Toutes leurs solutions qui satisfont à l'inégalité (13) et à l'inégalité

(49) $$q(x) < 1 \quad \text{pour chaque} \quad x \geqslant 0$$

sont les suivantes:

A. *D'après* (I.1.1), *quand* $p(0) > 0$, $\lambda > 0$, $\alpha = \sqrt{\varkappa^2 - \lambda^2} > 0$,

$$p(x) = \frac{\alpha}{\alpha \cosh \alpha x - \varkappa \sinh \alpha x}, \qquad q(x) = \frac{\lambda \sinh \alpha x}{\alpha \cosh \alpha x - \varkappa \sinh \alpha x},$$

$$r(x) = 1 - \frac{\alpha + \lambda \sinh \alpha x}{\alpha \cosh \alpha x - \varkappa \sinh \alpha x}.$$

B. *D'après* (I.1.2), *quand* $p(0)>0$, $\lambda>0$, $\varkappa+\lambda=0$,

(50) $$p(x) = \frac{1}{1-\varkappa x}, \quad q(x) = -\frac{\varkappa x}{1-\varkappa x}, \quad r(x) \equiv 0.$$

C. *D'après* (I.3), *quand* $p(0)>0$, $\lambda=0$,

(51) $$p(x) = e^{\varkappa x}, \quad q(x) \equiv 0, \quad r(x) = 1 - e^{\varkappa x}.$$

D. *D'après* (II.2), *quand* $p(0)=0$,

$$p(x) \equiv 0, \quad q(x) \equiv q, \quad r(x) \equiv 1 - q.$$

Une solution directe du système (2), (46)-(48) peut être obtenue d'une manière analogue à celle du § 1.3 pour le système (2)-(9), mais notons encore que pour obtenir les fonctions p et q dans les cas A, B, C (quand $p(0)=1$) l'équation (47) seule est suffisante (moyennant (13) et (49)). Nous en dérivons l'équation différentielle

(52) $$q'(x) = \lambda p^2(x).$$

D'autre part, d'après la symétrie de $q(x+y)$ en x et y on obtient l'égalité analogue à (24)

(53) $$2\varkappa q(x) = \lambda[p^2(x) - q^2(x) - 1].$$

Les fonctions p et q s'obtiennent des équations (52) et (53). Pour obtenir r il suffit alors d'utiliser (2) ((48) résulte de (2), (46), (47)).

Note ajoutée pendant la correction. Récemment nous avons pris connaissance d'un travail de M. R. M. Redheffer, *Novel uses of functional equations*, Journal of rational mechanics and analysis 3 (1954), p. 271-279, qui traite un problème analogue. On y trouve les équations (46) et (48) de C. Ryll-Nardzewski.

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Some remarks on the existence and uniqueness of solutions of the hyperbolic equation $\dfrac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}\right)$

by

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In this paper we prove some facts concerning the partial differential equation of the hyperbolic type

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

First, we prove an existence theorem for the case where the initia data are prescribed on two intersecting characteristics. The classical proof carried out by the method of successive approximation (Kamke [4][1]), p. 402) uses in an essential way the hypothesis that the function $f(x, y, z, p, q)$ satisfies a Lipschitz condition with respect to p, q and to z. Schauder ([6], p. 56) has proved the same assuming that the function f satisfies a Hölder condition with respect to x, y, z and a Lipschitz one with respect to p and q; the proof is based on the fixed-point theorem in Banach spaces (Schauder [7]). Recently Hartman and Wintner[2]) have shown that, for the existence of a solution, it suffices to suppose that the function f is continuous, bounded, and satisfies a Lipschitz condition only with respect to p and q. Perhaps the shortest way of proving this theorem is the use of the fixed-point theorem of Schauder. We give here a proof of the theorem of Hartman and Wintner, using quite elementary and standard methods; the application in the proof of a Banach space *via* the vector-valued functions is made only for the sake of brevity and may easily be omitted. The ideas of this proof are basic for the rest of the present paper.

Next we give a proof of continuous dependence of the solutions on the initial data and on the function f. Then we prove a category-

[1]) Numbers in brackets refer to the bibliography at the end of this paper.

[2]) [3], see also [2]; both papers were unavailable for the authors.

-theorem concerning the class of those functions f for which the equation has not a unique solution. We conclude with some remarks concerning the applicability of the method of successive approximations.

1. The hyperbolic equation

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with the initial conditions $z(x,c)=\sigma(x)$, $z(a,y)=\tau(y)$ is equivalent to the integral equation

$$z(x,y) = z_0(x,y) + \int_a^x \int_c^y f\left(u, v, z(u,v), \frac{\partial z(u,v)}{\partial x}, \frac{\partial z(u,v)}{\partial y}\right) du\, dv, \tag{I}$$

where $z_0(x,y)=\sigma(x)+\tau(y)-\sigma(a)$. We shall deal only with solutions having continuous partial derivatives of the first order.

The function $f(x,y,z,p,q)$ will be supposed to be defined in

$$Q_\infty:\quad a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad -\infty < z, p, q < \infty,$$

and the following sets will be used frequently:

$$Q_k:\quad a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad |z| \leqslant k, \quad |q| \leqslant k, \quad |p| \leqslant k,$$

$$S_k:\quad a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad |p| \leqslant k, \quad |q| \leqslant k,$$

$$R:\quad a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d.$$

It will be tacitly assumed that the functions $\sigma(x)$ and $\tau(y)$ are defined in $\langle a,b\rangle$ and $\langle c,d\rangle$ respectively, that they have continuous derivatives of the first order and that $\sigma(a)=\tau(b)$.

