Analytical characterization of a composed, non-homogeneous Poisson process

by

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I. The non-homogeneous Poisson process has been dealt with by Rényi [8], Prékopa [7], and Ryll-Nardzewski [9] and [10]. In the paper [10] the non-homogeneous Poisson process, whose realizations are of bounded variation, has been fully discussed. The aim of this paper is to give by means of analytical methods a general formulation of a composed non-homogeneous Poisson process.

II. Let us consider a stochastic process ξ_t defined in the closed time interval $[0,T]$. We shall denote by ξ_I the increment of ξ_t in the interval $I=[a,b]$, where $0\leqslant a<b\leqslant T$. Let $Q(x,I)$ be a function of an interval, given by the formulae

$$Q(x,I)=\begin{cases} P(\xi_I<x) & \text{for} \quad x<0, \\ -P(\xi_I\geqslant x) & \text{for} \quad x>0. \end{cases} \tag{1}$$

We shall prove the following

THEOREM. *Let us assume that:*

(i) *ξ_t is a process with independent increments,*

(ii) $\lim\limits_{|I|\to 0} P(\xi_I=0)=1$ (*$|I|$ denotes the length of the interval I*).

Then ξ_t is a composed Poisson process and the characteristic function $\varphi(s,I)$ of ξ_I is given by the formula

$$\log\varphi(s,I)=\int\limits_{x\neq 0}(e^{isx}-1)\,d_x\int\limits_I Q(x,J). \tag{2}$$

($\int\limits_I Q(x,J)$ denotes the Burkill integral of the function $Q(x,I)$).

The proof of the theorem will be preceeded by the proof of the following

LEMMA. *If assumptions* (i) *and* (ii) *are satisfied, the upper Burkill integral of the function $P(\xi_I\neq 0)$ is finite.*

Proof[1]). Let the interval I be the sum of non-overlapping inter-

[1]) In the proof of the lemma an idea of Kolmogorov is used ([5], p. 56-57).

vals $I_1,I_2,\ldots,I_n$ where the indices are so chosen that I_k and I_{k+1} ($k=1,2,\ldots,n-1$) have a common point. J_k will denote the interval obtained as the sum of $I_{k+1},\ldots,I_n$.

Assumption (ii) implies the existence of such a $\delta>0$, that if $|I|\leqslant\delta$, the relation

$$P(\xi_I\neq 0)=1-P(\xi_I=0)\leqslant 1/4 \tag{3}$$

holds. Let us now assume that the considered interval I satisfies (3); then, taking into account assumption (i), we obtain the following inequalities:

$$\begin{aligned} &\sum_{k=1}^n P(\xi_{I_k}\neq 0)P(\xi_{J_k}=0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0)\leqslant P(\xi_I\neq 0)\leqslant 1/4,\\ &\sum_{k=1}^n P(\xi_{I_k}\neq 0)P(\xi_{J_k}=0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0)\geqslant \tfrac{3}{4}\sum_{k=1}^n P(\xi_{I_k}\neq 0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0). \end{aligned} \tag{4}$$

These inequalities imply

$$\sum_{k=1}^n P(\xi_{I_k}\neq 0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0)\leqslant 1/3;$$

hence, taking into account formula (3), we obtain

$$\prod_{r=1}^n P(\xi_{I_r}=0)\geqslant P(\xi_I=0)-\sum_{k=1}^n P(\xi_{I_k}\neq 0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0)\geqslant 5/12. \tag{5}$$

This inequality, relation (3), in which I is replaced by J_k, and relation (4) imply

$$\frac{5}{16}\sum_{k=1}^n P(\xi_{I_k}\neq 0)\leqslant\sum_{k=1}^n P(\xi_{I_k}\neq 0)P(\xi_{J_k}=0)\prod_{r=1}^{k-1}P(\xi_{I_r}=0)\leqslant 1/4.$$

Thus for every division of the interval I with $|I|\leqslant\delta$ into $I_1,I_2,\ldots,I_n$ the following inequality holds:

$$\sum_{k=1}^n P(\xi_{I_k}\neq 0)\leqslant 4/5.$$

We have thus obtained the following inequality for the upper Burkill integral:

$$\overline{\int\limits_I} P(\xi_J\neq 0)\leqslant 4/5 \quad \text{for} \quad |I|\leqslant\delta. \tag{6}$$

Since the inequality

$$P(\xi_I \neq 0) \leqslant \sum_{k=1}^{n} P(\xi_{I_k} \neq 0)$$

evidently holds, we find that the function $P(\xi_I \neq 0)$ is monotone increasing. Consequently the upper Burkill integral

$$\overline{\int_I} P(\xi_J \neq 0)$$

is an additive function of an interval (cf. [6], p. 243, or [3], p. 16). From the last relation and from (6) the assertion of the lemma is obtained.

