On the distances between signals in the non-homogeneous Poisson stochastic process

by

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In this paper*) I investigate a special class of the non-homogeneous Poisson stochastic processes. I denote by the random variable $\omega(t)$ $(t>0)$ the number of signals in the half-open interval $(0,t\rangle$, *e. g.* the number of discharges in the Geiger-Müller counter (in this case t denotes time); $\omega(t)$ is therefore a non-negative, non-decreasing, continuous on the right and integral valued function of t. I put $\omega(0)=0$. Let us write also

$$Pr_k(t_1,t_2)=Pr\{\omega(t_2)-\omega(t_1)=k\}$$

for $0\leqslant t_1<t_2$ and $k=0,1,2,\ldots$; it is the probability of k signals coming in the interval $(t_1,t_2\rangle$.

Suppose that the process has the following properties:

(a) the random variables $\omega(t_2)-\omega(t_1),\ldots,\ \omega(t_{2l})-\omega(t_{2l-1})$ are independent for $0\leqslant t_1<t_2\leqslant\ldots\leqslant t_{2l-1}<t_{2l}$ $(l=2,3,\ldots)$ (process with independent increments);

$$\text{(b)}\quad Pr_k(t_1,t_2)=\frac{\left(\int_{t_1}^{t_2}a(t)\,dt\right)^k}{k!}\exp\Big(-\int_{t_1}^{t_2}a(t)\,dt\Big),$$

where $a(t)$ is a function defined for $t\geqslant 0$, non-negative and continuous.

The property (b) implies another two:

$$\text{(c)}\quad \lim_{\Delta t\to 0}\frac{1-Pr_0(t,t+\Delta t)}{\Delta t}=a(t);$$

$$\text{(d)}\quad \lim_{\Delta t\to 0}\frac{1-Pr_0(t,t+\Delta t)-Pr_1(t,t+\Delta t)}{\Delta t}=0.$$

It is easy to see from these formulae that $\omega(t)$ can possess only jumps equal to 1 (signals can come only singly).

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Besides the properties (a) and (b) I assume nearly everywhere in the paper (with the exception of Theorem 1) the following third property:

$$\text{(e)}\quad \int_0^\infty a(t)\,dt=\infty.$$

This assumption means that in the whole process there will be an infinite number of signals ($\omega(t)\to\infty$ for $t\to\infty$).

We obtain the *homogeneous* Poisson process from (a)-(e) setting $a(t)=a>0$.

The purpose of this paper is to investigate the *distances between signals* in the process defined by the properties (a), (b) and (e). In section 1 I give the general definitions and theorems concerning the distributions of those distances, in section 2 I prove the limit theorem for the mean value of the fraction of distances not greater than y $(y>0)$ among the initial n distances $(n=1,2,\ldots)$, and in section 3 I investigate the convergence of this mean value to a constant for $n\to\infty$.

The results obtained can be of importance in practical investigations of the non-homogeneous Poisson processes: knowing the distribution of the distances between signals it is possible to draw conclusions concerning the function $a(t)$. This method seems to be suitable when, for instance, the oscillation frequency of a periodic function $a(t)$ is of the same or greater order than the mean frequency of signals.

1. The general definitions and theorems. I call the distance between the k-th and the $(k+1)$-th signals (*i. e.* the jumps of the function $\omega(t)$, equal to 1) the *k-th distance between signals* $(k=1,2,\ldots)$.

DEFINITION 1. $L(y_1,\ldots,y_n;\Delta y_1,\ldots,\Delta y_n)$ for $n=1,2,\ldots$, $y_k\geqslant 0$ and $\Delta y_k>0$ $(k=1,2,\ldots,n)$ is the following event: in the whole process no less than $n+1$ signals appear and the i-th distance between signals has a value from the interval $(y_i,y_i+\Delta y_i\rangle$ (for all i from 1 to n); when we write $\Delta y_l=\infty$, it means that the l-th distance is greater than y_l.