THEOREM 1. *Let the function $f(x,y,z,p,q)$ be continuous and bounded in Q_∞:*

$$|f(x,y,z,p,q)| \leqslant M,$$

and let it satisfy a Lipschitz condition,

$$|f(x,y,z,p_1,q_1) - f(x,y,z,p_2,q_2)| \leqslant L_k(|p_1-p_2| + |q_1-q_2|),$$

in every set Q_k. Then there exists a function $z(x,y)$ in R having a continuous derivative $\partial^2 z/\partial x \partial y$ and satisfying the equation

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with the initial conditions $z(x,c)=\sigma(x)$, $z(a,y)=\tau(y)$.

Proof. Choose A such that

$$\max_{a\leqslant x\leqslant b} |\sigma(x)| + \max_{a\leqslant x\leqslant b} |\sigma'(x)| + \max_{c\leqslant y\leqslant d} |\tau(y)| + \max_{c\leqslant y\leqslant d} |\tau'(y)| \leqslant A,$$

and set $\alpha=b-a$, $\beta=d-c$, $k=A+M(\alpha+\beta+\alpha\beta)$.

We shall approximate the function $f(x,y,z,p,q)$ in R by appropriately regular functions. There exists a function $\omega(\delta)$ [3] such that

$$\lim_{\delta\to 0+} \omega(\delta) = 0$$

and

$$|f(x_1,y_1,z_1,p,q) - f(x_2,y_2,z_2,p,q)| \leqslant \omega(\max(|x_1-x_2|, |y_1-y_2|, |z_1-z_2|)) \tag{1}$$

for $|z|\leqslant k$, $|p|\leqslant k$, $|q|\leqslant k$. Let $\mathfrak{C}$ stand for the Banach space of continuous functions $\varphi=\varphi(u,v,p,q)$ defined in S_k. Then the mapping

$$t \to f(u,v,t,p,q) = F(t)$$

defines a vector-valued function from the interval Δ: $|t|\leqslant k$ to $\mathfrak{C}$. The norm in $\mathfrak{C}$ being defined as

$$\|\varphi\| = \max_{(u,v,p,q)\in S_k} |\varphi(u,v,p,q)|,$$

the inequality (1) implies that

$$\|F(t_1) - F(t_2)\| \leqslant \omega(|t_1-t_2|),$$

whence $F(t)$ is (strongly) continuous. Let $\mathfrak{K}$ denote the subset of $\mathfrak{C}$ composed of those functions $\varphi(u,v,p,q)$ which satisfy the conditions

$$|\varphi(u_1,v_1,p,q) - \varphi(u_2,v_2,p,q)| \leqslant L_k(|p_1-p_2| + |q_1-q_2|)$$

for every $(u,v)\in R$, and

$$|\varphi(u,v,p_1,q_1) - \varphi(u,v,p_2,q_2)| \leqslant \omega(\max(|u_1-u_2|, |v_1-v_2|))$$

for every (u_1,v_1,p,q), $(u_2,v_2,p,q)\in S_k$. The set $\mathfrak{K}$ is obviously convex, and the function $F(t)$ takes on values from $\mathfrak{K}$. Then there exists for every n a function $F_n(t)$ from Δ to $\mathfrak{K}$ such that $\|F_n(t)-F(t)\|\leqslant 1/n$.

$F_n(t)$ satisfies the Lipschitz condition $\|F_n(t_1)-F_n(t_2)\|\leqslant A_n|t_1-t_2|$, its modulus of continuity not exceeding three times that of $F(t)$; more precisely, $\|F_n(t_1)-F_n(t_2)\|\leqslant 3\omega(|t_1-t_2|)$.

To show this choose a $\delta>0$ such that $\omega(\delta)<1/n$, then divide the interval $|t|\leqslant k$ into equal subintervals of length less than δ. Let $t_0=$

[3]) The modulus of continuity $\omega(\delta)$ depends, of course, on k too. The constant k being fixed, it is unnecessary to point out this dependence in the notation.

$=-k<t_1<\ldots<t_s=k$ be such a partition. We define $F_n(t)$ as equal to $F(t)$ for $t=t_i$, and linear in every interval $t_{i-1}\leqslant t\leqslant t_i$, *i. e.*

$$F_n(t)=F(t_{i-1})+\frac{F(t_i)-F(t_{i-1})}{t_i-t_{i-1}}(t-t_{i-1})\quad \text{for}\quad t_{i-1}\leqslant t\leqslant t_i.$$

The function $F_n(t)$ satisfies the imposed conditions (in particular the last one follows from the fact that the linear interpolation does not increase more than three times the modulus of continuity[4])).

If $F_n(t)=f_n(u,v,t,p,q)$, then the above conditions give

$$\text{(2)}\qquad \max_{Q_k}|f_n(u,v,t,p,q)-f(u,v,t,p,q)|\leqslant 1/n,$$

$$|f_n(u,v,t_1,p,q)-f_n(u,v,t_2,p,q)|\leqslant\min\big(A_n|t_1-t_2|,\,3\omega(|t_2-t_1|)\big).$$

