3. Using the above Lemmata we can prove by the same method as in the paper of A. Alexiewicz the following

THEOREM 1. *If the set B is linear and analytic, then there exists a decomposition $T=e\cup h$ and a residual set R in X such that*

(a) *for every x and every $\varepsilon>0$ there exists a set e' such that $\mu(e\setminus e')<\varepsilon$ and $U(x)_{e'}\epsilon B$;*

(b) *for every $x\epsilon R$ and every set $h'\subset h$ of positive measure $U(x)_{h'}\bar{\epsilon}B$.*

THEOREM 2. *When the set B is analytic and satisfies the following condition:*

from $y_{e_n}\epsilon B$ $(n=1,2,\ldots)$ and $e=\bigcup_{n=1}^{\infty}e_n$ results $y_e\epsilon B$,

then there exists a decomposition $T=e\cup h$ and a residual set $R\subset X$ such that

(a₁) *$U(x)_e\epsilon B$ for every x,*

(a₂) *$U(x)_{h'}\bar{\epsilon}B$ for every $x\epsilon R$ and every set $h'\subset h$ of positive measure.*

References

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[3] C. Kuratowski, *Topologie I*, 2 éd., Warszawa 1948.

[4] S. Mazur und W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationen* (Erste Mitteilung), Studia Mathematica 5 (1934), p. 50-68.

[5] — *Grundlegende Eigenschaften der polynomischen Operationen* (Zweite Mitteilung), Studia Mathematica 5 (1934), p. 179-189.

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On the estimation of the norm of the n-linear symmetric operation

by

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Let X and Y be two Banach spaces. An operation $U(x_1,\ldots,x_n)$ from $\underbrace{X\times\ldots\times X}_{n}$ to Y is called *n-linear* if it is linear in each variable x_i separately. It is called *symmetric* if $U(x_1,\ldots,x_n)=U(x_{\pi_1},\ldots,x_{\pi_n})$ for every permutation $\pi_1,\ldots,\pi_n$ of the numbers $1,\ldots,n$. The operation $U(x_1,\ldots,x_n)$ being n-linear and symmetric, we call the operation $U(x)=U(x,x,\ldots,x)$ the *power of degree n*; $U(x_1,\ldots,x_n)$ is then called the *primitive* (or *polar*) operation of $U(x)$. Between the norms of these operations an inequality

$$\sup_{\|x_1\|\leqslant 1,\ldots,\|x_n\|\leqslant 1}\|U(x_1,\ldots,x_n)\|\leqslant B_n\sup_{\|x\|\leqslant 1}\|U(x)\|$$

holds, with B_n depending only on n. A. E. Taylor[1]) has shown that $B_n\leqslant n^n/n!$. We shall show that this estimation is the best possible.

Let $X=L$, $Y=R^1$ (the space of reals), $\Delta_k=\langle(k-1)/n,\ k/n\rangle$ $(k=1,2,\ldots,n)$. Let us consider the operation

$$U(x_1,\ldots,x_n)=\sum_{(\pi_1,\ldots,\pi_n)}\int_{\Delta_1}x_{\pi_1}(t)\,dt\ldots\int_{\Delta_n}x_{\pi_n}(t)\,dt,$$

the summation being extended over all permutations $\pi_1,\ldots,\pi_n$ of the numbers $1,\ldots,n$. This operation is obviously n-linear and symmetric.

We shall prove that

$$(*)\qquad \sup_{\|x_1\|\leqslant 1,\ldots,\|x_n\|\leqslant 1}\|U(x_1,\ldots,x_n)\|=\frac{n^n}{n!}\sup_{\|x\|\leqslant 1}\|U(x)\|.$$

Let

$$\|x_i\|=\int_0^1|x_i(t)|\,dt\leqslant 1$$

[1]) A. E. Taylor, *Additions to the Theory of Polynomials in Normed Linear Spaces*, The Tôhoku Math. Journal 44 (1938), p. 302-318, theorems 2.5 and 2.6.