THEOREM 1. *If the process has the properties* (a) *and* (b), *then*

$$Pr\{L(y_1,\ldots,y_n;\Delta y_1,\ldots,\Delta y_n)\}=\int_0^\infty a(x_1)\,dx_1\int_{x_1+y_1}^{x_1+y_1+\Delta y_1}a(x_2)\,dx_2\times\ldots$$

$$\ldots\times\int_{x_{n-1}+y_{n-1}}^{x_{n-1}+y_{n-1}+\Delta y_{n-1}}a(x_n)\,dx_n\int_{x_n+y_n}^{x_n+y_n+\Delta y_n}a(x_{n+1})-\exp\Big(\int_0^{x_{n+1}}a(t)dt\Big)\,dx_{n+1}.$$

Proof. In view of the properties (a) and (b) the probability that in the whole process there will be no less than $n+1$ signals and the i-th signal will appear in the interval $\langle t_i, t_i+\Delta t_i\rangle$ (for all i from 1 to $n+1$), where $0<t_1<t_1+\Delta t_1<\ldots<t_{n+1}<t_{n+1}+\Delta t_{n+1}$, is equal to

$$Pr_0(0,t_1)Pr_1(t_1,t_1+\Delta t_1)Pr_0(t_1+\Delta t_1,t_2)\ldots Pr_0(t_n+\Delta t_n,t_{n+1})\times$$
$$\times\{1-Pr_0(t_{n+1},t_{n+1}+\Delta t_{n+1})\}$$
$$=\Big(\exp\Big(-\int_0^{t_{n+1}}a(t)\,dt\Big)-\exp\Big(-\int_0^{t_{n+1}+\Delta t_{n+1}}a(t)\,dt\Big)\Big)\prod_{i=1}^{n}\int_{t_i}^{t_i+\Delta t_i}a(t)\,dt$$
$$=\int_{t_1}^{t_1+\Delta t_1}dx_1\ldots\int_{t_{n+1}}^{t_{n+1}+\Delta t_{n+1}}a(x_1)\ldots a(x_{n+1})\exp\Big(-\int_0^{x_{n+1}}a(t)\,dt\Big)\,dx_{n+1}.$$

The function which we integrate in the last expression is therefore the conditional probability density of the appearance of the i-th signal at the point x_i (for all i from 1 to $n+1$), where $0<x_1\ldots<x_{n+1}$, under the condition of the appearance in the whole process of no less than $n+1$ signals, multiplied by the probability of this last event. Integrating this function through all the values of x_i possible in the event $L(y_1,\ldots,y_n;\Delta y_1,\ldots,\Delta y_n)$, i. e. $x_{i-1}+y_{i-1}<x_i\leqslant x_{i-1}+y_{i-1}+\Delta y_{i-1}$ (for i from $n+1$ to 2) and $0<x_1<\infty$, we get for the probability of this event the expression given in Theorem 1.

DEFINITION 2. The random variable $\vartheta_k(y)$ $(k=1,2,\ldots,\ y>0)$ is a variable *equal to* 1 if the k-th distance between signals is not greater than y, or *equal to* 0 if either the k-th distance between signals is greater than y or less than $k+1$ signals appear in the whole process.

THEOREM 2. *If the process has the properties* (a), (b) *and* (e), *then the mean value of the random variable $\vartheta_k(y)$ (i. e. the probability that the k-th distance between signals will not be greater than y) is equal to*

$$E\{\vartheta_k(y)\}=1-\frac{1}{(k-1)!}\int_0^\infty a(x)\exp\Big(-\int_0^{x+y}a(t)\,dt\Big)\Big(\int_0^x a(t)\,dt\Big)^{k-1}dx.$$

