On the distances between signals in the non-homogeneous Poisson stochastic process

by

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In this paper I investigate a special class of the non-homogeneous Poisson stochastic processes. I denote by the random variable \( \omega(t) \) (\( t > 0 \)) the number of signals in the half-open interval \((0, t]\), e.g., the number of discharges in the Geiger-Müller counter (in this case \( t \) denotes time); \( \omega(t) \) is therefore a non-negative, non-decreasing, continuous on the right and integral valued function of \( t \). I put \( \omega(0) = 0 \). Let us write also

\[ P_k(t_1, t_2) = P\{\omega(t_2) - \omega(t_1) = k\} \]

for \( 0 \leq t_1 < t_2 \) and \( k = 0, 1, 2, \ldots \); it is the probability of \( k \) signals coming in the interval \((t_1, t_2)\).

Suppose that the process has the following properties:

(a) the random variables \( \omega(t_1) - \omega(t_2), \ldots, \omega(t_{t_2}) - \omega(t_{t_2-1}) \) are independent for \( 0 \leq t_1 < t_2 < \cdots < t_{t_2} < t_2 \) (\( t = 2, 3, \ldots \)) (process with independent increments);

(b) \[ P_k(t_1, t_2) = \frac{1}{k!} \exp\left(-\int_{t_1}^{t_2} a(t) dt\right), \]

where \( a(t) \) is a function defined for \( t \geq 0 \), non-negative and continuous.

The property (b) implies another two:

(c) \[ \lim_{dt \to 0} \frac{1 - P_k(t, t + dt)}{dt} = a(t); \]

(d) \[ \lim_{dt \to 0} \frac{1 - P_{k+1}(t, t + dt) - P_k(t, t + dt)}{dt} = 0. \]

It is easy to see from these formulae that \( \omega(t) \) can possess only jumps equal to 1 (signals can come only singly).

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Besides the properties (a) and (b) I assume nearly everywhere in the paper (with the exception of Theorem 1) the following third property:

(e) \[ \int_0^\infty a(t) dt = \infty. \]

This assumption means that in the whole process there will be an infinite number of signals \( \omega(t) \to \infty \) for \( t \to \infty \).

We obtain the homogeneous Poisson process from (a)-(e) setting \( a(t) = a > 0 \).

The purpose of this paper is to investigate the distances between signals in the process defined by the properties (a), (b) and (e). In section 1 I give the general definitions and theorems concerning the distributions of these distances, in section 2 I prove the limit theorem for the mean value of the fraction of distances not greater than \( y \) \((y > 0)\) among the initial \( n \) distances \((n = 1, 2, \ldots)\), and in section 3 I investigate the convergence of this mean value to a constant for \( n \to \infty \).

The results obtained can be of importance in practical investigations of the non-homogeneous Poisson processes: knowing the distribution of the distances between signals it is possible to draw conclusions concerning the function \( a(t) \). This method seems to be suitable when, for instance, the oscillation frequency of a periodic function \( a(t) \) is of the same or greater order than the mean frequency of signals.

1. The general definitions and theorems.

I call the distance between the \( k \)-th and the \((k+1)\)-th signals \((k, k+1)\) the \( k \)-th distance between signals \((k = 1, 2, \ldots)\).

Definition 1. \( L(y_1, y_2, \ldots, y_n; d_{y_1}, \ldots, d_{y_n}) \) for \( n = 1, 2, \ldots \), \( y_i \geq 0 \) and \( d_{y_i} > 0 \) \((i = 1, 2, \ldots, n)\) is the following event: in the whole process no less than \( n-1 \) signals appear and the \( i \)-th distance between signals has a value from the interval \((y_i, y_i + d_{y_i})\) \((for \ all \ i \ from \ 1 \ to \ n; \ when \ we \ write \ d_{y_i} = \infty, \ it \ means \ that \ the \ i \)-th distance is greater than \( y_i \).)