Since $F(t)\epsilon\mathfrak{K}$, we easily obtain, setting $B_n=A_n+L_k$,

$$|f_n(u,v,t_1,p_1,q_1)-f_n(u,v,t_2,p_2,q_2)|\leqslant B_n(|t_1-t_2|+|p_1-p_2|+|q_1-q_2|),$$

$$|f_n(u_1,v_1,t_1,p,q)-f_n(u_2,v_2,t_2,p,q)|\leqslant\omega(|u_1-u_2|)+\omega(|v_1-v_2|)+3\omega(|t_1-t_2|),$$

$$|f_n(u,v,t,p_1,q_1)-f_n(u,v,t,p_2,q_2)|\leqslant L_k(|p_1-p_2|+|q_1-q_2|).$$

Consider now the differential equation

$$\frac{\partial^2 z}{\partial x\partial y}=f_n\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial x}{\partial y}\right);$$

by a well known theorem[5]) there exists a function $z_n(x,y)$ satisfying this equation with initial conditions $z_n(x,c)=\sigma(x)$, $z_n(a,y)=\tau(y)$. Set for brevity

$$p_n(x,y)=\partial z_n(x,y)/\partial x,\quad q_n(x,y)=\partial z_n(x,y)/\partial y$$

then

$$z_n(x,y)=z_0(x,y)+\int_a^x\int_c^y f_n\big(u,v,z_n(u,v),p_n(u,v),q_n(u,v)\big)\,du\,dv,$$

$$\text{(3)}\qquad p_n(x,y)=\sigma'(x)+\int_c^y f_n\big(x,v,z_n(x,v),p_n(x,v),q_n(x,v)\big)\,dv,$$

$$q_n(x,y)=\tau'(y)+\int_a^x f_n\big(u,y,z_n(u,y),p_n(u,y),q_n(u,y)\big)\,du.$$

Obviously $|z_n(x,y)|\leqslant k$, $|p_n(x,y)|\leqslant k$, $|q_n(x,y)|\leqslant k$.

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[4]) If instead of $F_n(t)$ we would introduce the „Steklov functions"

$$F_n^*(t)=n\int_0^{1/n}F(t+u)\,du,$$

we might obtain even that the modulus of continuity would not increase at all.

[5]) Kamke [2], p. 402; we use the theorem in a slightly altered version.

We shall prove now that these functions are equicontinuous. For this purpose choose an integer l so that $l>\alpha L_k$, $l>\beta L_k$, and divide the interval R into l^2 congruent intervals Δ_{ij}:

$$a+(i-1)\frac{\alpha}{l}\leqslant x\leqslant a+i\frac{\alpha}{l},\qquad c+(j-1)\frac{\beta}{l}\leqslant y\leqslant c+j\frac{\beta}{l}.$$

It is sufficient to prove that the functions are equicontinuous with respect to every variable separately in every interval Δ_{ij}.

We prove this first for the interval Δ_{11}. It is trivial that

$$\text{(4)}\qquad \begin{aligned}|z_n(x_1,y)-z_n(x_2,y)|&\leqslant(A+M\beta)|x_1-x_2|,\\ |z_n(x,y_1)-z_n(x,y_2)|&\leqslant(A+M\alpha)|y_1-y_2|,\\ |p_n(x,y_1)-p_n(x,y_2)|&\leqslant M|y_1-y_2|,\\ |q_n(x_1,y)-q_n(x_2,y)|&\leqslant M|x_1-x_2|.\end{aligned}$$

There exists a function $\tilde\omega(\delta)$ tending to 0 as $\delta\to 0+$ and such that

$$|\sigma'(x_1)-\sigma'(x_2)|+|\tau'(y_1)-\tau'(y_2)|\leqslant\tilde\omega\big(\max(|x_1-x_2|,|y_1-y_2|)\big).$$

Now

$$\begin{aligned}&|p_n(x_1,y)-p_n(x_2,y)|\\ &\leqslant|\sigma'(x_1)-\sigma'(x_2)|+\Big|\int_y^c[f_n\big(x_1,v,z_n(x_1,v),p_n(x_1,v),q_n(x_1,v)\big)\\ &\qquad -f_n\big(x_2,v,z_n(x_2,v),p_n(x_2,v),q_n(x_2,v)\big)]\,dv\Big|\\ &\leqslant\tilde\omega(|x_1-x_2|)+\int_c^{c+\beta/l}[\omega(|x_1-x_2|)+3\omega\big(|z_n(x_1,v)-z_n(x_2,v)|\big)\\ &\qquad +L_k|p_n(x_1,v)-p_n(x_2,v)|+L_k|q_n(x_1,v)-q_n(x_2,v)|]\,dv\\ &\leqslant\tilde\omega(|x_1-x_2|)+\frac{\beta}{l}[\omega(|x_1-x_2|)+3\omega\big((A+M\beta)|x_1-x_2|\big)\\ &\qquad +L_k\max_v|p_n(x_1,v)-p_n(x_2,v)|+L_kM|x_1-x_2|],\end{aligned}$$

whence

$$\left(1-\frac{\beta}{l}L_k\right)\max_x|p_n(x_1,v)-p_n(x_2,v)|$$

$$\leqslant\tilde\omega(|x_1-x_2|)+\frac{\beta}{l}[\omega(|x_1-x_2|)+3\omega\big((A+M\beta)|x_1-x_2|\big)+L_kM|x_1-x_2|],$$

and similarly

$$\left(1-\frac{a}{l}L_k\right)\max_u |q_n(u,y_1)-q_n(u,y_2)|$$

$$\leqslant \tilde{\omega}(|y_1-y_2|)+\frac{a}{l}\left[\omega(|y_1-y_2|)+3\omega\big((A+Ma)|y_1-y_2|\big)+L_kM|y_1-y_2|\right].$$

Equicontinuity in Δ_{11} results from the fact that

$$1-\frac{a}{l}L_k>0, \quad 1-\frac{\beta}{l}L_k>0.$$

By the same argument we can now prove equicontinuity in the intervals $\Delta_{12},\Delta_{13},\ldots$ Continuing this process we prove, after l steps, equicontinuity in the strip $a\leqslant x\leqslant a+a/l$, $c\leqslant y\leqslant d$, and so on.

