Linear functionals over the space of functions continuous in an open interval

by

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1. Let $C\langle a,b\rangle$ be the Banach space of functions $x(t)$ continuous in the closed interval $\langle a,b\rangle$ with the norm

$$\sup_{\langle a,b\rangle}|x(t)|$$

and $C(a,b)$ the Banach space of functions $x(t)$ continuous and bounded in the open interval (a,b) with the norm

$$\|x\|=\sup_{(a,b)}|x(t)|.$$

Let us fix two sequences $t'_n\downarrow a$ and $t''\uparrow b$ such that $t'_1<t''_1$. We write

$$\|x\|_n=\sup_{\langle t'_n,t''_n\rangle}|x(t)|,\qquad \|x\|^*=\sum_{n=1}^{\infty}\frac{1}{2^n}\|x\|_n,\qquad d(x,y)=\|x-y\|^*.$$

Then the set of all continuous functions whose absolute value does not exceed 1 in (a,b) forms, with the distance $d(x,y)$, a Saks space ([4], p. 240). We shall denote this space by $K_s(a,b)$. In these definitions of spaces $C(a,b)$ and $K_s(a,b)$ the values a and b may also be infinite.

The object of our note is to establish the general form of a linear functional over the space $K_s(a,b)$ an to investigate the convergence conditions for sequences of such functionals. The functional ξ over $K_s(a,b)$ is called *linear* if it is continuous with respect to the distance $d(x,y)$ and if for every real λ_1,λ_2 the conditions $x_1,x_2,\lambda_1x_1+\lambda_2x_2\in K_s(a,b)$ imply $\xi(\lambda_1x_1+\lambda_2x_2)=\lambda_1\xi(x_1)+\lambda_2\xi(x_2)$.

In the space $C(a,b)$ we can introduce the definition of the limit in the following way. The sequence $\{x_n\}$ will be called *(l)-convergent to x* if the functions $x_n(t)$ are uniformly bounded and $\|x_n-x\|^*\to 0$, *i. e.* $x_n(t)\to x(t)$ almost uniformly in (a,b). One can easily prove that for each linear functional ξ over $K_s(a,b)$ there exists a uniquely determined functional η over $C(a,b)$ linear with respect to the (l)-convergence and such that, for every $x\in K_s(a,b)$, $\eta(x)=\xi(x)$. We have $\|x\|^*\leqslant\|x\|$, whence the functional η is also linear over $C(a,b)$ with respect to $\|x\|$. Thus we see that for each linear functional ξ over $K_s(a,b)$ there exists a uniquely determined linear functional η over $C(a,b)$ such that $\eta(x)=\xi(x)$ for $x\in K_s(a,b)$.

For the purposes of our note we point out the difference between the variation of a function in an open interval and in a closed one.

As the *variation* $\operatorname{var}_{(a,b)} y(t)$ of the function defined in the open interval (a,b) we shall introduce the limit

$$\lim_{\varepsilon\to 0+}\ \operatorname{var}_{\langle a+\varepsilon,b-\varepsilon\rangle}\ y(t)$$

if a and b are finite and the limit

$$\lim_{\varepsilon\to 0+}\ \operatorname{var}_{\langle -1/\varepsilon,1/\varepsilon\rangle}\ y(t)$$

for $a=-\infty$, $b=\infty$. In a similar way we define variation in intervals which are infinite on one side. This definition is obviously equivalent to the following one. We take a partition π of the interval (a,b)

$$\ldots<t_{-2}<t_{-1}<t_0<t_1<t_2<\ldots\qquad (t_{-n}\to a,\ t_n\to b\ \text{as}\ n\to\infty),$$

and write

$$\operatorname{var}_{(a,b)}y(t)=\sup_{\pi}\sum_{-\infty}^{+\infty}|y(t_n)-y(t_{n-1})|.$$

For sequences of uniformly bounded functions $y_n(t)$ with uniformly bounded variations in (a,b), the well known theorem of Helly on extracting of subsequences remains true.

The variations in the open interval (a,b) and the closed one $\langle a,b\rangle$ are connected by the following formula:

$$\operatorname{var}_{\langle a,b\rangle}y(t)=\operatorname{var}_{(a,b)}y(t)+|y(a)-y(a+0)|+|y(b)-y(b-0)|,$$

a,b and $\operatorname{var}_{\langle a,b\rangle}y(t)$ being finite.