Theorem 1. If the process has the properties (a) and (b), then

\[ P\{L(y_1, \ldots, y_n; d_{y_1}, \ldots, d_{y_n})\} = \int_0^\infty \mathcal{A}(x_1) dx_1 \int_0^\infty \mathcal{A}(x_2) dx_2 \times \ldots \]

\[ \times \int_0^\infty \mathcal{A}(x_{n-1}) dx_{n-1} \int_0^\infty \mathcal{A}(x_n) dx_n - \exp\left(\int_0^\infty a(t) dt\right) dx_{n+1}. \]

Proof. In view of the properties (a) and (b) the probability that in the whole process there will be no less than \( n-1 \) signals and the \( i \)-th
signal will appear in the interval \((t_i, t_i + d t_i)\) (for all \(i\) from 1 to \(n+1\)), where \(0 < t_i < t_{i+1} < \ldots < t_{n+1} < t_{n+1} + d t_{n+1}\), is equal to

\[
Pr_1(t_1, t_2, t_3, t_4, \ldots, t_n, t_{n+1}, t_{n+1} + d t_{n+1}) \times \left[1 - Pr_1(t_{n+1}, t_{n+1} + d t_{n+1})\right]
\]

\[
= \left(\exp\left(-\int_0^{t_{n+1}} a(t) dt\right) - \exp\left(-\int_0^{t_{n+1} + d t_{n+1}} a(t) dt\right)\right) \times \left[1 - \exp\left(-\int_0^{t_{n+1} + d t_{n+1}} a(t) dt\right)\right]
\]

The function which we integrate in the last expression is therefore the conditional probability density of the appearance of the \(i\)-th signal at the point \(x_i\) (for all \(i\) from 1 to \(n+1\)), where \(0 < x_1 < x_2 < \ldots < x_{n+1}\), under the condition of the appearance in the whole process of no less than \(n+1\) signals, multiplied by the probability of this last event. Integrating this function through all the values of \(x_i\) possible in the event \(L(y_1, \ldots, y_{n+1}; d y_1, \ldots, d y_{n+1})\), i.e. \(x_1, \ldots, x_{n+1} < x_1 < x_2 < \ldots < x_{n+1} + d x_{n+1}\) (for \(i\) from \(n+1\) to 2) and \(0 < x_{n+1} < \infty\), we get for the probability of this event the expression given in Theorem 1.

DEFINITION 2. The random variable \(\theta_k(y)\) (\(k=1, 2, \ldots, y>0\)) is a variable equal to 1 if the \(k\)-th distance between signals is not greater than \(y\), equal to 0 if either the \(k\)-th distance between signals is greater than \(y\) or less than \(k-1\) signals appear in the whole process.

THEOREM 2. If the process has the properties (a), (b) and (e), then the mean value of the random variable \(\theta_k(y)\) i.e. the probability that the \(k\)-th distance between signals will not be greater than \(y\) is equal to

\[
E[\theta_k(y)] = 1 - \frac{1}{(k-1)!} \int_0^y a(x) \exp\left(-\int_0^x a(t) dt\right) dx\]

Proof. Let \(y_i = 0\) for all \(i\) from 1 to \(k-1\), \(y_k = y\) and \(d y_k = \infty\) for all \(i\) from 1 to \(k\). Then

\[
E[\theta_k(y)] = 1 - Pr[L(y_1, \ldots, y_k; d y_1, \ldots, d y_k)]
\]

\[
= 1 - \int_0^y a(x) dx \int_{x_2}^x a(x_2) dx_2 \cdots \int_{x_{k-1}}^x a(x_{k-1}) dx_{k-1} \exp\left(-\int_0^{x_{k-1}} a(t) dt\right) dx_{k-1}
\]

\[
= 1 - \int_0^y a(x) dx \int_{x_2}^x a(x_2) dx_2 \cdots \int_{x_{k-1}}^x a(x_{k-1}) dx_{k-1} \int_{x_{k-2}}^x a(x_{k-2}) dx_{k-2} \exp\left(-\int_0^{x_{k-2}} a(t) dt\right) dx_{k-2}
\]