By a theorem of Arzelà there exists a sequence m_n such that

$$z_{m_n}(x,y)\rightrightarrows z(x,y),$$

$$p_{m_n}(x,y)\rightrightarrows p(x,y)=\frac{\partial z(x,y)}{\partial x},$$

$$q_{m_n}(x,y)\rightrightarrows q(x,y)=\frac{\partial z(x,y)}{\partial y}$$

uniformly in R; moreover $|z(x,y)|\leqslant k$, $|p(x,y)|\leqslant k$, $|q(x,y)|\leqslant k$. Now

$$f\big(x,y,z_{m_n}(x,y),p_{m_n}(x,y),q_{m_n}(x,y)\big)\rightrightarrows f\big(x,y,z(x,y),p(x,y),q(x,y)\big),$$

whence by (2)

$$f_{m_n}\big(x,y,z_{m_n}(x,y),p_{m_n}(x,y),q_{m_n}(x,y)\big)\rightrightarrows f\big(x,y,z(x,y),p(x,y),q(x,y)\big).$$

Passing to the limit in (3) with n replaced by m_n, we get

$$z(x,y)=z_0(x,y)+\int_a^x\int_c^y f\left(u,v,z(u,v),\frac{\partial z(u,v)}{\partial x},\frac{\partial z(u,v)}{\partial y}\right)du\,dv,$$

which completes the proof.

The same method enables us to prove the existence of a solution when $\sigma(x)$ and $\tau(y)$ are given on two lines parallel to the axes of coordinates and lying in the interior of R.

A similar procedure may be used to solve Cauchy's problem under the same hypotheses.

2. We shall now prove the existence of a solution of a special hyperbolic equation, namely

$$\frac{\partial^2 z}{\partial x\partial y}=f(x,y,z),$$

under weaker hypotheses.

THEOREM 2. *Let the function $f(x,y,z)$ defined for $a\leqslant x\leqslant b$, $c\leqslant y\leqslant d$, $-\infty<z<\infty$ be: 1° bounded, $|f(x,y,z)|\leqslant M$; 2° measurable in (x,y) for fixed z in a set dense in $(-\infty,\infty)$; 3° continuous in z for fixed x,y. Then there exists a continuous function $z(x,y)$ satisfying the equation*

$$\frac{\partial^2 z}{\partial x\partial y}=f(x,y,z)$$

almost everywhere, and such that $z(x,c)=\sigma(x)$, $z(a,y)=\tau(y)$.

Proof. It is sufficient to prove that there exists a continuous function satisfying the integral equation

$$z(x,y)=\sigma(x)+\tau(y)-\sigma(a)+\int_a^x\int_c^y f\big(u,v,z(u,v)\big)\,du\,dv.$$

By a theorem proved by the authors [6]) there exists a sequence of continuous functions $f_n(u,v,z)$ such that

(5) $$|f_n(u,v,z)|\leqslant M,\quad \lim_{n\to\infty}\max_{|z|\leqslant k}|f_n(u,v,z)-f(u,v,z)|=0$$

for almost any $(u,v)\in R$, where $k=A+M(a+\beta+a\beta)$ has the same meaning as above. By Theorem 1 the equation

(6) $$z_n(x,y)=z_0(x,y)+\int_a^x\int_c^y f_n\big(u,v,z_n(u,v)\big)\,du\,dv$$

has a solution $z_n(x,y)$, $|z_n(x,y)|\leqslant M$, and the functions $z_n(x,y)$ are obviously equicontinuous. By Arzelà's theorem there exists a subsequence $z_{m_n}(x,y)$ uniformly convergent in R to a continuous function $z(x,y)$. Then, by (5), $f_{m_n}\big(x,y,z_{m_n}(x,y)\big)-f\big(x,y,z_{m_n}(x,y)\big)$ converges to 0 almost everywhere in R. By Lebesgue's theorem on integration of sequences

$$\lim_{n\to\infty}\int_a^x\int_c^y \left|f_{m_n}\big(u,v,z_{m_n}(u,v)\big)-f\big(u,v,z_{m_n}(u,v)\big)\right|du\,dv=0,$$

[6]) [1], p. 415.

whence

$$\lim_{n\to\infty}\int_a^x\int_c^y f_{m_n}(u,v,z_{m_n}(u,v))\,du\,dv=\int_a^x\int_c^y f(u,v,z(u,v))\,du\,dv,$$

and by (6)

$$z(x,y)=z_0(x,y)+\int_a^x\int_c^y f(u,v,z(u,v))\,du\,dv.$$

3. It is obvious that under the conditions of Theorem 1 the solution of the differential equation is not necessarily unique. As an example may serve the equation

$$\frac{\partial^2 z}{\partial x\partial y}=9|z|^{2/3}\qquad (x\geqslant 0,\ y\geqslant 0)$$

with the initial conditions $z(x,0)=z(0,y)=0$, which has at least two solutions, $z_1(x,y)=0$, $z_2(x,y)=x^3y^3$. It is known that if $f(x,y,z,p,q)$ is supposed to satisfy the Lipschitz condition with respect to the variables z,u,v, then the uniqueness of the solution is guaranteed.