We shall denote the Stieltjes integral in the classical sense, *i. e.* in a closed interval $\langle a,b\rangle$, by

$$\int_a^b x(t)\,dy(t)$$

and the Stieltjes integral in an open interval (a,b) by

$$\int_{a+}^{b-}x(t)\,dy(t)=\lim_{\varepsilon_1,\varepsilon_2\to 0+}\int_{a+\varepsilon_1}^{b-\varepsilon_2}x(t)\,dy(t) \tag{1}$$

if a and b are finite and by

$$\int_{-\infty}^{+\infty}x(t)\,dy(t)=\lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{-1/\varepsilon_1}^{+1/\varepsilon_2}x(t)\,dy(t)$$

if they are infinite.

In a similar way we define the Stieltjes integral in intervals infinite on one side.

Assuming that $x(t)$ is continuous on the right in a and continuous on the left in b and that there exist limits $y(a+0)$, $y(b-0)$ and the integral (1), we have

$$\int_a^b x(t)\,dy(t)=\int_{a+}^{b-} x(t)\,dy(t)+x(b)[y(b)-y(b-0)]-x(a)[y(a)-y(a+0)]. \tag{2}$$

If $x(t)$ is continuous and bounded in (a,b) and $\operatorname{var}_{(a,b)} y(t)<\infty$, then there exists the integral (1) and

$$\Big|\int_{a+}^{b-} x(t)\,dy(t)\Big|\leqslant \sup_{(a,b)}|x(t)|\operatorname{var}_{(a,b)} y(t).$$

For Stieltjes integrals in an open interval the following theorem of Helly is true:

If the functions $y_n(t)$ satisfy the following conditions:

(α) $\sup_n \operatorname{var}_{(a,b)} y_n(t)<\infty$,

(β) *for every $\varepsilon>0$ there exist numbers $\delta_1,\delta_2\in(a,b)$ such that for each n*

$$\operatorname{var}_{(a,\delta_1)} y_n(t)<\varepsilon \quad \textit{and} \quad \operatorname{var}_{(\delta_2,b)} y_n(t)<\varepsilon,$$

(γ) *there exists a limit $\lim_{n\to\infty} y_n(t)=y(t)$ in (a,b),*

then the function $y(t)$ is of bounded variation in (a,b) and

$$\lim_{n\to\infty}\int_{a+}^{b-} x(t)\,dy_n(t)=\int_{a+}^{b-} x(t)\,dy(t)$$

for every function $x(t)$ continuous and bounded in (a,b). The ends a, b of the interval (a,b) may be finite or infinite.

We shall further need the following two known lemmata:

LEMMA 1. *Let us consider the class of continuous functions $x(t)$, of absolute value not exceeding 1 in (a,b), satisfying the following condition: there exists a number $\delta>0$ such that $x(t)=0$ for $t\in(a,a+\delta)\cup(b-\delta,b)$. If, for every function $x(t)$ belonging to this class, we have*

$$\int_{a+}^{b-} x(t)\,dy(t)=0,$$

the function $y(t)$ being of bounded variation in (a,b), then there exists a number c such that $y(t)=c$ for every t at which $y(t)$ is continuous on the left in (a,b).

LEMMA 2. *Let the functions $y_1(t)$ and $y_2(t)$ be of bounded variation in (a,b) and let $y_1(t)=y_2(t)$ in a set dense in (a,b). Then, for every function $x(t)$ continuous and bounded in (a,b), we have*

$$\int_{a+}^{b-} x(t)\,dy_1(t)=\int_{a+}^{b-} x(t)\,dy_2(t).$$

2. THEOREM 1. *The general form of a linear functional ξ over $K_s(a,b)$ is*

$$\xi(x)=\int_{a+}^{b-} x(t)\,dy(t),$$

with $y(t)$ satisfying the conditions:

(a) $\operatorname{var}_{(a,b)} y(t)<\infty$,

(b) *$y(t)$ is continuous on the left in (a,b),*

(c) *$y(t_0)=0$ (t_0 is a fixed point of the interval (t_1',t_1'')).*

This representation is unique, i. e. for every linear functional ξ over $K_s(a,b)$ there exists only one such function $y(t)$. The ends a, b of the interval (a,b) may be finite or infinite.

