A linear completely continuous operator is already continuous; this follows immediately from the above lemma.

A closed linear subspace $N \subset X$ is said to be an *invariant subspace* of U if $U(N) \subset N$. N is a proper invariant subspace if $(0) \neq N \neq X$.

THEOREM. *Let U be a linear completely continuous operator which maps the locally convex linear topological space X into itself. There exist proper invariant subspaces of U.*

Proof. On the basis of the above lemma we construct an auxiliary Banach space as follows.

Let $|x|$ be the pseudonorm chosen above. We divide the space X into classes and we say that x_1 *and* x_2 *belong to the same class* X if $|x_1 - x_2| = 0$. The set of all elements $x \epsilon X$ such that $|x| = 0$ constitutes the zero class. We denote by X^* the quotient space obtained, which is a linear normed space with the norm $|\mathfrak{x}| = |x|$. The transformation U defines a transformation $\mathfrak{y} = \mathfrak{U}(\mathfrak{x})$ in the space X^*, where $x \epsilon \mathfrak{x}$, $y \epsilon \mathfrak{y}$ and $y = U(x)$. It follows from the lemma that $\mathfrak{U}$ is a linear completely continuous transformation.

We denote by X' the completion of the space X^*. X' is a Banach space. The transformation $\mathfrak{U}$ can be extended to the whole space X'. It can easily be verified that the range of this extension is contained in X^*.

By the theorem of N. Aronszajn and K. T. Smith [2] there exists a proper invariant subspace $\mathfrak{N}$ of $\mathfrak{U}$. Denote by $\mathfrak{N}_1$ the intersection of the sets $\mathfrak{N}$ and X^*. $\mathfrak{N}_1 = \mathfrak{N}X^*$ is a linear set. Since $\mathfrak{U}(\mathfrak{N}) \subset X^*$ and $\mathfrak{U}(\mathfrak{N}) \subset \mathfrak{N}$, we have $\mathfrak{U}(\mathfrak{N}) \subset \mathfrak{N}_1$, hence $\mathfrak{U}(\mathfrak{N}_1) \subset \mathfrak{N}_1$. Denote by N_1 the union of all elements of all classes belonging to $\mathfrak{N}_1$, *i. e.* $x \epsilon N_1$ if there exists a class $\mathfrak{x} \epsilon \mathfrak{N}_1$ such that $x \epsilon \mathfrak{x}$. N_1 is evidently a linear set. It can immediately be verified that $U(N_1) \subset N_1$. Since the operator U is continuous we need only to take (in X) the closure N of N_1 and we find that $N = \overline{N}_1$ is a proper invariant subspace of U. This completes the proof.

References

[1] M. Altman, *On linear functional equations in locally convex linear topological spaces*, Stud. Math. 13 (1953), p. 194-207.

[2] N. Aronszajn and K. T. Smith, *Invariant subspaces of completely continuous operators*, Ann. of Math. 60 (1954), p. 345-350.

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On linear functional equations in (B_0)-spaces

by

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The well known classical Riesz-Schauder [3, 4] theory deals with a very important class of linear transformations in Banach spaces. The basic class of transformations considered in this theory consists of transformations of the form $H + U$, where H is an isomorphism onto and U is completely continuous. One of the principal properties of such a transformation is that it may be represented as the sum of an isomorphism onto and a finite-dimensional linear transformation. Hence every transformation of this class has a finite nullity (*i. e.* the space of all characteristic elements is finite-dimensional).

The following natural question arises:

Is the powerful algebraic method of F. Riesz strongly connected exclusively with the above class of transformations?

Is it possible to extend this method to a wider class of linear transformations, including, for instance, also some transformations with infinite nullity?

In the present paper an attempt is made to generalize the Riesz-Schauder theory in the above mentioned direction. The generalization given here is extended to a class of linear transformations which includes the subclass of all projections.

Let $\mathfrak{X}$ be a fixed (B_0)-space (*i. e.* a linear complete metric space with a topology determined by a sequence of pseudo-norms (see [2]).

We shall say that a *linear transformation T having its domain and range in $\mathfrak{X}$ possesses the property* (R_μ) if there exists a non-negative integer μ such that $T^{\mu+1}(x) = 0$ implies $T^\mu(x) = 0$.

