Note on relatively complete B_0-spaces *

by

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Let X be a linear space. A functional $|x|$ is called *pseudonorm* if

(1) (a) $|x+y| \leqslant |x|+|y|$, (b) $|tx| = |t|\,|x|$.

Let $I=\{x \in X : |x|=0\}$. We say that the pseudonorm x is *complete* if the space X/I [1]) is complete with the distance function $\varrho(x,y)=|x-y|$. We also say that X is *completely relative to* $|x|$.

Let $(|x|_k)$, $k=1,2,\ldots$, be a sequence of pseudonorms in X. If

(2) $|x|_k=0$ $(k=1,2,\ldots)$ implies $x=0$,

then the formula

$$\varrho(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|x-y|_k}{1+|x-y|_k}$$

defines a distance in X. The linear metric space X with the distance function thus defined is denoted in the following by $X(|x|_k)$. The space $X(|x|_k)$ is called *B_0-space* if it is complete with the distance function $\varrho(x,y)$. In other words the space $X(|x|_k)$ is a B_0-space if and only if

(3) *$x_n \in X$, $|x_n-x_m|_k \to 0$ as $n,m\to\infty$, $k=1,2,\ldots$, implies that there exists an $x_0 \in X$ such that $|x_n-x_0|_k\to 0$ $(k=1,2,\ldots)$.*

Let X be a linear space, $(|x|_k)$ and $(|x|_k^*)$ $(k=1,2,\ldots)$ being two sequences of pseudonorms in X. We say that the sequences of pseudonorms $(|x|_k)$ and $(|x|_k^*)$ are *equivalent* and we write $(|x|_k)\sim(|x|_k^*)$ if

(4) *$|x_n|_k\to 0$ $(k=1,2,\ldots)$ is equivalent to $|x_n|_k^*\to 0$ $(k=1,2,\ldots)$ for every sequence $x_n \in X$.*

In general, for a given B_0-space $X(|x|_k)$ there exist many sequences of pseudonorms $(|x|_k^*)$ such that $(|x|_k)\sim(|x|_k^*)$. The question arises [2]),

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* S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires I, II*, Studia Mathematica 10 (1948), p. 182-208; 13 (1953), p. 137-179.

[1]) If X and L are linear spaces ($L\subset X$), then the symbol X/L denotes the linear space of all equivalence classes $a=x\in X: x-a\in L$ modulo L.

[2]) This problem has been kindly suggested to us by Prof. A. Alexiewicz.

whether it is possible to choose a sequence of pseudonorms $(|x|_k^*)$ in such a way that all $|x|_k^*$ $(k=1,2,\ldots)$ are complete. The answer is negative and the counter-example is a B_0-space Z of functions analytic in the unit circle, with the pseudonorms

$$|a|_k = \sup_{z\in S_k} |a(z)|, \quad \text{where} \quad S_k = \left\{z: |z| < \frac{k-1}{k}\right\}.$$

In fact, let us suppose that $(|x|_n^*)$ is a sequence of complete pseudonorms in Z equivalent to the sequence $(|x|_k)$. We put

$$\|x\|_n = \sup(|x|_1^*, |x|_2^*, \ldots, |x|_n^*).$$

The sequence $(\|x\|_n)$ is equivalent to $(|x|_k)$, and $\|x\|_n \leqslant \|x\|_{n+1}$. There are increasing sequences of natural numbers (k_n) and (l_n) such that $k_n \leqslant l_n \leqslant k_{n+1}$, and sequences of positive numbers (K_n) and (L_n) such that $K_n|x|_{k_n} \leqslant L_n\|x\|_{l_n} \leqslant K_{n+1}|x|_{k_{n+1}}$.

All the $|x|_n$ are norms, and therefore $\|x\|_n$ are also norms. By a theorem of Banach the $\|x\|_n$ $(n=1,2,\ldots)$ are equivalent to one another. This is a contradiction because Z is not a Banach space. We have seen that there is no sequence of complete pseudonorms in Z.

DEFINITION. A B_0-space $X(|x|_k)$ is called *relatively complete* if there exists a sequence of complete pseudonorms $(|x|_k^*)$, $k=1,2,\ldots$, such that $(|x|_k)\sim(|x|_k^*)$.