Proof. The sufficiency of conditions (a)-(c) is clear and the uniqueness follows from lemma 1.

Necessity. Let us extend the given linear functional ξ over $K_s(a,b)$ to a functional η over $C(a,b)$ linear with respect to the (l)-convergence and, for a given integer n, let ζ_n denote a linear functional over $C\langle t_n',t_n''\rangle$ such that $\zeta_n(x)=\eta(x)$ for every $x\in C(a,b)$ satisfying the condition

$$x(t)=0 \quad \text{for} \quad t\in(a,t_n'\rangle\cup\langle t_n'',b). \tag{3}$$

Then, by a theorem of Riesz, there exists a function $\bar{y}_n(t)$ of bounded variation in $\langle t_n',t_n''\rangle$, continuous on the left in (t_n',t_n''), and such that $\bar{y}_n(t_0)=0$ (t_0 fixed) and that for every $x\in C(a,b)$ satisfying condition (3) we have

$$\eta(x)=\zeta_n(x)=\int_{t_n'}^{t_n''} x(t)\,d\bar{y}_n(t)=\int_{a+}^{b-} x(t)\,d\bar{y}_n(t).$$

We put here $\bar{y}_n(t)=0$ for $t\in(a,t_n')\cup(t_n'',b)$.

Since n is an arbitrary integer, the application of lemma 1 gives $\bar{y}_n(t)=\bar{y}_{n+1}(t)$ for $t\in(t_n',t_n'')$. If we put $y(t)=\bar{y}_n(t)$ for $t\in(t_n',t_n'')$, we have

$$\eta(x)=\int_{a+}^{b-} x(t)\,dy(t) \tag{4}$$

for every $x\in C(a,b)$ satisfying condition (3).

Let us write

$$y_n(t)=\begin{cases} y(t) & \text{for} \quad t\in\langle t_n', t_n''\rangle, \\ y(t_n') & \text{for} \quad t\in(a, t_n'), \\ y(t_n'') & \text{for} \quad t\in(t_n'', b). \end{cases}$$

Now we shall prove that $\lim\limits_{n\to\infty}\operatorname*{var}\limits_{(a,b)} y_n(t)<\infty$. It suffices to notice that

$$\sup_{x\in K_s(a,b)}\Big|\int_{t_n'}^{t_n''} x(t)\,dy(t)\Big|=\operatorname*{var}_{(a,b)} y_n(t).$$

The continuity on the left of $y_n(t)$ implies the existence of a function $\bar{x}_n(t)$ in $C(a,b)$ satisfying the following conditions: there exists a number $\delta_n>0$ such that $\bar{x}_n(t)=0$ for $t\in(a,a+\delta_n)\cup(b-\delta_n,b)$, $|\bar{x}_n(t)|\leqslant 1$ and

$$|\eta(\bar{x}_n)|=\Big|\int_{a+}^{b-}\bar{x}_n(t)\,dy(t)\Big|\geqslant\operatorname*{var}_{(a,b)} y_n(t)-\frac{1}{n}.$$

Since the sequence $|\eta(\bar{x}_n)|$ is bounded,

$$\operatorname*{var}_{(a,b)} y(t)=\lim_{n\to\infty}\operatorname*{var}_{(a,b)} y_n(t)<\infty.$$

Now, given $x\in C(a,b)$, let us take the continuous function

$$x_n(t)=\begin{cases} x(t) & \text{for} \quad t\in\langle t_n', t_n''\rangle, \\ 0 & \text{for} \quad t\in(a, t_{n+1}'\rangle\cup\langle t_{n+1}'', b), \\ \text{linear in intervals } (t_{n+1}', t_n') \text{ and } (t_n'', t_{n+1}''). \end{cases}$$

Then

$$\Big|\eta(x_n)-\int_{a+}^{b-}x(t)\,dy(t)\Big|=\Big|\int_{a+}^{b-}[x_n(t)-x(t)]\,dy(t)\Big|$$
$$\leqslant 2\sup_{(a,b)}|x(t)|\,[\operatorname*{var}_{(a,t_n')} y(t)+\operatorname*{var}_{(t_n'',b)} y(t)]\to 0 \quad \text{as} \quad n\to\infty.$$