Proof. Let $y_i=0$ (for all i from 1 to $k-1$), $y_k=y$ and $\Delta y_i=\infty$ (for all i from 1 to k). Then

$$E\{\vartheta_k(y)\}=1-Pr\{L(y_1,\ldots,y_k;\Delta y_1,\ldots,\Delta y_k)\}$$
$$=1-\int_0^\infty a(x_1)\,dx_1\int_{x_1}^\infty a(x_2)\,dx_2\ldots\int_{x_{k-1}}^\infty a(x_k)\,dx_k\int_{x_k+y}^\infty a(x_{k+1})\exp\Big(-\int_0^{x_{k+1}}a(t)\,dt\Big)\,dx_{k+1}$$
$$=1-\int_0^\infty a(x_1)\,dx_1\int_{x_1}^\infty a(x_2)\,dx_2\ldots\int_{x_{k-2}}^\infty a(x_{k-1})\,dx_{k-1}\int_{x_{k-1}}^\infty a(x_k)\exp\Big(-\int_0^{x_k+y}a(t)\,dt\Big)\,dx_k.$$

The region defined by the inequalities

$$0<x_1<\infty,\quad x_1<x_2<\infty,\quad\ldots,\quad x_{k-1}<x_k<\infty$$

can also be defined by the inequalities

$$0<x_k<\infty,\quad 0<x_{k-1}<x_k,\quad\ldots,\quad 0<x_1<x_2.$$

Therefore

$$E\{\vartheta_k(y)\}=1-\int_0^\infty a(x_k)\exp\Big(-\int_0^{x_k+y}a(t)\,dt\Big)\,dx_k\int_0^{x_k}a(x_{k-1})\,dx_{k-1}\ldots\int_0^{x_2}a(x_1)\,dx_1$$
$$=1-\frac{1}{(k-1)!}\int_0^\infty a(x_k)\exp\Big(-\int_0^{x_k+y}a(t)\,dt\Big)\Big(\int_0^{x_k}a(x_1)\,dx_1\Big)^{k-1}dx_k,$$

q. e. d.

DEFINITION 3. The random variable $\Theta_n(y)$ $(n=1,2,\ldots,\ y>0)$ is

$$\Theta_n(y)=\frac{1}{n}\sum_{k=1}^{n}\vartheta_k(y).$$

If no less than $n+1$ signals appeared in the whole process (as always happens in the case of a process with property (e)), then $\Theta_n(y)$ is the fraction of distances not greater than y among the initial n distances.

THEOREM 3. *If the process has the properties* (a), (b) *and* (e), *then*

$$E\{\Theta_n(y)\}=1-\frac{1}{n}\int_0^\infty a(x)\exp\Big(-\int_0^{x+y}a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k dx.$$

Proof. In view of the theorem for the mean value of the sum of random variables we obtain

$$E\{\Theta_n(y)\}=\frac{1}{n}\sum_{k=1}^{n}E\{\vartheta_k(y)\};$$

from this formula and Theorem 2 we deduce Theorem 3.

For the *homogeneous* process, i. e. for $a(t)=a>0$, we obtain from Theorems 1, 2 and 3 respectively the known formulae:

$$(1)\qquad Pr\{L(y_1,\ldots,y_n;\Delta y_1,\ldots,\Delta y_n)\}=\exp\Big(-a\sum_{k=1}^{n}y_k\Big)\prod_{k=1}^{n}(1-e^{-a\Delta y_k}),$$

$$(2)\qquad E\{\vartheta_k(y)\}=1-e^{-ay},$$

$$(3)\qquad E\{\Theta_n(y)\}=1-e^{-ay}.$$

2. The limit theorem for $E\{\Theta_n(y)\}$.

DEFINITION 4. For the process with properties (a), (b) and (e) I write

$$S_n(y) = 1 - \frac{1}{n}\int_0^{\tau_n} a(x)\exp\Big(-\int_x^{x+y} a(t)\,dt\Big)\,dx,$$

where $n=1,2,\ldots$, $y>0$; τ_n is one of the solutions of the equation

$$\int_0^{\tau_n} a(t)\,dt = n.$$

If the last equation has more than one solution, then these solutions compose a closed interval in which $a(t)=0$; the choice of τ_n from this interval has therefore no significance for the value of $S_n(y)$.

It is easy to see from Definition 4 that $0<S_n(y)<1$.