The purpose of this note is to find a certain necessary and sufficient condition for a B_0-space to be relatively complete.

Remark. Let $X(|x|_k)$ be a B_0-space. Then putting

$$|x|_k^* = \sup(|x|_1, |x|_2, \ldots, |x|_k)$$

we have $|x|_k \leqslant |x|_k^* \leqslant |x|_{k+1}^*$, and $(|x|_k)\sim(|x|_k^*)$.

LEMMA 1. *If L is a linear closed subset of a B_0-space $X(|x|_k)$, $|x|_k \leqslant |x|_{k+1}$, then X/L with usual operations and pseudonorms,*

$$|\mathfrak{X}|_n = \inf_{x\in\mathfrak{X}} |x|_n, \quad \mathfrak{X}\in X/L, \tag{5}$$

is a B_0-space.

Proof. We have only to show that

(a) $|\mathfrak{X}|_n = \inf_{x\in\mathfrak{X}} |x|_n = 0$ $(n=1,2,\ldots)$ *implies* $\mathfrak{X}=L$,

(b) *X/L is complete.*

(a) Let $|\mathfrak{X}|_n=0$, $n=1,2,\ldots$ For $n=1$ there exists a sequence $x_{k,1}$ $(k=1,2,\ldots)$, $x_{k,1}\in\mathfrak{X}$, such that $|x_{k,1}|_1\to 0$. Therefore there is a number k_1 such that $k>k_1$ implies $|x_{k,1}|_1<1$.

Now we can choose inductively for every n a sequence $x_{k,n}\in\mathfrak{X}$ and an increasing sequence of numbers $k_1<k_2<\ldots<k_n$ such that $k>k_n$ implies $|x_{k,n}|<1/n$.

The sequence

$$(y_m) = (x_{k_1+1,1}, \ldots, x_{k_2,1}, x_{k_2+1,2}, \ldots, x_{k_n,n-1}, x_{k_n+1,n}, \ldots)$$

is such that $y_m\to 0$ in $X(|x|_k)$ and $y_m\in\mathfrak{X}$. Since the set $\mathfrak{X}\subset X$ is closed, it contains 0, and thus $\mathfrak{X}=L$.

(b) Let $|\mathfrak{X}_p-\mathfrak{X}_q|_n\to 0$, $p,q\to\infty$ $(n=1,2,\ldots)$.

There exists an increasing sequence k_n such that $p,q\geqslant k_n$ implies $|\mathfrak{X}_p-\mathfrak{X}_q|_n<1/2^n$. Thus we have $|\mathfrak{X}_{k_n}-\mathfrak{X}_{k_{n+1}}|<1/2^n$.

Let $x_{k_1}\in\mathfrak{X}_{k_1}$. There exist $x_1\in\mathfrak{X}_{k_1}$ and $y_1\in\mathfrak{X}_{k_2}$ such that $|x_1-y_1|_1<1/2$. We put $x_{k_2}=x_{k_1}+y_1-x_1\in\mathfrak{X}_{k_2}$. Then $|x_{k_1}-x_{k_2}|_1=|x_1-y_1|_1<1/2$.

There exists $x_2\in\mathfrak{X}_{k_2}$ and $y_2\in\mathfrak{X}_{k_3}$ such that $|x_2-y_2|<1/2^2$ and for $x_{k_3}=x_{k_2}+y_2-x_2\in\mathfrak{X}_{k_3}$ we have $|x_{k_2}-x_{k_3}|_2=|x_2-y_2|<1/2^2$.

In this way we may choose by induction a sequence $x_{k_n}\in\mathfrak{X}_{k_n}$ such that $|x_{k_n}-x_{k_{n+1}}|_n<1/2^n$.

Since the sequence of pseudonorms $|x|_n$ is monotonely increasing, $|x_n|\leqslant|x|_{n+1}$, we conclude that there is $x\in X$ such that $x_{k_n}\to x$ and consequently $\mathfrak{X}_{k_n}\to\mathfrak{X}=[x]\in X/L$ [3]). We find that $\mathfrak{X}_n\to\mathfrak{X}$ and X/L is complete.