On the other hand $\|x_n-x\|^*\to 0$, whence $\xi(x_n)\to\xi(x)$. Since $\xi(x_n)=\eta(x_n)$, we have

$$\xi(x)=\int_{a+}^{b-}x(t)\,dy(t).$$

Remark 1. The connection between the general form of a linear functional over $C\langle a,b\rangle$ and $K_s(a,b)$ gives equality (2). Then, in the subspace of functions $x(t)$ in $C\langle a,b\rangle$ having the limits $x(a+0)=x(b-0)=0$, the general form of a linear functional is given by (4). However, it is easily seen that in the space $C(a,b)$ the general form of a linear functional is not given by (4). Let us consider, in $C(a,b)$, the subspace of functions $x(t)$ having the limit $x(b-0)$ and let us define the linear functional $\eta(x)=x(b-0)$ over this subspace. This functional, extended to $C(a,b)$, cannot be written in the form (4).

Remark 2. It is easy to prove that the norm of a functional in $C(a,b)$ linear with respect to the (l)-convergence, given by formula (4), is equal to

$$\|\eta\|=\operatorname*{var}_{(a,b)} y(t),$$

the function $y(t)$ satisfying the conditions (a)-(c).

Remark 3. Theorem 1 obviously implies the following necessary and sufficient condition of weak convergence of a sequence $x_n\in K_s(a,b)$ to $x\in K_s(a,b)$: $x_n(t)\to x(t)$ for each $t\in(a,b)$.

3. Now we shall prove the following theorem on the convergence of sequences of linear functionals over $K_s(a,b)$, generalizing theorem 2 in [4], p. 213:

THEOREM 2. *The sequence of linear functionals over* $K_s(a,b)$

$$\xi_n(x)=\int_{a+}^{b-}x(t)\,dy_n(t),$$

$y_n(t)$ satisfying the conditions (a)-(c) *of theorem* 1, *is convergent for every $x\in K_s(a,b)$ (hence for every $x\in C(a,b)$) if and only if*

(α) $\sup\limits_{n}\operatorname*{var}\limits_{(a,b)} y_n(t)<\infty$,

(β) *for every $\varepsilon>0$ there exist numbers $\delta_1,\delta_2\in(a,b)$ such that for each n*

$$\operatorname*{var}_{(a,\delta_1)} y_n(t)<\varepsilon \quad \textit{and} \quad \operatorname*{var}_{(\delta_2,b)} y_n(t)<\varepsilon,$$

(γ) *there exists a function $y(t)$ of bounded variation in (a,b) such that for an arbitrary subsequence $y_{n_i}(t)$ there exist an enumerable set A and a subsequence $y_{n_{i_k}}(t)$ convergent to $y(t)$ for $t\in(a,b)-A$.*

The conditions (α) (γ) *being satisfied, we have*

$$\lim_{n\to\infty}\xi_n(x)=\int_{a+}^{b-}x(t)\,dy(t). \tag{5}$$

The ends a, b of the interval (a,b) may be finite or infinite.

Proof. Sufficiency. Applying (γ) and Helly's theorem, we extract from an arbitrary subsequence $y_{n_i}(t)$ a subsequence $y_{n_{i_k}}(t)\to\bar{y}(t)$, $\bar{y}(t)=y(t)$ in $(a,b)-A$, A being enumerable. By the conditions (α), (β) we can apply Helly's theorem on Stieltjes integrals. We get

$$\xi_{n_{l_k}}(x)\to\int_{a+}^{b-}x(t)\,d\bar{y}(t);$$

hence, by lemma 2,

$$\xi_{n_{i_k}}(x)\to\int_{a+}^{b-}x(t)\,dy(t),$$

which proves the convergence and formula (5).

Necessity. The convergence of the sequence $\xi_n(x)$ for $x\in K_s(a,b)$ implies its convergence for all $x\in C(a,b)$. The convergence of the sequence of functionals $\eta_n(x)=\xi_n(x)$ over $C(a,b)$ gives the condition $\sup_n\|\eta_n\|<\infty$, *i. e.*, by remark 2 to theorem 1, the condition (α).