THEOREM 4. *If the process has the properties* (a), (b) *and* (e), *then*

$$|E\{\Theta_n(y)\} - S_n(y)| < 1/\sqrt{2\pi n}.$$

Proof. In view of Definition 4 and Theorem 3 we obtain

$$S_n(y) - E\{\Theta_n(y)\} = \frac{1}{n}\int_{\tau_n}^{\infty} a(x)\exp\Big(-\int_0^{x+y} a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k dx + \tag{4}$$

$$+\frac{1}{n}\int_0^{\tau_n} a(x)\exp\Big(-\int_x^{x+y} a(t)\,dt\Big)\Big\{\exp\Big(-\int_0^x a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k - 1\Big\}\,dx.$$

For the first part of the right side of this formula we have the inequalities

$$0 < \frac{1}{n}\int_{\tau_n}^{\infty} a(x)\exp\Big(-\int_0^{x+y} a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k dx \tag{5}$$

$$< \frac{1}{n}\int_{\tau_n}^{\infty} a(x)\exp\Big(-\int_0^{x} a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k dx$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k!}\int_{\tau_n}^{\infty} e^{-v(x)}v(x)^k\,dv(x),$$

where

$$v(x) = \int_0^x a(t)\,dt.$$

Since, for $k=0,1,2,\ldots$,

$$\frac{1}{k!}\int_{\tau_n}^{\infty} e^{-v(x)}v(x)^k\,dv(x)$$

$$= -e^{-v(x)}\frac{v(x)^k}{k!}\Big|_{\tau_n}^{\infty} + \frac{1}{(k-1)!}\int_{\tau_n}^{\infty} e^{-v(x)}v(x)^{k-1}\,dv(x) = \ldots = e^{-n}\sum_{l=0}^{k}\frac{n^l}{l!},$$

we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k!}\int_{\tau_n}^{\infty} e^{-v(x)}v(x)^k\,dv(x) = \frac{e^{-n}}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{n^l}{l!} \tag{6}$$

$$= \frac{e^{-n}}{n}\sum_{l=0}^{n-1}\sum_{k=l}^{n-1}\frac{n^l}{l!} = \frac{e^{-n}}{n}\sum_{l=0}^{n-1}(n-l)\frac{n^l}{l!}$$

$$= \frac{e^{-n}}{n}\Big(\sum_{l=0}^{n-1}\frac{n^{l+1}}{l!} - \sum_{l=0}^{n-2}\frac{n^{l+1}}{l!}\Big) = \frac{n^n}{n!}e^{-n}.$$

For the second part of the right side of formula (4) we have the inequalities

$$0 > \frac{1}{n}\int_0^{\tau_n} a(x)\exp\Big(-\int_x^{x+y} a(t)\,dt\Big)\times \tag{7}$$

$$\times\Big\{\exp\Big(-\int_0^x a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k - 1\Big\}\,dx$$

$$> \frac{1}{n}\int_0^{\tau_n} a(x)\Big\{\exp\Big(-\int_0^x a(t)\,dt\Big)\sum_{k=0}^{n-1}\frac{1}{k!}\Big(\int_0^x a(t)\,dt\Big)^k - 1\Big\}\,dx$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k!}\int_0^{\tau_n} e^{-v(x)}v(x)^k\,dv(x) - 1.$$

Since, for $k=0,1,2,\ldots$,

$$\frac{1}{k!}\int_0^{\tau_n} e^{-v(x)}v(x)^k\,dv(x)$$

$$=-e^{-v(x)}\frac{v(x)^k}{k!}\Big|_0^{\tau_n}+\frac{1}{(k-1)!}\int_0^{\tau_n} e^{-v(x)}v(x)^{k-1}\,dv(x)=\ldots=-e^{-n}\sum_{l=0}^{k}\frac{n^l}{l!}+1,$$

we obtain

$$(8)\qquad \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k!}\int_0^{\tau_n} e^{-v(x)}v(x)^k\,dv(x)-1=-\frac{e^{-n}}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{n^l}{l!}=-\frac{n^n}{n!}e^{-n}.$$

From (4)-(8) we deduce

$$|E\{\Theta_n(y)\}-S_n(y)|<\frac{n^n}{n!}e^{-n},$$

and the inequality

$$\frac{n^n}{n!}e^{-n}<\frac{1}{\sqrt{2\pi n}}$$

completes the proof.