LEMMA 2. *Let $X(|x|_k)$ be a B_0-space, (a_n) a sequence of positive real numbers, and*

$$X_{(a_n)} = \left\{x\in X: \sup_n a_n|x|_n<\infty\right\}.$$

The set $X_{(a_n)}$ is a Banach space $X_{(a_n)}(\|x\|)$ with the norm $\|x\|=\sup_n a_n|x|_n$.

Proof. It can easily be seen that $X_{(a_n)}(\|x\|)$ is a B^*-space, and thus it remains to show that $X_{(a_n)}(\|x\|)$ is complete. Let (x_p) be a Cauchy sequence in $X_{(a_n)}(\|x\|)$, i. e. $\|x_p-x_q\|\to 0$ as $p,q\to\infty$. Then $|x_p-x_q|_n\to 0$ $(n=1,2,\ldots)$. Hence there exists an $x\in X$ such that $|x_p-x|_n\to 0$, $p\to\infty$, $n=1,2,\ldots$ We have

$$|x_p-x|_k \leqslant |x_p-x_q|_k + |x_q-x|_k$$

and

$$a_k|x_p-x|_k \leqslant \|x_p-x_q\| + a_k|x_q-x|_k.$$

Given $\varepsilon>0$ there exists a K such that $p,q\geqslant K$ implies $\|x_p-x_q\|\leqslant\varepsilon/2$. There exists also a $q_k\geqslant K$ such that $a_k|x_{q_k}-x|_k<\varepsilon/2$, and thus

$$a_k|x_p-x|_k \leqslant \|x_p-x_{q_k}\| + a_k|x_{q_k}-x|_k \leqslant \varepsilon.$$

[3]) See footnote [1]) on page 267.

Hence for $p\geqslant K$ we have $\|x_p-x\|\leqslant\varepsilon$. Finally we notice that $x\in X_{(a_n)}$ because it is a difference of two elements, x_p and x_p-x, of $X_{(a_n)}$.

Let $X(|x|_k)$ be a B_0-space, $|x_k|\leqslant|x|_{k+1}$, and $(a_{n,m})$ a double sequence of positive numbers. We put

$$|x|_{n,m}=\inf\{|x+y|_n\colon |y|_m=0\},$$

$$\tilde{X}_{(a_{n,m})}=\{x\in X\colon \sup_n a_{n,m}|x|_{n,m}<\infty\quad (m=1,2,\ldots)\}. \tag{6}$$

On the linear space $\tilde{X}_{(a_{n,m})}$ we can define the pseudonorms as follows:

$$\|x\|_m=\sup_n a_{n,m}|x|_{n,m}\qquad (m=1,2,\ldots).$$

LEMMA 3. *The linear space $\tilde{X}_{(a_{n,m})}(\|x\|_m)$ is a B_0-space.*

Proof. We must show that (a) $\|x\|_m=0$ $(m=1,2,\ldots)$ implies $x=0$, (b) $\tilde{X}_{(a_{n,m})}$ is complete.

(a) It follows from $\|x\|_m=0$ $(m=1,2,\ldots)$ that $\inf\{|x+y|_n\colon |y|_m=0\}=0$ for $n,m=1,2,\ldots$ In particular, $\inf\{|x+y|_m\colon |y|_m=0\}=|x|_m=0$ $(m=1,2,\ldots)$ and it follows that $x=0$.

(b) Let $\|x_p-x_q\|\to 0$, $p,q\to\infty$, $n=1,2,\ldots$ Then $|x_p-x_q|_n\to 0$ $(n=1,2,\ldots)$. Since X is complete, there exists $x\in X$ such that $|x_p-x|_n\to 0$ $(n=1,2,\ldots)$ and $|x_p-x|_{n,k}\to 0$ as $p\to\infty$, $k,n=1,2,\ldots$ (see (6)).