The region defined by the inequalities

\[
0 < x_1 < \infty, \quad x_1 < x_2 < \infty, \quad \ldots, \quad x_{n-1} < x_n < \infty
\]

can also be defined by the inequalities

\[
0 < x_1 < \infty, \quad 0 < x_{n-1} < x_n, \quad \ldots, \quad 0 < x_n < \infty
\]

Therefore

\[
E[\theta_n(y)] = 1 - \frac{1}{(n-1)!} \int_0^y a(x) \exp\left(-\int_0^x a(t) dt\right) dx
\]

\[
= 1 - \frac{1}{(k-1)!} \int_0^y a(x) \exp\left(-\int_0^x a(t) dt\right) dx
\]

q. e. d.

DEFINITION 3. The random variable \(\theta_n(y)\) (\(n=1, 2, \ldots, y>0\)) is

\[
\theta_n(y) = \frac{1}{n} \sum_{i=1}^n \theta_i(y)
\]

If no less than \(n+1\) signals appeared in the whole process (as always happens in the case of a process with property (e)), then \(\theta_n(y)\) is the fraction of distances not greater than \(y\) among the first \(n\) distances.

THEOREM 3. If the process has the properties (a), (b) and (e), then

\[
E[\theta_n(y)] = 1 - \frac{1}{n} \int_0^y a(x) \exp\left(-\int_0^x a(t) dt\right) dx
\]

Proof. In view of the theorem for the mean value of the sum of random variables we obtain

\[
E[\theta_n(y)] = \frac{1}{n} \sum_{i=1}^n E[\theta_i(y)]
\]

from this formula and Theorem 2 we deduce Theorem 3.

For the homogeneous process, i.e. for \(a(t) = a > 0\), we obtain from Theorems 1, 2 and 3 respectively the known formulas:

1. \(Pr[L(y_1, \ldots, y_n; d y_1, \ldots, d y_n)] = \exp\left(-a \sum_{i=1}^n y_i \prod_{i=1}^n (1 - e^{-a y_i})\right)\)

2. \(E[\theta_i(y)] = 1 - e^{-a y_i}\)

3. \(E[\theta_n(y)] = 1 - e^{-a y_n}\)
2. The limit theorem for $E[\Theta_n(x)]$.

Definition 4. For the process with properties (a), (b) and (c) I write

$$S_n(y) = 1 - \frac{1}{n!} \int_0^y a(x) \exp \left( - \int_0^x a(t) dt \right) dx,$$

where $n = 1, 2, \ldots, y > 0$; $\tau_n$ is one of the solutions of the equation

$$\int_0^{\tau_n} a(t) dt = n.$$

If the last equation has more than one solution, then these solutions constitute an interval in which $a(t) = 0$; the choice of $\tau_n$ from this interval has therefore no significance for the value of $S_n(y)$.

It is easy to see from Definition 4 that $0 < S_n(y) < 1$.

Theorem 4. If the process has the properties (a), (b) and (c), then

$$E[\Theta_n(x)] - S_n(y) = 1,$$

we obtain

$$\frac{1}{n!} \int_0^y a(x) \exp \left( \int_0^x a(t) dt \right) dx = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!} \int_0^y a(x) \exp \left( \int_0^x a(t) dt \right)^k dx.$$

Proof. In view of Definition 4 and Theorem 3 we obtain

$$S_n(y) - E[\Theta_n(x)] = \frac{1}{n} \int_0^y a(x) \exp \left( - \frac{a(x) \tau_n}{2} \right) \sum_{k=0}^{n-1} \frac{1}{k!} \left( \int_0^x a(t) dt \right)^k dx +$$

$$+ \frac{1}{n} \int_0^y a(x) \exp \left( - \frac{a(x) \tau_n}{2} \right) \left( \exp \left( - \frac{a(x) \tau_n}{2} \right) \sum_{k=0}^{n-1} \frac{1}{k!} \left( \int_0^x a(t) dt \right)^k - 1 \right) dx.$$