From Theorem 4 follows the uniform convergence of the sequence of functions $E\{\Theta_n(y)\}$ to the sequence of functions $S_n(y)$ in the half-axis $y>0$.

Putting $a(t)=a>0$ in Definition 4 and comparing the obtained function $S_n(y)$ with the function $E\{\Theta_n(y)\}$ given by formula (3), it is easy to prove that for the *homogeneous* Poisson process $S_n(y)=E\{\Theta_n(y)\}$ for all n and y.

3. Convergence of the sequence $E\{\Theta_n(y)\}$. For the homogeneous process, in view of formula (3), $E\{\Theta_n(y)\}$ is independent of n; it is, however, not true in the general case and the problem arises for which $a(t)$ the sequence $E\{\Theta_n(y)\}$ converges if $n\to\infty$.

Theorem 4 implies the following obvious

COROLLARY 1. *If the process has the properties* (a), (b) *and* (e), *then, for each* $y>0$, *the necessary and sufficient condition for the existence of* $\lim_{n\to\infty} E\{\Theta_n(y)\}$ *is that of* $\lim_{n\to\infty} S_n(y)$; *if these limits exist, they are equal.*

This corollary enables us to replace the investigation of the convergence of the sequence $E\{\Theta_n(y)\}$ by such of the convergence of the sequence $S_n(y)$, which we shall now consider. Since we have defined $S_n(y)$ only for the processes with properties (a), (b) and (e) we shall not assume *explicite* in the following that the process has those properties.

LEMMA. *The necessary and sufficient condition for the existence, for some* $y>0$, *of the limit* $\lim_{n\to\infty} S_n(y)$ *is that of the limit*

$$\lim_{z\to\infty}\left(1-\frac{\int_0^z a(x)\exp\left(-\int_x^{x+y}a(t)\,dt\right)dx}{\int_0^z a(t)\,dt}\right);$$

if these limits exist, they are equal.

Proof. Let[1])

$$n(z)=\left[\int_0^z a(t)\,dt\right];$$

for $z\geqslant\tau_1$, *i. e.* $n(z)\geqslant 1$, we have

$$(9)\qquad \frac{\int_0^z a(x)\exp\left(-\int_x^{x+y}a(t)\,dt\right)dx}{\int_0^z a(t)\,dt}=\frac{n(z)}{\int_0^z a(t)\,dt}\,\frac{\int_0^{\tau_{n(z)}} a(x)\exp\left(-\int_x^{x+y}a(t)\,dt\right)dx}{n(z)}+$$

$$+\frac{\int_{\tau_{n(z)}}^{z} a(x)\exp\left(-\int_x^{x+y}a(t)\,dt\right)dx}{\int_0^z a(t)\,dt}.$$

If $z\to\infty$, then $n(z)\to\infty$ successively throughout all natural numbers; since

$$\lim_{z\to\infty}\frac{n(z)}{\int_0^z a(t)\,dt}=1,\qquad \lim_{z\to\infty}\frac{\int_{\tau_{n(z)}}^{z} a(x)\exp\left(-\int_x^{x+y}a(t)\,dt\right)dx}{\int_0^z a(t)\,dt}=0,$$

we deduce our Lemma from formula (9).

THEOREM 5. *If, for some* $y>0$ *and* $C'\geqslant 0$, *the limit*

$$\lim_{n\to\infty} S_n(y)$$

exists in the process defined by the function $a(t+C')$, *then this limit exists and is the same in the process defined by the function* $a(t+C)$, *for each* $C\geqslant 0$.

[1]) By the symbol $[x]$ I denote, in this proof and in the proof of Theorem 8, the function *"entier x"*.