Let us denote by $z(x,y,\xi,\eta,\sigma,\tau,f)$ the solution of the equation

$$\frac{\partial^2 z}{\partial x\partial y}=f\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)$$

with the initial conditions

$$z(x,\eta)=\sigma(x),\qquad z(\xi,y)=\tau(y),\qquad (\sigma(\xi)=\tau(\eta));$$

then $z(x,y,\xi,\eta,\sigma,\tau,f)$ is an operation (in general multi-valued) defined in the space of points (ξ,η,σ,τ). We shall prove that this operation is continuous. To be precise we introduce some functional spaces. By $\mathfrak{A}$ we shall denote the space of quadruples $(\xi,\eta,\sigma,\tau)=\mathfrak{z}$ where $(\xi,\eta)\in R$, $\sigma=\sigma(x)$ and $\tau=\tau(y)$ are two functions with continuous derivatives of first order, defined respectively for $a\leqslant x\leqslant b$, $c\leqslant y\leqslant d$ and such that $\sigma(\xi)=\tau(\eta)$. If we define the distance of two elements $\mathfrak{z}_1,\mathfrak{z}_2$ as

$$\varrho(\mathfrak{z}_1,\mathfrak{z}_2)=|\xi_1-\xi_2|+|\eta_1-\eta_2|+\max_{a\leqslant x\leqslant b}|\sigma_1(x)-\sigma_2(x)|$$
$$+\max_{a\leqslant x\leqslant b}|\sigma_1'(x)-\sigma_2'(x)|+\max_{c\leqslant y\leqslant d}|\tau_1(y)-\tau_2(y)|+\max_{c\leqslant y\leqslant d}|\tau_1'(y)-\tau_2'(y)|,$$

$\mathfrak{A}$ becomes a complete metric space. $\mathfrak{B}$ will stand for the space of continuous and bounded functions $f(x,y,z,p,q)$ in Q_∞. With the norm

$$\|f\|=\sup_{(x,y,z,p,q)\in Q_\infty}|f(x,y,z,p,q)|,$$

$\mathfrak{B}$ becomes a Banach space. Finally, $\mathfrak{C}$ will denote the space of functions

$z=z(x,y)$ continuous together with the partial derivatives of the first order; the norm being defined as

$$\|z\|=\max_{(x,y)\in R}|z(x,y)|+\max_{(x,y)\in R}\left|\frac{\partial z(x,y)}{\partial x}\right|+\max_{(x,y)\in R}\left|\frac{\partial z(x,y)}{\partial y}\right|,$$

$\mathfrak{C}$ is also a Banach space.

We shall prove that $z(x,y,\xi,\eta,\sigma,\tau,f)$ is a continuous operation from $\mathfrak{A}\times\mathfrak{B}$ to $\mathfrak{C}$ at every point where $z(x,y)$ is uniquely determined.

THEOREM 3. *Let $f(x,y,z,p,q)$ satisfy the hypotheses of theorem 1 and let $\|f_n-f\|\to 0$. Let $z_n(x,y)$ be a solution in R of the equation*

$$\frac{\partial^2 z}{\partial x\partial y}=f_n\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)$$

with initial values $z(x,\eta_n)=\sigma_n(x)$, $z(\xi_n,y)=\tau_n(y)$ [7]*. Let the solution $z(x,y)$ of the equation*

$$\frac{\partial^2 z}{\partial x\partial y}=f\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)$$

with initial condition $z(x,\eta)=\sigma(x)$, $z(\xi,y)=\tau(y)$ be unique.

If $\mathfrak{z}_n=(\xi_n,\eta_n,\sigma_n,\tau_n)\to(\xi,\eta,\sigma,\tau)$, then $\|z_n-z\|\to 0$.

Proof. Since $|f(x,y,z,p,q)|\leqslant M$, we have $|f_n(x,y,z,p,q)|\leqslant 2M$ for almost all n's.

Let $A\geqslant\varrho(\mathfrak{z}_n,\mathfrak{z}_0)$ where $\mathfrak{z}_0=(a,c,0,0)$, and set $k\geqslant(A+2M)(\alpha+\beta+\alpha\beta)$. Then

$$|z_n(x,y)|\leqslant k,\qquad |p_n(x,y)|=\left|\frac{\partial z_n(x,y)}{\partial x}\right|\leqslant k,\qquad |q_n(x,y)|=\left|\frac{\partial z_n(x,y)}{\partial y}\right|\leqslant k.$$

We shall prove again that the functions $z_n(x,y)$, $p_n(x,y)$, $q_n(x,y)$ are equicontinuous. We prove this first for the interval Δ_{11}. The equicontinuity of $z_n(x,y)$ and of $p_n(x,y)$ with respect to y and of $q_n(x,y)$ with respect to x is obvious. Now let $\varepsilon>0$ be chosen arbitrarily. Then $\|f_n-f\|<\varepsilon$ for $n\geqslant N$. Approximating the function f_n by f and using a similar argument to that used in the proof of Theorem 1, we obtain in Δ_{11}

$$\max_v|p_n(x_1,v)-p_n(x_2,v)|$$
$$\leqslant|\sigma_n'(x_1)-\sigma_n'(x_2)|+\frac{\beta}{l}\left[\omega(|x_1-x_2|)+\omega\big((A+M\beta)|x_1-x_2|\big)\right.$$
$$\left.+L_k\max_v|p_n(x_1,v)-p_n(x_2,v)|+L_kM|x_1-x_2|\right]+\frac{\beta}{l}\cdot 2\varepsilon$$

[7]) Note that existence of $z_n(x,y)$ in R must be assumed explicitely, for $f\in\mathfrak{B}$ alone does not imply the existence of a solution.

for $n \geqslant N$, whence the equicontinuity of the functions $p_n(x,y)$ with respect to x follows from the equicontinuity of the functions $\sigma_n'(x)$. Similarly, $q_n(x,y)$ are equicontinuous with respect to y in Δ_{11}. Following the device of the proof of theorem 1 we can successively prove the equicontinuity in the intervals $\Delta_{12}, \Delta_{13}, \ldots$ and so on.