For the first part of the right side of this formula we have the inequalities

$$0 < \frac{1}{n} \int_0^y a(x) \exp \left( - \frac{a(x) \tau_n}{2} \right) \sum_{k=0}^{n-1} \frac{1}{k!} \left( \int_0^x a(t) dt \right)^k dx$$

$$+ \frac{1}{n} \int_0^y a(x) \exp \left( - \frac{a(x) \tau_n}{2} \right) \sum_{k=0}^{n-1} \frac{1}{k!} \left( \int_0^x a(t) dt \right)^k dx$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{k!} \int_0^y a(x) \exp \left( - \frac{a(x) \tau_n}{2} \right) dx,$$

where

$$\psi(x) = \frac{1}{2} \int_0^x a(t) dt.$$
Since, for \( k = 0, 1, 2, \ldots \),
\[
\frac{1}{k!} \int_{k} e^{-\eta v(x)} v(x)^k \, dv(x)
\]
\[-= -e^{-\eta} \frac{v(x)}{k!} + \frac{1}{(k-1)!} \int_{k} e^{-\eta v(x)} v(x)^{k-1} \, dv(x) = \ldots = -e^{-\eta} \frac{n^n}{n!} + 1,
\]
we obtain
\[
\frac{1}{n} \sum_{k=0}^{n} \frac{1}{k!} \int_{k} e^{-\eta v(x)} v(x)^k \, dv(x) - 1 = -e^{-\eta} \frac{n^n}{n!} \sum_{k=0}^{n} \frac{k^n}{n!} = -\frac{n^n}{n!} e^{-\eta}.
\]

From (8) we deduce
\[
|E[\theta_n(y)] - S_n(y)| \leq \frac{n^n}{n!} e^{-\eta},
\]
and the inequality
\[
\frac{n^n}{n!} e^{-\eta} < \frac{1}{\sqrt{2\pi n}}
\]
completes the proof.

From Theorem 4 follows the uniform convergence of the sequence of the function \( E[\theta_n(y)] \) to the sequence of functions \( S_n(y) \) in the half-axis \( y > 0 \).

Putting \( a(t) = a > 0 \) in Definition 4 and comparing the obtained function \( S_n(y) \) with the function \( E[\theta_n(y)] \) given by formula (3), it is easy to prove that for the homogenous Poisson process \( S_n(y) = E[\theta_n(y)] \) for all \( n \) and \( y \).

3. Convergence of the sequence \( E[\theta_n(y)] \). For the homogenous process, in view of formula (3), \( E[\theta_n(y)] \) is independent of \( n \); it is, however, not true in the general case and the problem arises for which \( a(t) \) the sequence \( E[\theta_n(y)] \) converges if \( n \rightarrow \infty \).

Theorem 4 implies the following obvious

**COROLLARY 1.** If the process has the properties (a), (b) and (e), then, for each \( y > 0 \), the necessary and sufficient condition for the existence of \( \lim E[\theta_n(y)] \) is that of \( \lim S_n(y) \); if these limits exist, they are equal.

This corollary enables us to replace the investigation of the convergence of the sequence \( E[\theta_n(y)] \) by such of the convergence of the sequence \( S_n(y) \), which we shall now consider. Since we have defined \( S_n(y) \) only for the processes with properties (a), (b) and (e) we shall not assume explicitly in the following that the process has those properties.