Proof. In virtue of the Lemma and by transformations of integrals we get

$$\lim_{n\to\infty} S_n(y)_{a(t+C')} = 1-\lim_{z\to\infty}\frac{\int\limits_0^z a(x+C')\exp\Big(-\int\limits_x^{x+y} a(t+C')\,dt\Big)\,dx}{\int\limits_0^z a(t+C')\,dt}$$

$$=1-\lim_{z\to\infty}\frac{\int\limits_{C'-C}^{z+C'-C} a(x+C)\exp\Big(-\int\limits_x^{x+y} a(t+C)\,dt\Big)\,dx}{\int\limits_{C'-C}^{z+C'-C} a(t+C)\,dt}$$

$$=1-\lim_{z\to\infty}\frac{\int\limits_0^{z+C'-C} a(x+C)\exp\Big(-\int\limits_x^{x+y} a(t+C)\,dt\Big)\,dx}{\int\limits_0^{z+C'-C} a(t+C)\,dt}$$

$$=\lim_{n\to\infty} S_n(y)_{a(t+C)},$$

q. e. d.

THEOREM 6. *If, for some* $y>0$, $\lim_{n\to\infty} S_n(y)$ *exists in the process defined by the function* $a(t)$, *then this limit exists and is the same in the process defined by the function* $a(t)+b(t)$, *where* $b(t)$ *is any continuous function, defined for* $t\geqslant 0$, *such that* $a(t)+b(t)\geqslant 0$ *for all* $t\geqslant 0$ *and the integral*

$$\int\limits_0^\infty |b(t)|\,dt$$

is finite.

Proof. Denote by M_C and m_C respectively the upper and the lower limits of the function

$$f(x)=\int\limits_x^{x+y} b(t)\,dt$$

for $x\geqslant C$, where $C\geqslant 0$; for any $z\geqslant C\geqslant \tau_1$ we have

$$e^{-M_C}\frac{\int\limits_C^z a(x)\exp\Big(-\int\limits_x^{x+y} a(t)\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt} < \frac{\int\limits_0^z a(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt}$$

$$\leqslant e^{-m_C}\frac{\int\limits_C^z a(x)\exp\Big(-\int\limits_x^{x+y} a(t)\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt}+\frac{\int\limits_0^C a(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt}.$$

If $z\to\infty$, the first and the last expressions converge in virtue of the Lemma to

$$e^{-M_C}\{1-\lim_{n\to\infty} S_n(y)_{a(t)}\}\quad\text{and}\quad e^{-m_C}\{1-\lim_{n\to\infty} S_n(y)_{a(t)}\}$$

respectively. Since C is arbitrary and, as follows from the existence of the integral $\int\limits_0^\infty |b(t)|\,dt$, $\lim\limits_{C\to\infty} M_C=\lim\limits_{C\to\infty} m_C=0$, we obtain

$$(10)\qquad 1-\lim_{n\to\infty} S_n(y)_{a(t)}=\lim_{z\to\infty}\frac{\int\limits_0^z a(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt}.$$

We also have

$$(11)\qquad \Big|\int\limits_0^z b(t)\,dt\Big|\leqslant\int\limits_0^z |b(t)|\,dt\leqslant\int\limits_0^\infty |b(t)|\,dt$$

and

$$(12)\qquad \Big|\int\limits_0^z b(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx\Big|$$

$$\leqslant\int\limits_0^z |b(x)|\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx\leqslant\int\limits_0^\infty |b(x)|\,dx.$$

From (10), (11), (12) and the Lemma we deduce

$$1-\lim_{n\to\infty} S_n(y)_{a(t)}$$

$$=\lim_{z\to\infty}\frac{\int\limits_0^z a(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx+\int\limits_0^z b(x)\exp\Big(-\int\limits_x^{x+y}\{a(t)+b(t)\}\,dt\Big)\,dx}{\int\limits_0^z a(t)\,dt+\int\limits_0^z b(t)\,dt}$$

$$=1-\lim_{n\to\infty} S_n(y)_{a(t)+b(t)},$$ q. e. d.