Proof of the theorem. From assumptions (i) and (ii) it follows (Doob [1], III, § 4, VIII, § 7) that ξ_I has an infinitely divisible distribution, having a characteristic function given by Lévy's formula

$$\log \varphi(s, I) = i\gamma(I)s - \frac{\sigma^2(I)s^2}{2} + \int_{-\infty}^{0} \left(e^{isx} - 1 - \frac{isx}{1+x^2}\right) dM(x, I) + \int_{0}^{\infty} \left(e^{isx} - 1 - \frac{isx}{1+x^2}\right) dN(x, I), \tag{7}$$

where $\gamma(I)$ and $\sigma^2(I)$ are constants, $M(x,I)$ and $N(x,I)$ are non-decreasing functions (of the argument x) in the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively, and where the relations

$$M(-\infty, I) = N(\infty, I) = 0, \qquad \int_{-1}^{0} x^2\, dM(x, I) + \int_{0}^{1} x^2\, dN(x, I) < \infty$$

hold.

As formula (7) depends only on values of functions $M(x,I)$ and $N(x,I)$ in continuity points, we shall consider these functions for x being continuity points.

Now let the interval I be a sum of non-overlapping intervals I_{n1}, $I_{n2}, \ldots, I_{nn}$. We shall denote by $F_{nk}(x)$ the distribution function $P(\xi_{I_{nk}} < x)$. Assumption (ii) implies that for each $\varepsilon > 0$ the relation

$$\max_{1 \leqslant k \leqslant n} P(|\xi_{I_{nk}}| > \varepsilon) \to 0$$

holds as $\max\limits_{1 \leqslant k \leqslant n} |I_{nk}| \to 0$. The following equality is thus evidently true:

$$\xi_I = \xi_{I_{n1}} + \xi_{I_{n2}} + \ldots + \xi_{I_{nn}}.$$

Let us now observe that, according to a known theorem of Gnedenko ([2], § 25), the constant $\sigma^2(I)$ and the functions $M(x,I)$ and $N(x,I)$ can be determined from the following formulae as $\max\limits_{1 \leqslant k \leqslant n} |I_{nk}| \to 0$:

$$\sigma^2(I) = \lim_{\varepsilon \to 0+} \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} \left[\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x\, dF_{nk}(x) \Big)^2 \right], \tag{8}$$

$$M(x, I) = \lim_{n \to \infty} \sum_{k=1}^{n} F_{nk}(x) \qquad (x < 0), \tag{9}$$

$$N(x, I) = \lim_{n \to \infty} \sum_{k=1}^{n} \big(F_{nk}(x) - 1\big) \qquad (x > 0). \tag{10}$$

From formula (8) the inequality

$$\sigma^2(I) \leqslant \lim_{\varepsilon \to 0+} \varepsilon^2 \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} P(\xi_{I_{nk}} \neq 0) \leqslant \lim_{\varepsilon \to 0+} \varepsilon^2 \overline{\int_I} P(\xi_J \neq 0)$$

is obtained, and thus — in virtue of the lemma — the equality

$$\sigma^2(I) = 0 \tag{11}$$

holds.

Formulae (1), (9) and (10) imply that for $x \neq 0$ the function $Q(x,I)$ is integrable in the sense of Burkill and the equalities

$$\int_I Q(x, J) = \begin{cases} M(x, I) & \text{for} \quad x < 0, \\ N(x, I) & \text{for} \quad x > 0. \end{cases} \tag{12}$$

From the inequality

$$\Big| \int_I Q(x, J) \Big| \leqslant \overline{\int_I} P(\xi_J \neq 0) \qquad \text{for} \quad x \neq 0, \tag{13}$$

from the lemma and from formula (12) it follows that the functions $M(x,I)$ and $N(x,I)$ are of bounded variation (of the argument x). Thus, taking into account formulae (11) and (12), we can write formula (7) in the following way:

$$\log \varphi(s, I) = i\gamma_0(I)s + \int_{x \neq 0} (e^{isx} - 1)\, d_x \int_I Q(x, J), \tag{14}$$

where

$$\gamma_0(I) = \gamma(I) - \int_{x \neq 0} \frac{x}{1+x^2}\, d_x \int_I Q(x, J). \tag{15}$$

For $\tau>0$ such that $-\tau$ and τ are continuity points of $M(x,I)$ and $N(x,I)$ respectively let us write

$$\gamma(\tau,I)=\gamma(I)+\int_{-\tau}^{0}\frac{x^3}{1+x^2}\,dM(x,I)+\int_{0}^{\tau}\frac{x^3}{1+x^2}\,dN(x,I)-$$
$$-\int_{-\infty}^{-\tau}\frac{x}{1+x^2}\,dM(x,I)-\int_{\tau}^{\infty}\frac{x}{1+x^2}\,dN(x,I).$$