and write

$$a_{ik} = \int_{\Delta_k} |x_i(t)|\,dt.$$

Then

$$0 \leqslant \sum_{k=1}^{n} a_{ik} \leqslant 1$$

and

$$\|U(x_1,\ldots,x_n)\| = |U(x_1,\ldots,x_n)| \leqslant \sum_{(\pi_1,\ldots,\pi_n)} \prod_{k=1}^{n} \int_{\Delta_k} |x_{\pi_k}(t)|\,dt$$

$$= \sum_{(\pi_1,\ldots,\pi_n)} \prod_{k=1}^{n} a_{\pi_k k} \leqslant \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \leqslant 1.$$

On the other hand choose

$$x_k(t) = \begin{cases} n & \text{for } t \in \Delta_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\|x_k\| = 1$ and $U(x_1,\ldots,x_n) = 1$. Hence

$$\sup_{\|x_1\|\leqslant 1,\ldots,\|x_n\|\leqslant 1} \|U(x_1,\ldots,x_n)\| = 1.$$

By definition of the operation $U(x_1,\ldots,x_n)$ it easily follows that

$$U(x) = U(x,\ldots,x) = n! \prod_{k=1}^{n} \int_{\Delta_k} x(t)\,dt.$$

Let $\|x\| \leqslant 1$; since

$$\|x\| = \sum_{k=1}^{n} \int_{\Delta_k} |x(t)|\,dt,$$

we obtain

$$\|U(x)\| = |U(x,\ldots,x)| \leqslant n! \prod_{k=1}^{n} \int_{\Delta_k} |x(t)|\,dt \leqslant n! \left(\frac{1}{n} \sum_{k=1}^{n} \int_{\Delta_k} |x(t)|\,dt\right)^n \leqslant \frac{n!}{n^n} \|x\|.$$

On the other hand, $x(t) = 1$ implies $U(x,\ldots,x) = n!/n^n$, whence

$$\sup_{\|x\|\leqslant 1} \|U(x)\| = n!/n^n.$$

Thus the equality (*) is true.

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On quotient-fields generated by pseudonormed rings

by

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I. A linear ring R over the field of complex numbers as scalars with unit element is called a *pseudonormed ring* if R is a B_0-space (Mazur and Orlicz [4], p. 185) with respect to a sequence of pseudonorms $\| \|_1, \| \|_2, \ldots$ *submultiplicative*, *i. e.* satisfying the condition $\|xy\|_k \leqslant \|x\|_k \|y\|_k$ for every $x, y \in R$.

In this paper we shall deal exclusively with commutative pseudonormed rings without divisors of zero.

Let $Q(R)$ be the quotient-field obtained from the pseudonormed ring R (Van der Waerden [7], p. 46-49). J. G.-Mikusiński has introduced by means of the convergence in the ring R a convergence in the field $Q(R)$ (which will be called the *M-convergence*) as follows: the sequence $a_1, a_2, \ldots$ $(a_n \in Q(R))$ converges to $a \in Q(R)$ (which will be denoted by $M\text{-}\lim_{n\to\infty} a_n = a$) if there exists $x \in R$ $(x \neq 0)$ such that $xa_n \in R$ $(n = 1, 2, \ldots)$, $xa \in R$ and the sequence $xa_1, xa_2, \ldots$ converges to xa in the ring R, *i. e.* $\lim_{n\to\infty} \|xa_n - xa\|_k = 0$ $(k = 1, 2, \ldots)$ (Mikusiński [5], p. 62). It is easily seen that $Q(R)$ with the M-convergence is an L-space of Fréchet (Kuratowski [3], p. 84) and that the addition and the ring- and scalar-multiplication in $Q(R)$ are continuous with respect to the M-convergence.

The M-convergence in $Q(R)$ is called *topological* if there exists in $Q(R)$ a topology satisfying the first axiom of countability (Kuratowski [3], p. 33), such that the convergence according to this topology is equivalent to the M-convergence, and $Q(R)$ is a topological field with respect to this topology (we suppose here that the inverse of an element is a continuous function). In connection with the study of the M-convergence in the field of operators of Mikusiński (Urbanik [6]) arose the problem of determining those pseudonormed rings for which the M-conevrgence in $Q(R)$ is topological. In this paper we shall prove that

If the M-convergence in the field $Q(R)$ is topological, then the pseudonormed ring R is isomorphic and homeomorphic with the field of complex numbers (topologized as usual).