By Arzelà's theorem every sequence of indices m_i contains a partial one, n_i, such that

$$z_{n_i}(x,y) \rightrightarrows \bar{z}(x,y), \quad p_{n_i}(x,y) \rightrightarrows \bar{p}(x,y), \quad q_{n_i}(x,y) \rightrightarrows \bar{q}(x,y)$$

uniformly in R and

$$\bar{p}(x,y) = \frac{\partial \bar{z}(x,y)}{\partial x}, \quad \bar{q}(x,y) = \frac{\partial \bar{z}(x,y)}{\partial y}.$$

Passing to the limit in the equation

$$z_{n_i}(x,y) = \sigma_{n_i}(x) + \tau_{n_i}(y) - \sigma_{n_i}(\xi_{n_i}) + \int_{\xi_{n_i}}^{x}\int_{\eta_{n_i}}^{y} f_{n_i}\big(u,v,z_{n_i}(u,v),p_{n_i}(u,v),q_{n_i}(u,v)\big)\,du\,dv$$

and taking into account the fact that $z(x,y)$ is uniquely determined, we get $\bar{z}(x,y) = z(x,y)$. It follows easily that

$$z_n(x,y) \rightrightarrows z(x,y), \quad \frac{\partial z_n(x,y)}{\partial x} \rightrightarrows \frac{\partial z(x,y)}{\partial x}, \quad \frac{\partial z_n(x,y)}{\partial y} \rightrightarrows \frac{\partial z(x,y)}{\partial y},$$

i. e. $\|z_n - z\| \to 0$.

Now we shall prove that non-uniqueness of the solution is in some sense a rare case.

Let $\mathfrak{D}$ denote the space of continuous and bounded functions $f(x,y,z,p,q)$ in Q_∞, satisfying the conditions

$$|f(x_1,y_1,z_1,p,q) - f(x_2,y_2,z_2,p,q)| \leqslant \omega\big(\max(|x_1-x_2|,|y_1-y_2|,|z_1-z_2|)\big),$$

where $\omega(\delta) \to 0$ as $\delta \to 0+$,

$$|f(x,y,z,p_1,q_1) - f(x,y,z,p_2,q_2)| \leqslant L_k(|p_1-p_2| + |q_1-q_2|)$$

for $(x,y,z,p_1,q_1) \in Q_k$, $(x,y,z,p_2,q_2) \in Q_k$. With the distance $\varrho(f_1,f_2) = \|f_1 - f_2\|$, where

$$\|f\| = \sup_{(x,y,z,p,q)\in Q_\infty} |f(x,y,z,p,q)|,$$

$\mathfrak{D}$ becomes a complete metric space.

THEOREM 4. *The set $\mathfrak{S}$ of those $(\xi,\eta,\sigma,\tau,f) \in \mathfrak{A}\times\mathfrak{D}$ for which the equation* (I) *has at least two different solutions in R is of Baire's first category in the space $\mathfrak{A}\times\mathfrak{D}$* [8]).

Proof. Let us denote by $\Delta(x,y,\xi,\eta,\sigma,\tau,f)$ the supremum of the numbers $z_1(x,y) - z_2(x,y)$ where z_1 and z_2 are solutions of the equation (I). Let (ξ_n,η_n) denote a sequence of points of R, dense in R. Then let Ω_{MNpq} denote the set of those $(\xi,\eta,\sigma,\tau,f) \in \mathfrak{A}\times\mathfrak{D}$ for which

1° $|f(x,y,z,p,q)| \leqslant M$,

2° $\max\limits_{a\leqslant x\leqslant b}|\sigma(x)| + \max\limits_{a\leqslant x\leqslant b}|\sigma'(x)| + \max\limits_{c\leqslant y\leqslant d}|\tau(y)| + \max\limits_{c\leqslant y\leqslant d}|\tau'(y)| \leqslant N$,

3° $\Delta(\xi_p,\eta_p,\xi,\eta,\sigma,\tau,f) \geqslant 1/q$.

The sets Ω_{MNpq} are closed. Indeed, let the elements $(\xi_n,\eta_n,\sigma_n,\tau_n,f_n) \in \Omega_{MNpq}$ converge to (ξ,η,σ,τ,f). Then $\xi_n \to \xi$, $\eta_n \to \eta$ and $\sigma_n(x) \rightrightarrows \sigma(x)$, $\sigma_n'(x) \rightrightarrows \sigma'(x)$ uniformly in $\langle a,b\rangle$, $\tau_n(y) \rightrightarrows \tau(y)$, $\tau_n'(y) \rightrightarrows \tau'(y)$ uniformly in $\langle c,d\rangle$. By 3° there exist functions $z_n^{(1)}(x,y)$, $z_n^{(2)}(x,y)$ satisfying the equation

$$(7)\qquad z_n^{(i)}(x,y) = \sigma_n(x) + \tau_n(y) - \sigma_n(\xi_n) + \int_{\xi_n}^{x}\int_{\eta_n}^{y} f_n\left(u,v,z_n^{(i)}(u,v),\frac{\partial z_n^{(i)}(u,v)}{\partial x},\frac{\partial z_n^{(i)}(u,v)}{\partial y}\right)du\,dv$$

and such that

$$(8)\qquad z_n^{(1)}(\xi_p,\eta_p) - z_n^{(2)}(\xi_p,\eta_p) \geqslant \frac{1}{q} - \frac{1}{n}.$$