**THEOREM 5.** If, for some \( y > 0 \) and \( C \geq 0 \), the limit
\[
\lim_{n \rightarrow \infty} S_n(y) = \text{exists in the process defined by the function } a(t+C), \text{ then this limit exists and is the same in the process defined by the function } a(t+C), \text{ for each } C \geq 0.
\]

---

\( \text{Lemma. The necessary and sufficient condition for the existence, for some } y > 0, \text{ of the limit } \lim_{n \rightarrow \infty} S_n(y) \text{ is that of the limit}
\]
\[
\lim_{n \rightarrow \infty} \left( 1 - \frac{\int a(x) \exp \left( -\int a(t) \, dt \right) \, dx}{\int a(t) \, dt} \right);
\]

\( \text{if these limits exist, they are equal.}
\]

**Proof.** Let
\[
n(x) = \left[ \int_{0}^{x} a(t) \, dt \right],
\]

for \( x \geq x_0 \), i.e. \( n(x) \geq 1 \), we have
\[
\int_{0}^{x} a(x) \exp \left( -\int_{0}^{x} a(t) \, dt \right) \, dx = n(x) \int_{0}^{x} a(x) \exp \left( -\int_{0}^{x} a(t) \, dt \right) \, dx + \int_{0}^{x} a(x) \exp \left( -\int_{0}^{x} a(t) \, dt \right) \, dx.
\]

If \( x \rightarrow \infty \), then \( n(x) \rightarrow \infty \) successively throughout all natural numbers; since
\[
\lim_{x \rightarrow \infty} \frac{n(x)}{\int_{0}^{x} a(t) \, dt} = 1, \quad \lim_{x \rightarrow \infty} \frac{\int_{0}^{x} a(x) \exp \left( -\int_{0}^{x} a(t) \, dt \right) \, dx}{\int_{0}^{x} a(t) \, dt} = 0,
\]
we deduce our Lemma from formula (9).
Proof. In virtue of the Lemma and by transformations of integrals we get
\[
\lim_{n \to \infty} S_n(y)_{y_0+C_1} = 1 - \lim_{n \to \infty} \frac{\int_{y_0}^{y_1} a(x+C) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx}{\int_{y_0}^{y_1} a(t+C) \, dt}.
\]
\[
= 1 - \lim_{n \to \infty} \frac{\int_{y_0}^{y_1} a(x+C) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx}{\int_{y_0}^{y_1} a(t+C) \, dt}.
\]
\[
= 1 - \lim_{n \to \infty} \frac{\int_{y_0}^{y_1} a(x+C) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx}{\int_{y_0}^{y_1} a(t+C) \, dt}.
\]
\[
= 1 - \lim_{n \to \infty} \frac{\int_{y_0}^{y_1} a(x+C) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx}{\int_{y_0}^{y_1} a(t+C) \, dt}.
\]
q.e.d.

Theorem 6. If, for some \( y > 0 \), \( \lim_{n \to \infty} S_n(y) \) exists in the process defined by the function \( a(t) \), then this limit exists and is the same in the process defined by the function \( a(t) + b(t) \), where \( b(t) \) is any continuous function, defined for \( t \geq 0 \), such that \( a(t) + b(t) \geq 0 \) for all \( t \geq 0 \) and the integral \( \int_{-\infty}^{\infty} |b(t)| \, dt \) is finite.

Proof. Denote by \( M_C \) and \( m_C \) respectively the upper and the lower limits of the function
\[
f(x) = \int_{x_0}^{x+C} b(t) \, dt
\]
for \( x \geq x_0 \), where \( C \geq 0 \); for any \( x \geq x_0 \), we have
\[
e^{-m_C C} \int_{x}^{x+C} a(x) \exp \left( - \int_{s}^{x+C} b(t) \, dt \right) \, dx \leq \int_{x}^{x+C} a(x) \exp \left( - \int_{s}^{x+C} b(t) \, dt \right) \, dx \leq e^{-M_C C} \int_{x}^{x+C} a(x) \exp \left( - \int_{s}^{x+C} b(t) \, dt \right) \, dx.
\]