Now let us take an arbitrary sequence $\delta_n\in(a,b)$, $\delta_n\uparrow b$, and let us consider in $C(\delta_n,b)$ a function $x_n(t)$, $|x_n(t)|\leqslant 1$, such that

$$\Big|\int_{\delta_n+}^{b-}x_n(t)\,dy_n(t)\Big|\geqslant\operatorname*{var}_{(\delta_n,b)}y_n(t)-\frac{1}{n}.$$

Choosing $\delta_n'<\delta_n$ such that $\operatorname*{var}_{\langle\delta_n',\delta_n\rangle}y_n(t)<1/n$ and defining the continuous function

$$\bar{x}_n(t)=\begin{cases}x_n(t) & \text{for}\quad t\in(\delta_n,b),\\ 0 & \text{for}\quad t\in(a,\delta_n'),\\ \text{linear in the interval }\langle\delta_n',\delta_n\rangle,\end{cases}$$

we get

$$|\xi_n(\bar{x}_n)|=\Big|\int_{a+}^{b-}\bar{x}_n(t)\,dy_n(t)\Big|\geqslant\operatorname*{var}_{(\delta_n,b)}y_n(t)-\frac{2}{n}.$$

Since $\|\bar{x}_n\|^*\to 0$, $\xi_n(\bar{x}_n)\to 0$ ([4], p. 209, theorem B). Hence we have get the second part of condition (β). The first part may be proved in a similar way.

Now let two sequences of indices, $\{n_i\}$ and $\{m_i\}$, be given. Applying (α) and Helly's extraction-theorem we find subsequences $\{n_{i_k}\}$ and $\{m_{i_k}\}$ such that $y_{n_{i_k}}(t)\to y(t)$ and $y_{m_{i_k}}(t)\to\bar{y}(t)$ in (a,b). This and the convergence of the sequence $\xi_n(x)$ imply that for arbitrary $x\in K_s(a,b)$ we have

$$\int_{a+}^{b-}x(t)\,dy(t)=\int_{a+}^{b-}x(t)\,d\bar{y}(t).$$

Hence, by lemma 1, $\bar{y}(t)=y(t)$ at all points of the simultaneous continuity of both functions, *i. e.* at $(a,b)-A$, A being enumerable. This proves condition (γ).

Remark 1. Condition (γ) can be replaced by others. Supposing that (α) and (β) are satisfied we easily get the equivalence of (γ) to each of the following conditions ([1], p. 148-155):

(γ') the sequence $y_n(t)$ converges asymptotically in (a,b) to a function $y(t)$ of bounded variation in (a,b);

(γ'') there exists a function $y(t)$ of bounded variation in (a,b) such that

$$\lim_{n\to\infty}\int_a^b|y_n(t)-y(t)|\,dt=0;$$

(γ''') there exists a function $y(t)$ of bounded variation in (a,b) such that for each $(\alpha,\beta)\subset(a,b)$

$$\lim_{n\to\infty}\int_\alpha^\beta y_n(t)\,dt=\int_\alpha^\beta y(t)\,dt.$$

Remark 2. If the functions $y_n(t)$ are non-decreasing, condition (γ) in theorem 2 can be replaced by the following condition ([2]):

(γ^{IV}) there exists a non-decreasing function $y(t)$ such that $y_n(t)\to y(t)$ at all points of continuity of $y(t)$ in (a,b).

It is clear that then the functions $y(t)$ and $\bar{y}(t)$ in the necessity part of the proof of theorem 2 have the same points of continuity and discontinuity.

4. The problems considered above may be generalized. Instead of the Banach space $C(a,b)$ of functions continuous and bounded in an open interval (a,b) one can consider the Banach space $C(G)$ of functions $x(t)$ continuous and bounded in a certain fixed open set G with the norm $\|x\|=\sup_G|x(t)|$. Let $G=\bigcup_\nu(a_\nu,b_\nu)$, the intervals (a_ν,b_ν) being disjoint, and let us take for each integer ν two fixed sequences $t'_{n\nu}\downarrow a_\nu$ and $t''_{n\nu}\uparrow b_\nu$ such that $t'_{1\nu}<t''_{1\nu}$. Put

$$\|x\|_{n\nu}=\sup_{\langle t'_{n\nu},t''_{n\nu}\rangle}|x(t)|$$

and arrange the numbers $\|x\|_{n\nu}$ in a sequence $\|x\|_1,\|x\|_2,\dots$ Writing

$$\|x\|^*=\sum_{n=1}^{\infty}\frac{1}{2^n}\|x\|_n,$$

we can define the Saks space $K_s(G)$ in the same way as the space $K_s(a,b)$. The suitable (l)-convergence in $C(G)$ is equivalent to the uniform boundedness and almost uniform convergence of the sequence $x_n(t)$.