We have

$$|x_p-x|_{n,k}\leqslant|x_p-x_q|_{n,k}+|x_q-x|_{n,k},$$

$$a_{n,k}|x_p-x|_{n,k}\leqslant\|x_p-x_q\|_k+a_{n,k}|x_q-x|_{n,k}.$$

Given $\varepsilon>0$, there is a K such that $p,q\geqslant K$ implies $\|x_p-x_q\|\leqslant\varepsilon/2$. There is also a $q_k=K$ such that $a_{n,k}|x_q-x|_{n,k}\leqslant\varepsilon/2$. We have

$$a_{n,k}|x_p-x|_{n,k}\leqslant\|x_p-x_{q_k}\|_k+a_{n,k}|x_q-x|_{n,k}\leqslant\varepsilon$$

for all $p\geqslant K$, and hence $\|x_p-x\|_k\leqslant\varepsilon$ for $p\geqslant K$.

Since $x_p-x\in\tilde{X}_{(a_{n,m})}$ and $x_p\in\tilde{X}_{(a_{n,m})}$, we have $x=x_p-(x_p-x)\in\tilde{X}_{(a_{n,m})}$. Thus $x_p\to x\in\tilde{X}_{(a_{n,m})}$ and so $\tilde{X}_{(a_{n,m})}$ is complete.

LEMMA 4. *Let $X(|x|_n)$ be a B_0-space, $|x|_n\leqslant|x|_{n+1}$. If there exists another sequence $(|x|_n^*)$, $|x|_n^*\leqslant|x|_{n+1}^*$, of complete pseudonorms such that $(|x|_n)\sim(|x|_n^*)$, then there exists a sequence $(a_{n,k})$ of positive numbers and an increasing sequence of natural numbers (m_k), so that*

$$\sup_n a_{n,k}|x|_{n,k}\leqslant|x|_{m_k}^*,$$

where $|x|_{n,k}$ is defined by (6).

Proof. We notice that there exist numbers m_n and $M_n>0$ such that $|x|_n\leqslant M_n|x|_{m_n}^*$.

We can choose m_n in such a way that they form an increasing sequence. Now we consider the space X/I_k, $I_k=\{x\colon |x|_k=0\}$. The elements of X/I_k are denoted by $\mathfrak{X}^k$. We introduce in X/I_k the pseudonorms

$$\|\mathfrak{X}^k\|_{n,k}=\inf\{|x|_n\colon x\in\mathfrak{X}^k\}, \tag{7}$$

$$\|\mathfrak{X}^k\|_{n,k}^*=\inf\{|x|_n^*\colon x\in\mathfrak{X}^k\}. \tag{8}$$

Each of the pseudonorms $\|\mathfrak{X}^k\|_{n,k}$ and $\|\mathfrak{X}^k\|_{m,k}^*$ is a norm on X/I_k for $n\geqslant k$ and $m\geqslant m_k$. Since X is completely relative to $|x|_m^*$ $(m=1,2,\ldots)$, the space X/I_k is complete under the norm $\|\mathfrak{X}^k\|_{m,k}^*$ for $m\geqslant m_k$.

Hence by the theorem of Banach the norms $\|\mathfrak{X}^k\|_{m,k}^*$ for $m\geqslant m_k$ are equivalent to one another, and there exists a sequence of numbers $(N_{m,k})$ such that $\|\mathfrak{X}^k\|_{m,k}^*\leqslant N_{m,k}\|\mathfrak{X}^k\|_{m_k,k}^*$ $(m\geqslant m_k)$.

Since $\|\mathfrak{X}^k\|_{n,k}\leqslant M_n\|\mathfrak{X}^k\|_{m_n,k}^*$, it follows that $\|\mathfrak{X}^k\|_{n,k}\leqslant M_nN_{m_n,k}\|\mathfrak{X}^k\|_{m_k,k}^*$ $(n\geqslant k)$, and we have

$$|x|_{n,k}\leqslant M_nN_{m_k,k}|x|_{m_k}^*\qquad (n\geqslant k),$$

where $|x|_{n,k}$ is defined by (6). Putting $a_{n,k}=1/M_nN_{m_n,k}$ for $n\geqslant k$ and $a_{1,k}=a_{2,k}=\ldots=a_{k-1,k}=1/M_k$ we have

$$\sup_n a_{n,k}|x|_{n,k}\leqslant|x|_{m_k}^*.$$

THEOREM. *A B_0-space $X(|x|_n)$ is relatively complete for $x\in X$ if and only if there exists a sequence of positive numbers $(a_{n,k})$ such that*

$$\sup_n a_{n,k}|x|_{n,k}<\infty\qquad (k=1,2,\ldots), \tag{9}$$

where $|x|_{n,k}=\inf\{|x+y|_n\colon |y|_k=0\}$.