If \( s \to \infty \), the first and the last expressions converge in virtue of the Lemma to
\[
e^{-m_C C} \left[ 1 - \lim_{n \to \infty} S_n(y)_{y_0} \right] \quad \text{and} \quad e^{-m_C C} \left[ 1 - \lim_{n \to \infty} S_n(y)_{y_0} \right]
\]
respectively. Since \( C \) is arbitrary and, as follows from the existence of the integral \( \int_{0}^{\infty} |b(t)| \, dt \), \( \lim_{C \to \infty} M_C = \lim_{C \to \infty} m_C = 0 \), we obtain
\[
1 - \lim_{n \to \infty} S_n(y)_{y_0} = \lim_{n \to \infty} \int_{0}^{\infty} a(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx
\]
(10)
\[
\int_{0}^{\infty} a(t+C) \, dt
\]
\[
= 1 - \lim_{n \to \infty} S_n(y)_{y_0} = \lim_{n \to \infty} \int_{0}^{\infty} a(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx
\]
\[
\int_{0}^{\infty} a(t+C) \, dt
\]
\[
= 1 - \lim_{n \to \infty} S_n(y)_{y_0} = \lim_{n \to \infty} \int_{0}^{\infty} a(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx
\]
\[
\int_{0}^{\infty} a(t+C) \, dt
\]
We also have
\[
\left| \int_{0}^{\infty} b(t) \, dt \right| \leq \int_{0}^{\infty} |b(t)| \, dt \leq \int_{0}^{\infty} |b(t)| \, dt
\]
and
\[
\left| \int_{0}^{\infty} b(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx \right| \leq \int_{0}^{\infty} |b(x)| \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx \leq \int_{0}^{\infty} |b(x)| \, dx.
\]
From (10), (11), (12) and the Lemma we deduce
\[
1 - \lim_{n \to \infty} S_n(y)_{y_0} = \lim_{n \to \infty} \int_{0}^{\infty} a(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx \int_{0}^{\infty} b(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx
\]
\[
= \lim_{n \to \infty} \int_{0}^{\infty} a(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx \int_{0}^{\infty} b(x) \exp \left( - \int_{s}^{x+C} a(t+C) \, dt \right) \, dx
\]
\[
= 1 - \lim_{n \to \infty} S_n(y)_{y_0+C_1}, \quad \text{q.e.d.}
\]

Theorem 7. If, for some \( y > 0 \), the limit
\[
\lim_{n \to \infty} S_n(y)
\]
exists in the process defined by the function \( a(t) \), then also, for each \( C > 0 \), the limit
\[
\lim_{n \to \infty} S_n(y+C)
\]
equal to the first, exists in the process defined by the function \( Ca(t) \).
Proof. In virtue of the Lemma and by transformations of integrals we obtain
\[
\lim_{n \to \infty} S_n(y)_{n0} = 1 - \lim_{n \to \infty} \int_0^\infty \exp \left( - \int_0^x a(t) \, dt \right) \, dx
\]
\[
= 1 - \lim_{n \to \infty} \int_0^\infty \frac{Ca(c_n)}{\frac{x^{p+q}}{\int \frac{g(x)}{g(y)}}} \exp \left( - \int_0^x a(t) \, dt \right) \, dx
\]
\[
= \lim_{n \to \infty} S_n(y) \int \frac{g(x)}{g(y)} \exp \left( - \int_0^x a(t) \, dt \right) \, dx
\]
q.e.d.

Theorem 8. If, for some \( y > 0 \), the limit
\[
\lim_{n \to \infty} \int_0^y a(t) \, dt = A_y
\]
exists (in particular we may have \( A_y = \infty \)), then the function \( a(t) \) has the mean value
\[
a = \lim_{n \to \infty} \frac{1}{n} \int_0^n a(t) \, dt = \frac{A_y}{y},
\]
and the limit
\[
\lim_{n \to \infty} S_n(y) = 1 - e^{-A_y}
\]
exists.