THEOREM 7. *If, for some* $y>0$, *the limit*

$$\lim_{n\to\infty} S_n(y)$$

exists in the process defined by the function $a(t)$, *then also, for each* $C>0$, *the limit*

$$\lim_{n\to\infty} S_n(y/C),$$

equal to the first, exists in the process defined by the function $Ca(Ct)$.

Proof. In virtue of the Lemma and by transformations of integrals we obtain

$$\lim_{n\to\infty} S_n(y)_{a(t)} = 1 - \lim_{z\to\infty} \frac{\int_0^z a(x)\exp\left(-\int_x^{x+y} a(t)\,dt\right)dx}{\int_0^z a(t)\,dt}$$

$$= 1 - \lim_{z\to\infty} \frac{\int_0^{z/C} Ca(Cx)\exp\left(-\int_x^{x+y/C} Ca(Ct)\,dt\right)dx}{\int_0^{z/C} Ca(Ct)\,dt} = \lim_{n\to\infty} S_n(y/C)_{Ca(Ct)},$$

q. e. d.

THEOREM 8. *If, for some $y>0$, the limit*

$$\lim_{z\to\infty} \int_z^{z+y} a(t)\,dt = A_y$$

exists (in particular we may have $A_y=\infty$), then the function $a(t)$ has the mean value

$$a = \lim_{z\to\infty} \frac{1}{z}\int_0^z a(t)\,dt = \frac{A_y}{y},$$

and the limit

$$\lim_{n\to\infty} S_n(y) = 1 - e^{-ay}$$

exists.

Proof. For $z \geqslant y$ we have

$$\frac{\sum_{k=1}^{[z/y]} \int_{(k-1)y}^{ky} a(t)\,dt}{\{[z/y]+1\}y} \leqslant \frac{1}{z}\int_0^z a(t)\,dt \leqslant \frac{\sum_{k=1}^{[z/y]+1} \int_{(k-1)y}^{ky} a(t)\,dt}{[z/y]y}.$$

Because of the convergence of the sequence

$$C_k = \int_{(k-1)y}^{ky} a(t)\,dt \qquad (k=1,2,\ldots)$$

to A_y, the sequence of the arithmetical means of the sequence C_k also converges to this limit; therefore, if $z\to\infty$, we deduce from the inequalities obtained that the limit

$$\lim_{z\to\infty} \frac{1}{z}\int_0^z a(t)\,dt = \frac{A_y}{y}$$

exists.

Let us consider the function $g(t) = a(t) + e^{-t} > 0$. By virtue of de l'Hospital's rule and the Lemma we get

$$e^{-A_y} = \lim_{z\to\infty} \exp\left(-\int_z^{z+y} a(t)\,dt + e^{-z}(e^{-y}-1)\right) = \lim_{z\to\infty} \exp\left(-\int_z^{z+y} g(t)\,dt\right)$$

$$= \lim_{z\to\infty} \frac{g(z)\exp\left(-\int_z^{z+y} g(t)\,dt\right)}{g(z)} = \lim_{z\to\infty} \frac{\int_0^z g(x)\exp\left(-\int_x^{x+y} g(t)\,dt\right)dx}{\int_0^z g(t)\,dt}$$

$$= 1 - \lim_{n\to\infty} S_n(y)_{g(t)},$$

and since $\int_0^\infty e^{-t}\,dt = 1$, we have from Theorem 6

$$e^{-A_y} = 1 - \lim_{n\to\infty} S_n(y)_{a(t)},$$

which completes the proof.

Theorem 8 implies the following obvious

COROLLARY 2. *If the limit $\lim_{t\to\infty} a(t) = a$ exists (in particular we may have $a=\infty$), then a is the mean value of function $a(t)$ and for each $y>0$ the limit*

$$\lim_{n\to\infty} S_n(y) = 1 - e^{-ay}$$

exists.

THEOREM 9. *If the function $a(t)$ is periodic with period T, then for each $y>0$ the limit*

$$\lim_{n\to\infty} S_n(y) = 1 - \frac{\int_0^T a(x)\exp\left(-\int_x^{x+y} a(t)\,dt\right)dx}{\int_0^T a(t)\,dt}$$

exists.