Proof. The necessity follows directly from lemma 4. We are going to prove the sufficiency.

Let $(a_{n,k})$ be a double sequence of positive numbers such that (9) for all $x\in X$. By lemma 3 X is a B_0-space with the sequence of pseudonorms $\|x\|_k=\sup_n a_{n,k}|x|_{n,k}$.

We shall show that $X(\|x\|_k)$ is relatively complete. For fixed k_0 we consider the space

$$Y(|\mathfrak{X}|_n)=X(|x|_n)/I_{k_0},\qquad I_{k_0}=\{x\colon |x|_{k_0}=0\},$$

which is a B_0-space by lemma 1. The space $Y_{(a_{n,k_0})}$ is complete with the norm

$$\|\mathfrak{X}\|=\sup_n a_{n,k_0}|x|_{n,k}\qquad (x\in\mathfrak{X})$$

by lemma 2, and it is identical with the space

$$X(\|x\|_k)/J_{k_0}, \quad \text{where} \quad J_{k_0}=\{x:\|x\|_{k_0}=0\},$$

with the norm $\|\mathfrak{X}\|_{k_0}=\|x\|_{k_0}$ $(x\in\mathfrak{X})$.

Thus $X(\|x\|_k)$ is relatively complete.

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Sur un problème du calcul de probabilité (II)*

(Mouvement d'une molécule sur plusieures droites parallèles)

par

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Dans ce travail nous nous occupons d'une généralisation des problèmes introduits dans la partie I[1]): elle consiste à considérer le mouvement d'une molécule le long de plusieures droites parallèles, avec possibilité de saut d'une droite sur l'autre.

Dans le § 2.1 nous précisons le problème et les hypothèses nécessaires. Les deux paragraphes suivants jouent un rôle auxiliaire: dans le § 2.2 nous donnons quelques lemmes algébriques, le § 2.3 complète la note I par quelques lemmes employés dans la suite. Les §§ 2.4-2.6 contiennent les résultats principaux du travail. Le § 2.7 est un résumé de ces résultats. La numération des formules fait suite à celle de la note I.

2.1. Problèmes. $\Pi_1,\Pi_2,\ldots,\Pi_n$ sont des droites parallèles. Sur chaque droite se trouve dispersée une substance dont la masse est traitée du point de vue mathématique comme mesure. Ainsi, une mesure μ_i est supposée définie sur la droite Π_i $(i=1,2,\ldots,n)$, satisfaisant aux trois conditions: 1° elle est finie sur chaque intervalle, 2° elle est infinie sur chaque demi-droite, 3° elle est une fonction continue de l'intervalle.

Sur le système de droites $\Pi_1,\Pi_2,\ldots,\Pi_n$ se meut une molécule qui peut changer de direction de sa marche, sauter d'une droite sur l'autre (mais seulement dans une direction orthogonale aux droites), ou être arrêtée. Nous supposons qu'une molécule arrêtée ne peut plus entrer en mouvement.

Considérons un système de segments $I_1,I_2,\ldots,I_n$, où $I_i\subset\Pi_i$, qui sont des projections orthogonales l'un de l'autre. Posons $x_1=\mu_1(I_1)$, $x_2=\mu_2(I_2),\ldots,x_n=\mu_n(I_n)$. La molécule qui se meut sur le système $\Pi_1,\Pi_2,\ldots,\Pi_n$ peut entrer dans le système $I_1,I_2,\ldots,I_n$ du côté droit

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[1]) J. Mycielski et S. Paszkowski, *Sur un problème du calcul de probabilité (I) (Le mouvement d'une molécule sur une droite)*, Studia Mathematica, ce volume, p. 188-200. Dans la suite ce travail sera cité comme note I.