Thus, taking into account formula (12), we get

$$\gamma(\tau,I)=\gamma(I)-\int_{|x|>\tau}\frac{x}{1+x^2}\,d_x\int_I Q(x,J)+\int_{0<|x|\leqslant\tau}\frac{x^3}{1+x^2}\,d_x\int_I Q(x,J). \tag{16}$$

In virtue of a theorem of Gnedenko (see [2], pp. 90, 91 and 132) the following equality is obtained:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|<\tau}x\,dF_{nk}(x)=\gamma(\tau,I)\quad\text{for}\quad\tau>0, \tag{17}$$

as $\max\limits_{1\leqslant k\leqslant n}|I_{nk}|\to 0$.

We get further from (13)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|<\tau}x\,dF_{nk}(x)\leqslant\tau\overline{\int_I}P(\xi_J\neq 0),$$

$$\Big|\int_{0<x\leqslant\tau}\frac{x^3}{1+x^2}\,d_x\int_I Q(x,J)\Big|$$
$$\leqslant\tau\Big|\int_I Q(\tau,J)-\int_I Q(+0,J)\Big|+\tau\Big|\int_I Q(-0,J)-\int_I Q(-\tau,J)\Big|$$
$$\leqslant 4\tau\overline{\int_I}P(\xi_J\neq 0).$$

These relations and formulae (16) and (17) give the inequality

$$\Big|\gamma(I)-\int_{|x|>\tau}\frac{x}{1+x^2}\,d_x\int_I Q(x,J)\Big|\leqslant 5\tau\overline{\int_I}P(\xi_J\neq 0).$$

In the same way the inequality

$$\Big|\int_{0<|x|\leqslant\tau}\frac{x}{1+x^2}\,d_x\int_I Q(x,J)\Big|\leqslant 4\tau\overline{\int_I}P(\xi_J\neq 0)$$

is obtained. Finally, from the last two relations and from formula (15) the inequality

$$|\gamma_0(I)|\leqslant 9\tau\overline{\int_I}P(\xi_J\neq 0)$$

follows; since τ may be arbitrarily small this relation and the lemma imply the equality $\gamma_0(I)=0$.

The assertion of the theorem follows at once from formula (14) and from the last equality.

III. Remark. For $x\neq 0$ put

$$q(x,t)=\lim\frac{Q(x,I)}{|I|}, \tag{18}$$

as I contracts to a fixed point t. Since for a fixed $x\neq 0$ the function of an interval $Q(x,I)$ is non-negative or non-positive (cf. the definition 1), then, as $Q(x,I)$ is integrable in the sense of Burkill, we find that it is a function of restricted bounded variation (see [6], p. 246). Hence assumptions (i) and (ii) imply that limit (18) exists almost everywhere and the function $q(x,t)$ is integrable in the sense of Lebesgue with respect to t (see [6], p. 265, 266; [3], p. 17). Moreover the following equality holds:

$$\int_I Q(x,J)=\int_I q(x,t)\,dt+\mu(x,I),$$

where μ is for $x\neq 0$ a singular function of an interval (see [6], p. 265, [4], p. 18). If the function $Q(x,I)$ is absolutely continuous, formula (2) will take the form

$$\log\varphi(s,I)=\int_{x\neq 0}(e^{isx}-1)\,d_x\int_I q(x,t)\,dt.$$

This remark (without proof) has been published in the Bulletin de l'Acad. Pol. des Sci., Cl. III, vol. III, 3, by the present authors. (There, in theorem 1, the assumption that convergence (1) is uniform with respect to x has been omitted).

COROLLARIES. (α) In addition to (i) and (ii) let us assume that

(iii) *the increments ξ_I can assume only the values of a countable set of real numbers $0,x_1,x_2,\ldots$ ($x_k\neq 0$), which set is independent of the special choice of I.*

Let

$$W_x(I)=P(\xi_I=x). \tag{19}$$

We shall show that

$$(20)\qquad \int_I Q(x,J) = \begin{cases} \sum\limits_{x_k<x} \int\limits_I W_{x_k}(J) & \text{for} \quad x<0, \\ -\sum\limits_{x_k\geqslant x} \int\limits_I W_{x_k}(J) & \text{for} \quad x>0. \end{cases}$$

We shall prove (20) only for the case of $x<0$, since in the case of $x>0$ the proof is quite similar.