Proof. In view of the periodicity of the functions which we integrate, for every $y>0$ the limits

$$\lim_{z\to\infty} \frac{1}{z}\int_0^z a(x)\exp\left(-\int_x^{x+y} a(t)\,dt\right)dx = \frac{1}{T}\int_0^T a(x)\exp\left(-\int_x^{x+y} a(t)\,dt\right)dx,$$

$$\lim_{z\to\infty} \frac{1}{z}\int_0^z a(t)\,dt = \frac{1}{T}\int_0^T a(t)\,dt > 0,$$

exist. Therefore we obtain by virtue of the Lemma

$$1-\frac{\int\limits_0^T a(x)\exp\left(-\int\limits_x^{x+y} a(t)\,dt\right)dx}{\int\limits_0^T a(t)\,dt}=1-\lim_{z\to\infty}\frac{z^{-1}\int\limits_0^z a(x)\exp\left(-\int\limits_x^{x+y} a(t)\,dt\right)dx}{z^{-1}\int\limits_0^z a(t)\,dt}$$

$$=\lim_{n\to\infty} S_n(y),\quad \text{q. e. d.}$$

S. Hartman has proved (by a somewhat different method from the one employed here) the following

THEOREM 10. *If the function $a(t)$ is, for $t\geqslant 0$, equal to a certain uniformly almost periodic function, then for every $y>0$ the limit $\lim\limits_{n\to\infty} S_n(y)$ exists.*

Proof. Denote by $a_0(t)$ the uniformly almost periodic function which is equal to $a(t)$ for $t\geqslant 0$. Because of its being uniformly almost periodic the function $a_0(t)$ is bounded. Hence, there exists a number $M>0$, such that $-M\leqslant a_0(t)\leqslant M$ for all t. We have also

$$\int\limits_x^{x+y} a_0(t)\,dt=\int\limits_0^y a_0(t)\,dt+\int\limits_0^x\{a_0(t+y)-a_0(t)\}\,dt.$$

Since

$$-My\leqslant\int\limits_x^{x+y} a_0(t)\,dt\leqslant My,$$

the integral $\int\limits_0^x\{a_0(t+y)-a_0(t)\}\,dt$ is the bounded indefinite integral of a uniformly almost periodic function, *i. e.* it is a uniformly almost periodic function of x. Therefore $\int\limits_x^{x+y} a_0(t)\,dt$ is also a uniformly almost periodic function of x.

Since

$$\exp\left(-\int\limits_x^{x+y} a_0(t)\,dt\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\int\limits_x^{x+y} a_0(t)\,dt\right)^k,$$

with the series uniformly convergent for all x in view of the boundedness of $\int\limits_x^{x+y} a_0(t)\,dt$, $\exp\left(-\int\limits_x^{x+y} a_0(t)\,dt\right)$ is a uniformly almost periodic function of x.

In virtue of the mean value theorem for uniformly almost periodic functions, the limits

$$\lim_{z\to\infty}\frac{1}{z}\int\limits_0^z a_0(x)\exp\left(-\int\limits_x^{x+y} a_0(t)\,dt\right)dx$$

and

$$\lim_{z\to\infty}\frac{1}{z}\int\limits_0^z a_0(t)\,dt$$

exist; moreover the last limit is greater than zero, because for each $\varepsilon>0$ there exists $l>\tau_1$, such that

$$\int\limits_{(k-1)l}^{kl} a_0(t)\,dt>1-\varepsilon\tau_1$$

for all natural k. Therefore the limit

$$\lim_{z\to\infty}\frac{z^{-1}\int\limits_0^z a_0(x)\exp\left(-\int\limits_x^{x+y} a_0(t)\,dt\right)dx}{z^{-1}\int\limits_0^z a_0(t)\,dt}$$

exists, and in view of the Lemma we deduce the existence of the limit $\lim\limits_{n\to\infty} S_n(y)$, q. e. d.

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