t \epsilon I_0\}$ be PJ. Then for each point $t \epsilon I$ and for each $\varepsilon>0$ there exists such an $a(t,\varepsilon)>0$ that

$$P(x_{t-a}=x_t=x_{t+a})>1-\varepsilon\,. \tag{17}$$

Suppose now that relation (16) is not satisfied. Then there exists such a sequence $\{I_n\}$ of intervals that

$$\lim_{n\to\infty}|I_n|\to 0, \qquad a(I_n)>q>0\,.$$

We can thus choose a subsequence $\{I_{n_k}\}$ contracting to a point $t_0 \epsilon I$,

[1]) The existence of $A(I)$ can easily be deduced from the continuity of $a(I)$ proved below (see formula (16)), if we take into account that $a(I)$ is a semi-additive function and (6) holds (see [5], p. 168).

and for this point t_0 there will exist no $a(t_0,q)$ satisfying relation (17) for $\varepsilon=q$. Thus relation (16) holds.

Let us now observe that $A(I)$ is non-negative, additive (see [1], p. 283) and, as follows from (16) (see [5], p. 167), a continuous function of an interval, and thus $A(I)$ is of bounded variation. Thus the derivative of $A(I)$ exists almost everywhere in I and — according to a theorem of Saks (see [6], p. 221) — the same holds for $a(I)$, and moreover the derivatives of $a(I)$ and $A(I)$ are almost everywhere equal. Since $A(I)$ is non-negative, relation (8) is obtained.

Theorem 1 is thus proved.

Theorem 2. *Let the process* $\{x_t, t \epsilon I_0\}$ *be separable, without fixed discontinuity points and let* $a(I)$ *be an absolutely*[2]) *continuous function of an interval. Then the assertions of theorem* 1 *hold and moreover the relation*

$$\int_I Q(t)\,dt=A(I) \tag{18}$$

is satisfied.

Proof. Let the assumptions of the theorem be satisfied. A theorem of Burkill ([1], p. 287) implies then that relation (6) is true. The assumptions of theorem 1 are thus satisfied and consequently so are all its assertions. On the other hand, from a theorem of Burkill (see [1], p. 289) it follows that $A(I)$ is then also an absolutely continuous function of an interval and thus equality (18) holds. Theorem 2 is thus proved.

The following corollary from theorem 2 holds:

Corollary. *If the process* $\{x_t, t \epsilon I_0\}$ *is separable without fixed discontinuity points, and if* $a(I)$ *satisfies the Lipschitz condition, i. e. there exists such a constant* $K>0$ *that for all* $I\subset I_0$ *the inequality*

$$a(I)<K|I| \tag{19}$$

holds, then the assertions of theorem 2 *are true.*

The proof of this corollary follows directly from theorem 2 and from a theorem of Burkill (see [1], p. 287) stating that if $a(I)$ satisfies the Lipschitz condition, it is an absolutely continuous function of an interval.

The probabilistic sense of $Q(t)$ is the following: $Q(t)$ is the density of the mathematical expectation $\mathrm{E}\,\xi(I)$ at those points $t \epsilon I$ at which the derivatives of $a(I)$ and $A(I)$ are equal. In the particular case where the

[2]) The function $a(I)$ of an interval is *absolutely continuous* if to every $\varepsilon>0$ there corresponds such a $\vartheta>0$ that the relation $\sum_k |I_k|<\vartheta$ implies the relation $\sum_k a(I_k)<\varepsilon$, where $\{I_k\}$ is a partition of I in non-overlapping intervals.

process considered is homogeneous in time and $a(I)$ is an absolutely continuous function of an interval, the relation $Q(t)=Q=$ constant holds. Relation (18) will then be of the form

$$A(I)=Q|I|. \tag{20}$$

Consequently, for $|I|=1$ the equality $A(I)=Q$ is obtained. Thus Q is the expectation $\mathrm{E}\,\xi(I)$ during a time-unit.

A condition for the existence of a finite $\mathrm{E}\,\xi(I)$ for Markov processes with a finite number of states, similar to (6) (but not identical with it) has been given by Dobrusin ([2], p. 543).

It is easily seen that the converse to theorem 1 is not true. One can give examples of separable stochastic processes without fixed discontinuity points for which $\mathrm{E}\,\xi(I)<\infty$, and yet relation (6) does not hold. The following theorem is in a certain sense a converse to theorem 1:

THEOREM 3. *Let the stochastic process $\{x_t, t\in I_0\}$ be separable without fixed discontinuity points and let it be purely jumping. If $\mathrm{E}\,\xi(I)$ exists and is finite, relation* (6) *holds and consequently relation* (7) *holds also.*

Proof. Let the assumptions of the theorem hold. Let $\{\tau_j\}$ and $\{\tau_{nk}\}$ be defined as above. The same reasoning as that used in the proof of theorem 1 leads to the assertion of this theorem.

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On a theorem of Mazur and Orlicz

by

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In [1], Mazur and Orlicz have proved an interesting theorem concerning the existence of functionals fulfilling certain inequalities. The original proof of the authors has been simplified by Sikorski in [2]. Because of the importance of this result for applications, the reader will excuse our returning to this theorem once more. It is the object of the present remark to reduce the proof of the theorem of Mazur and Orlicz to the classical theorem of Banach. The resulting proof is both simple and short.

Let X be a linear space, let $\omega(x)$ be a real function defined on X. The function ω will be called a *Banach functional* on X, if

1° $\omega(x_1+x_2)\leqslant\omega(x_1)+\omega(x_2)$ for every $x_1,x_2\in X$,

2° $\omega(\lambda x)=\lambda\omega(x)$ for every $x\in X$ and every $\lambda\geqslant 0$.

THEOREM. *Let X be a linear space, let T be an abstract set. Let $x(t)$ be a mapping of T into X, let $\beta(t)$ be a real function on T. Let $\omega(x)$ be a Banach functional defined on X.*

A necessary and sufficient condition for the existence of an additive and homogeneous functional $f(x)$ defined on X and fulfilling

3° *$f(x)\leqslant\omega(x)$ for every $x\in X$,*

4° *$\beta(t)\leqslant f(x(t))$ for every $t\in T$,*

is the following one:

for every finite sequence $t_1,\dots,t_n\in T$ and for arbitrary non-negative numbers $\lambda_1,\dots,\lambda_n$

5° $\sum\lambda_i\beta(t_i)\leqslant\omega\left(\sum\lambda_i x(t_i)\right)$.

Proof. The necessity being evident, we shall prove the sufficiency only. Suppose that 5° is fulfilled. For every $x\in X$ let us put

$$\tilde{\omega}(x)=\inf\left[\omega\left(x+\sum\lambda_i x(t_i)\right)-\sum\lambda_i\beta(t_i)\right]$$