Using the same notation as in the proof of the lemma, we get from assumption (i) for $x\neq 0$ the inequality

$$P(\xi_I=x) \geqslant \sum_{k=1}^{n} P(\xi_{I_k}=x)P(\xi_{J_k}=0)\prod_{r=1}^{k-1} P(\xi_{I_r}=0).$$

Hence, taking into account formulae (3) and (5), we obtain for every division $I_1, I_2, \ldots, I_n$ of I such that $|I|<\delta$ the inequality

$$P(\xi_I=x) \geqslant \frac{5}{16}\sum_{k=1}^{n} P(\xi_{I_k}=x).$$

Finally, from assumption (iii) follows for $x<0$

$$\sum_{k=1}^{n} P(\xi_{I_k}<x) = \sum_{x_r<x}\sum_{k=1}^{n} P(\xi_{I_k}=x_r) \leqslant \sum_{\substack{r=1\\x_r<x}}^{N}\sum_{k=1}^{n} P(\xi_{I_k}=x_r) + \tfrac{16}{5}\sum_{r=N}^{\infty} P(\xi_I=x_r).$$

Let us now observe that in virtue of the lemma the upper Burkill integrals

$$\overline{\int_I} W_{x_r}(J) \qquad (r=1,2,\ldots)$$

and the Burkill integral $\int_I Q(x,J)$ exist. Hence, taking into account (19), we obtain for each N, as $\max\limits_{1\leqslant k\leqslant n}|I_k|\to 0$, the inequality

$$\int_I Q(x,J) \leqslant \sum_{\substack{r=1\\x_r<x}}^{N} \overline{\int_I} W_{x_r}(J) + \sum_{r=N}^{\infty} P(\xi_I=x_r).$$

As $N\to\infty$, the last inequality implies the inequality

$$\int_I Q(x,J) \leqslant \sum_{x_r<x} \overline{\int_I} W_{x_r}(J);$$

since the opposite inequality is evident, we get

$$\int_I Q(x,J) = \sum_{x_r<x} \overline{\int_I} W_{x_r}(J).$$

In the same way it can be shown that

$$\int_I Q(x,J) = \sum_{x_r<x} \underline{\int_I} W_{x_r}(J).$$

These two equalities give at once formula (20) for $|I|\leqslant\delta$ and consequently, the Burkill integral being an additive function, this formula is true for arbitrary intervals I.

Then, in virtue of equality (20), formula (2) will take the form

$$\log\varphi(s,I) = \sum_{k=1}^{\infty}(e^{ix_k s}-1)\int_I W_{x_k}(J).$$

This result has been obtained by Prékopa ([6], p. 318).

(β) In addition to (i) and (ii) let us assume that the process is *homogeneous*, *i. e.* that

(iv) *the distribution function $P(\xi_I<x)$ depends only on $|I|$ and x.*

From the assumed homogeneity it follows that limit (18) exists for every t $(0\leqslant t\leqslant T)$ and does not depend on t. The convergence in (18) is uniform with respect to t. Hence the limit

$$q(x) = \lim_{|I|\to 0}\frac{Q(x,I)}{|I|} \qquad (x\neq 0)$$

exists and the equality

$$\int_I Q(x,J) = q(x)|I|$$

holds. In this case formula (2) will be of the form

$$\log\varphi(s,I) = |I|\int_{x\neq 0}(e^{isx}-1)\,dq(x).$$

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Reçu par la Rédaction le 21. 8. 1955

A general bilinear vector integral

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Since the time of the introduction of the Lebesgue integral, several types of extensions and generalizations have been studied. We shall be concerned with two such generalizations in the present paper.

The first extension is in the direction of integration when both the function to be integrated and the measure take values in a relatively general vector space[1]). This paper considers the case that there is a continuous bilinear "multiplication" defined on the product of the vector spaces in which the function and the measure take their values, the product lying in a (possibly different) vector space. The integral discussed here possesses many of the properties of the usual Lebesgue integral; in particular, we show that the well-known Vitali and Bounded Convergence theorems remain valid in this generality, while the natural extension of the Lebesgue Dominated Convergence theorem fails. The second extension is in the direction of replacing the usual requirement of countable additivity of the measure by the assumption of finite additivity. It was shown by Hildebrandt [20] and Fichtenholz and Kantorovitch [13] that this may be done for bounded functions, but some recent work of Dunford and Schwartz [12] demonstrates that it is also possible for unbounded functions, provided that almost everywhere convergence is replaced by convergence in measure.

The structure of the present paper is as follows: sections 1 and 2 introduce the basic terminology and elementary properties; section 3, the principal section, develops the general integral with respect to an additive set function. In section 4 the assumption of countable additivity is imposed and the main results of section 3 are recast in this light. Finally, in section 5 comparisons are made with other integrals. It is found that certain cases of the countably additive integral presented here reduce to (a) the Lebesgue integral, (b) the second Dunford [9] integral of vector functions with respect to a scalar measure (which includes the Bochner

[1]) Such integrals arise naturally in the definition of the concept of work, and in Ampère's law.