An identical argument to that used in the proof of theorem 1 shows that the functions $z_n^{(1)}(x,y)$ and $z_n^{(2)}(x,y)$ are equicontinuous together with the partial derivatives of the first order. By Arzelà's theorem there exists a sequence of indices such that

$$z_{n_k}^{(1)}(x,y) \rightrightarrows z^{(1)}(x,y), \quad \frac{\partial z_{n_k}^{(1)}(x,y)}{\partial x} \rightrightarrows \frac{\partial z^{(1)}(x,y)}{\partial x}, \quad \frac{\partial z_{n_k}^{(1)}(x,y)}{\partial y} \rightrightarrows \frac{\partial z^{(1)}(x,y)}{\partial y},$$

$$z_{n_k}^{(2)}(x,y) \rightrightarrows z^{(2)}(x,y), \quad \frac{\partial z_{n_k}^{(2)}(x,y)}{\partial x} \rightrightarrows \frac{\partial z^{(2)}(x,y)}{\partial x}, \quad \frac{\partial z_{n_k}^{(2)}(x,y)}{\partial y} \rightrightarrows \frac{\partial z^{(2)}(x,y)}{\partial y}.$$

Passing to the limit in (7) we see that $z^{(1)}(x,y)$ and $z^{(2)}(x,y)$ satisfy the equation

[8]) For ordinary equations this was proved in [5].

$$z^{(i)}(x,y) = \sigma(x) + \tau(y) - \sigma(\xi) + \int_{\xi}^{x}\int_{\eta}^{y} f\left(u,v,z^{(i)}(u,v), \frac{\partial z^{(i)}(u,v)}{\partial x}, \frac{\partial z^{(i)}(u,v)}{\partial y}\right) du\, dv,$$

and by (8)

$$z^{(1)}(\xi_p,\eta_p) - z^{(2)}(\xi_p,\eta_p) \geqslant 1/q,$$

whence $(\xi,\eta,\sigma,\tau,f) \in \Omega_{MNpq}$. The sets Ω_{MNpq} are non-dense. For, suppose that Ω_{MNpq} is dense in a sphere S with centre $(\xi_0,\eta_0,\sigma_0,\tau_0,f_0)$ and radius r. Then we can approximate the function f_0 with arbitrary accuracy by a function $\tilde{f}$ satisfying the Lipschitz condition with respect to z and belonging to $\mathfrak{D}$[9]). Hence we may suppose that $\tilde{f} \in S$. However, for $\tilde{f}$, the differential equation has a unique solution, and therefore $\tilde{f} \in \Omega_{MNpq}$. The theorem follows from the identity

$$\mathfrak{S} = \bigcup_{M=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcup_{p=1}^{\infty}\bigcup_{q=1}^{\infty} \Omega_{MNpq}.$$

4. The process of successive approximation is in general not successful if the function $f(x,y,z,p,q)$ satisfies the hypotheses of Theorem 1. Although the functions

$$z_0(x,y) = \sigma(x) + \tau(y) - \sigma(a),$$

. . . .

$$z_n(x,y) = z_0(x,y) + \int_{a}^{x}\int_{c}^{y} f\left(u,v,z_{n-1}(u,v), \frac{\partial z_{n-1}(u,v)}{\partial x}, \frac{\partial z_{n-1}(u,v)}{\partial y}\right) du\, dv$$

. . . .

are all well-defined, the sequence is not necessarily convergent.

THEOREM 5. *Let the function $f(x,y,z,p,q)$ satisfy the hypotheses of Theorem* 1 *and let the equation* (I) *have a unique solution. Then the following statements are equivalent*:

(a) *the sequence $z_n(x,y)$ converges in R*;

(b) *the functions $z_n(x,y)$ and their partial derivatives of the first order converge uniformly in R*;

(c) $|z_n(x,y) - z_{n+1}(x,y)| \rightrightarrows 0$ *uniformly in R*[10]);

(d) *the functions $z_n(x,y)$ converge in R to a solution of* (I).

[9]) See the footnote [4]).

[10]) This condition seems to play a role in investigation of uniqueness conditions.

Proof. 1° (a)⇒(b). Let $A, M, l, \omega(\delta)$ and $\tilde{\omega}(\delta)$ have the same meaning as in the proof of theorem 1 and write

$$p_n(x,y) = \frac{\partial z_n(x,y)}{\partial x}, \quad q_n(x,y) = \frac{\partial z_n(x,y)}{\partial y}.$$

We shall prove first that the functions $z_n(x,y)$, $p_n(x,y)$, $q_n(x,y)$ are equicontinuous. For this purpose we divide the interval R into intervals Δ_{ij} as in the proof of theorem 1 and first prove equicontinuity in Δ_{11}. Again, we prove equicontinuity in every variable separately.

The formulae (4) are valid in the case which we are considering now. Further

$$|p_n(x_1,y) - p_n(x_2,y)| \leqslant |\sigma'(x_1) - \sigma'(x_2)| + \int_{c}^{y} [f(x_1,v,z_{n-1}(x_1,v),p_{n-1}(x_1,v),q_{n-1}(x_1,v)) - f(x_2,v,z_{n-1}(x_2,v),p_{n-1}(x_2,v),q_{n-1}(x_2,v))]\, dv.$$

We shall prove that in Δ_{11}

$$\max_v |p_n(x_1,v) - p_n(x_2,v)| \leqslant \left(1 - \frac{\beta}{l} L_k\right)^{-1} \times \left\{\tilde{\omega}(|x_1-x_2|) + \frac{\beta}{l}\left[\omega(|x_1-x_2|) + \omega\big((A+M\beta)|x_1-x_2|\big) + L_k M|x_1-x_2|\right]\right\}. \tag{9}$$