Proof. For \( \varepsilon > y \) we have
\[
\sum_{k=0}^{[\varepsilon/y]} \frac{1}{\varepsilon - \varepsilon/n} \int_{\varepsilon/n}^{\varepsilon/k+1} a(t) \, dt \leq \frac{1}{y} \int_0^\varepsilon a(t) \, dt \leq \frac{1}{y} \int_0^{y/n} a(t) \, dt.
\]
Because of the convergence of the sequence
\[
C_k = \int_{\varepsilon - \varepsilon/n}^{\varepsilon/k} a(t) \, dt \quad (k = 1, 2, \ldots)
\]
to \( A_y \), the sequence of the arithmetical means of the sequence \( C_k \) also converges to this limit; therefore, if \( \varepsilon \to \infty \), we deduce from the inequalities obtained that the limit
\[
\lim_{n \to \infty} \int_0^y a(t) \, dt = \frac{A_y}{y}
\]
exists.
exist. Therefore we obtain by virtue of the Lemma
\[
\frac{\int a(t) \exp \left( \int_{-\infty}^y a(t) \, dt \right) \, dx}{\int a(t) \, dt} = 1 - \lim_{k \to \infty} \frac{\int_0^1 a(t) \exp \left( \int_{-\infty}^y a(t) \, dt \right) \, dx}{\int_0^1 a(t) \, dt}
\]
\[
= \lim_{k \to \infty} S_k(y), \text{ q.e.d.}
\]

S. Hartman has proved (by a somewhat different method from the one employed here) the following

**Theorem 10.** If the function \(a(t)\) is, for \(t \geq 0\), equal to a certain uniformly almost periodic function, then for every \(y > 0\) the limit \(\lim_{k \to \infty} S_k(y)\) exists.

**Proof.** Denote by \(a_k(t)\) the uniformly almost periodic function which is equal to \(a(t)\) for \(t \geq 0\). Because of its being uniformly almost periodic, the function \(a_k(t)\) is bounded. Hence, there exists a number \(M > 0\), such that \(-M \leq a_k(t) \leq M\) for all \(t\). We have also
\[
\int_{-\infty}^y \frac{\int a_k(t) \, dt}{\int a(t) \, dt} = \int_{-\infty}^y a_k(t) \, dt + \frac{\int \left[ a_k(t+y) - a_k(t) \right] \, dt}{\int a(t) \, dt}.
\]

Since
\[
-My \leq \int_{-\infty}^y a_k(t) \, dt \leq My,
\]
the integral \(\int \left[ a_k(t+y) - a_k(t) \right] \, dt\) is the bounded indefinite integral of a uniformly almost periodic function, i.e. it is a uniformly almost periodic function of \(x\). Therefore \(\int_{-\infty}^y a_k(t) \, dt\) is also a uniformly almost periodic function of \(x\).

Since
\[
\exp \left( - \int_{-\infty}^y a_k(t) \, dt \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \int_{-\infty}^y a_k(t) \, dt \right)^k,
\]
with the series uniformly convergent for all \(x\) in view of the boundedness of \(\int a_k(t) \, dt\), \(\exp \left( - \int_{-\infty}^y a_k(t) \, dt \right)\) is a uniformly almost periodic function of \(x\).

In virtue of the mean value theorem for uniformly almost periodic functions, the limits
\[
\lim_{k \to \infty} \frac{1}{\int_0^1 a_k(x) \exp \left( - \int_{-\infty}^y a_k(t) \, dt \right) \, dx}
\]
and
\[
\lim_{k \to \infty} \frac{\int_{-\infty}^y a_k(t) \, dt}{\int_{-\infty}^y a_k(t+y) \, dt} > 1 - \varepsilon \tau_1
\]
for all natural \(k\). Therefore the limit
\[
\lim_{k \to \infty} \frac{\int_{-\infty}^y a_k(t) \, dt}{\int_{-\infty}^y a_k(t+y) \, dt}
\]
exists, and in view of the Lemma we deduce the existence of the limit \(\lim_{k \to \infty} S_k(y)\), q.e.d.

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