The general form of a linear functional over $K_s(G)$ is

$$\xi(x) = \sum_\nu \int_{a_\nu +}^{b_\nu -} x(t)\, dy_\nu(t),$$

where the functions $y_\nu(t)$ satisfy conditions (a)-(c) of theorem 1 and

$$\sum_\nu \operatorname*{var}_{(a_\nu, b_\nu)} y_\nu(t) < \infty.$$

One may also establish the convergence conditions for sequences of functionals over $K_s(G)$. We shall formulate the following theorem, constituting a generalization of theorem 1 in [4], p. 210:

Let $y_n(t)$ be a sequence of functions of bounded variation in $\langle a,b\rangle$, continuous on the left in (a,b) and continuous at a fixed point $t_0 \in (a,b)$, $y_n(t_0)=0$. Then the sequence

$$\int_a^b x(t)\, dy_n(t)$$

is convergent for each function $x(t)$ continuous and bounded in $\langle a,t_0)\cup \cup (t_0,b\rangle$ if and only if

(α) $\sup_n \operatorname*{var}_{\langle a,b\rangle} y_n(t) < \infty$,

(β) *for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each n*

$$\operatorname*{var}_{\langle t_0-\delta, t_0+\delta\rangle} y_n(t) < \varepsilon,$$

(γ) *there exists a function $y(t)$ of bounded variation in $\langle a,b\rangle$ such that, for an arbitrary subsequence $y_{n_i}(t)$, there exists an enumerable set A and a subsequence $y_{n_{i_k}}(t)$ convergent to $y(t)$ for $t \in \langle a,b\rangle - A$.*

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Le calcul opérationnel d'intervalle fini

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1. Dans un travail antérieur [2], j'ai introduit des opérateurs de Heaviside comme des fractions f/g, où f et g sont des fonctions continues dans l'intervalle $0 \leqslant t \leqslant \infty$ et la „division" est entendue comme l'opération inverse au produit de composition. J'ai montré que l'ensemble de ces opérateurs est plus riche que celui fourni par la transformation de Laplace, ce qui permet à de nouvelles applications, en particulier dans le domaine des équations à dérivées partielles (cf. [4]).

Une théorie analogue pour l'intervalle fini $0 \leqslant t \leqslant T$ est le sujet de cet article. Ce calcul embrassera donc par exemple la fonction $\{\operatorname{tg}(2t/\pi T)\}$ $(0 \leqslant t < T)$, qui n'est susceptible d'aucune interprétation dans le calcul opérationnel d'intervalle infini. Le nouveau calcul permettra d'obtenir, dans la théorie des équations à dérivées partielles, des théorèmes plus forts, comme nous le verrons dans la suite.

Le calcul opérationnel d'intervalle infini peut parfois être remplacé par la transformation de Laplace,

$$\mathfrak{L}f = \int_0^\infty e^{-st} f(t)\, dt,$$

lorsqu'on se borne à des fonctions *transformables*. Ceci est possible grâce au théorème fondamental de Borel disant que le produit de composition fg de deux fonctions f et g se transforme en produit ordinaire:

$$\mathfrak{L}(fg) = \mathfrak{L}f \cdot \mathfrak{L}g. \tag{1}$$

Il est cependant important de remarquer qu'une dualité pareille d'interprétation est impossible pour le calcul opérationnel d'intervalle fini, car la transformation finie

$$\mathfrak{L}f = \int_0^T e^{-st} f(t)\, dt$$

ne jouit plus de la propriété (1). La transformation de Laplace ne peut donc servir de base pour le calcul opérationnel d'intervalle fini, même lorsqu'on se borne à des fonctions transformables.