For $n=0$ this is obvious. Suppose the inequality to be valid for $n-1$. If

$$\max_v |p_n(x_1,v) - p_n(x_2,v)| \leqslant \max_v |p_{n-1}(x_1,v) - p_{n-1}(x_2,v)|,$$

then (9) is obviously satisfied. If

$$\max_v |p_n(x_1,v) - p_n(x_2,v)| > \max_v |p_{n-1}(x_1,v) - p_{n-1}(x_2,v)|,$$

then

$$\begin{aligned} |p_n(x_1,y) - p_n(x_2,y)| &\leqslant \omega(|x_1-x_2|) + \int_{c}^{c+\beta/l} [\omega(|x_1-x_2|) + \omega(|z_{n-1}(x_1,v) - z_{n-1}(x_2,v)|) \\ &\quad + L_k|p_{n-1}(x_1,v) - p_{n-1}(x_2,v)| + L_k|q_{n-1}(x_1,v) - q_{n-1}(x_2,v)|]\, dv \\ &\leqslant \tilde{\omega}(|x_1-x_2|) + \frac{\beta}{l}[\omega(|x_1-x_2|) + \omega\big((A+M\beta)|x_1-x_2|\big) \\ &\quad + L_k \max_v |p_n(x_1,v) - p_n(x_2,v)| + L_k M|x_1-x_2|] \end{aligned}$$

for every y, whence (9) follows immediately. Similarly we can prove the equicontinuity of $q_n(x,y)$ with respect to y. Global equicontinuity in R may now be proved by following the procedure described in the proof of theorem 1.

By Arzelà's theorem every sequence m_i of indices contains a subsequence n_i for which the functions $z_{n_i}(x,y)$, $p_{n_i}(x,y)$, $q_{n_i}(x,y)$ converge uniformly in R. By (a) $z_{n_i}(x,y)\rightrightarrows z(x,y)$, whence also $p_{n_i}(x,y)\rightrightarrows \partial z(x,y)/\partial x$, $q_{n_i}(x,y)\rightrightarrows \partial z(x,y)/\partial y$. The sequence m_i being arbitrary, we get $z_n(x,y)\rightrightarrows z(x,y)$, $p_n(x,y)\rightrightarrows \partial z(x,y)/\partial x$, $q_n(x,y)\rightrightarrows \partial z(x,y)/\partial y$.

2° (b)$\Rightarrow$(c). Trivial.

3° (d)$\Rightarrow$(a). Trivial.

4° (b)$\Rightarrow$(d). Since we have $z_n(x,y)\rightrightarrows z(x,y)$, $\partial z_n(x,y)/\partial x\rightrightarrows \partial z(x,y)/\partial x$, $\partial z_n(x,y)/\partial y\rightrightarrows \partial z(x,y)/\partial y$ uniformly in R, (d) is obtained by passing to the limit in the formula defining the function $z_{n+1}(x,y)$.

5° (c)$\Rightarrow$(a). As shown in 1°, the functions $z_n(x,y)$, $p_n(x,y)$, $q_n(x,y)$ are equicontinuous. Hence every sequence of indices m_i contains a subsequence n_i such that

$$z_{n_i}(x,y)\rightrightarrows z(x,y),\qquad p_{n_i}(x,y)\rightrightarrows \frac{\partial z(x,y)}{\partial x},\qquad q_{n_i}(x,y)\rightrightarrows \frac{\partial z(x,y)}{\partial y}$$

uniformly in R. By (c) $z_{n_i+1}(x,y)\rightrightarrows z(x,y)$.

We shall prove that also

$$p_{n_i+1}(x,y)\rightrightarrows p(x,y)=\frac{\partial z(x,y)}{\partial x},\qquad q_{n_i+1}(x,y)\rightrightarrows q(x,y)=\frac{\partial z(x,y)}{\partial y}.$$

Indeed, suppose the contrary. Then there exists, for example, a subsequence r_i of n_i, $\varepsilon>0$, and $(x_i,y_i)\in R$ such that $|p_{r_i+1}(x_i,y_i)-p(x_i,y_i)|\geqslant\varepsilon$.

By equicontinuity r_i contains a subsequence s_i for which

$$z_{s_i+1}(x,y)\rightrightarrows \bar z(x,y),\qquad p_{s_i+1}(x,y)\rightrightarrows \frac{\partial \bar z(x,y)}{\partial x},\qquad q_{s_i+1}(x,y)\rightrightarrows \frac{\partial \bar z(x,y)}{\partial y}.$$

Since the sequence s_i is extracted from n_i, we get

$$p(x,y)=\frac{\partial \bar z(x,y)}{\partial x},$$

whence $p_{s_i+1}(x,y)\rightrightarrows p(x,y)$, which is impossible.

Passing to the limit in the equation defining $z_{n_i+1}(x,y)$, we see that $z(x,y)$ satisfies the equation (I), whence it is uniquely determined. The sequence m_i being arbitrary, this implies in turn that $z_n(x,y)\to z(x,y)$.

References

[1] A. Alexiewicz and W. Orlicz, *On a theorem of C. Carathéodory*, Annales Polonici Mathematici 1 (1955), p. 414-417.

[2] R. Conti, *Sul problema di Darboux per l'equazione* $z_{xy}=f(x,y,z,z_x,z_y)$, Annali di Univ. di Ferrara, Ser. VII (N. S.) 2 (1953), p. 129-140.

[3] Ph. Hartman and A. Wintner, *On hyperbolic partial differential equation*, American Journ. of Math. 74 (1952), p. 834-864.

[4] E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig 1930.

[5] W. Orlicz, *Zur Theorie der Differentialgleichung* $y'=f(x,y)$, Bull. de l'Acad. Polonaise des Sciences, Série A (1932), p. 221-228.

[6] J. P. Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Zeitschrift 26 (1925), p. 47-65.

[7] — *Der Fixpunktsatz in Funktionalräumen*, Studia Math. 2 (1930), p. 171-180.

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