Notice that if a linear transformation T possesses property (R_μ), then $T^n(x) = 0$ implies $T^\mu(x) = 0$ for $n > \mu$. We shall say that a linear transformation T is of *finite order* if it possesses property (R_μ) for some μ. The least non-negative integer μ is called the *order of the transformation T*. If T is of order μ then $T^n(x) = 0$ implies $T^\mu(x) = 0$ for $n > \mu$, and for $n < \mu$ there exists an element $x \epsilon \mathfrak{X}$ such that $T^{n+1}(x) = 0$, but $T^n(x) \neq 0$.

Consider a class of linear (*i. e.* additive and continuous) transformations T of $\mathfrak{X}$ into itself which possess the following properties:

1° the transformation T is of order μ;

2° the transformation $\overline{T}$ is of order $\bar{\mu}$, where $\overline{T}$ denotes the adjoint of T;

3° the range of the transformation T^k is closed for $k=1,2,\ldots,\bar{\mu}+1$.

In the sequel I denotes the identical mapping of $\mathfrak{X}$ and L_n the range of the transformation T^n.

THEOREM 1. *There exists a non-negative integer ν such that $L_n=L_\nu$ for $n>\nu$ and $L_{n+1}\neq L_n$ for $n<\nu$. Moreover $\nu=\mu=\bar{\mu}$, where μ and $\bar{\mu}$ are defined by properties* 1° *and* 2°.

Proof. It follows from 2° and 3° that $L_n=L_{\bar{\mu}}$ for $n>\bar{\mu}$; otherwise there would exist a linear functional f such that $f(x)=0$ for $x\in L_n$ and $f(x_0)\neq 0$ for some element x_0 of $L_{\bar{\mu}}$, *i. e.* $\overline{T}^n(f)=0$ and $\overline{T}^{\bar{\mu}}(f)\neq 0$. Since the equation $T(x)=y$ has for arbitrary $y\in L_{\bar{\mu}}$ a solution $x\in L_{\bar{\mu}}$, therefore, arguing as in the classical Riesz theory ([3], theorem 6, p. 85) we infer by 1° that there exists only one solution in $L_{\bar{\mu}}$. On the other hand, by the same Riesz' argument we conclude that the number ν coincides with that defined in property 1°.

The element $x\in X$ is called a *null-element* if it is a solution of the equation $T^\nu(x)=0$. The element y of the form $y=T^\nu(x)$, where $x\in\mathfrak{X}$, is called a *kernel-element*. The following definitions are equivalent:

The element x is said to be a *null-element* if it fulfils all equations $T^n(x)=0$ beginning with a positive integer. The element y is said to be a *kernel-element* if the equation $y=T^n(x)$ is solvable for an arbitrary positive integer.

G_ν denotes the set of all null-elements of T. L_ν denotes the set of all kernel-elements of T.

THEOREM 2. *Every element of $\mathfrak{X}$ can be represented as the sum of a null-element and a kernel-element in only one manner.*

Proof. This results from theorem 1 and from 1° in exactly the same way as in the case of the classical Riesz theory ([3], theorem 8, p. 87)

THEOREM 3. *There exists a unique linear transformation $T^{(0)}$, which maps every kernel-element into itself and every null-element into* 0. *$T^{(0)}$ maps every element of $\mathfrak{X}$ into a kernel-element and $I-T^{(0)}$ maps every element into a null-element. Moreover, $T^{(0)2}=T^{(0)}$ and $TT^{(0)}=T^{(0)}T$.*

Proof. For every element $x\in\mathfrak{X}$ we have, by theorem 2, $x=x_1+(x-x_1)$, where x_1 is a kernel-element and $x-x_1$ is a null-element.

Denote by $T^{(0)}$ the mapping $x\to x_1$. It is obvious that $T^{(0)}(x)=x$, if x is a kernel-element, $T^{(0)}(x)=0$ if x is a null-element and, for every $x\in X$, $T^{(0)}(x)$ is a kernel-element and $(I-T^{(0)})X$ is a null-element. Since the linear manifolds G_ν and L_ν are closed, it follows from theorem 2 that the space X is a direct sum of the subspaces G_ν and L_ν. By a theorem of Banach ([1], theorem 7, p. 41) the transformation $T^{(0)}$ is linear. By theorem 2 it is sufficient to prove that $TT^{(0)}=T^{(0)}T$ for the null-elements and for the kernel-elements. The transformation T maps every kernel-element into a kernel-element and every null-element into a null-element, hence $TT^{(0)}(x)=T(x)$ and $T^{(0)}T(x)=T(x)$ if x is a kernel-element. If x is a null-element, then $T^{(0)}T(x)=0$ and $TT^{(0)}(x)=0$.

THEOREM 4. *The transformation $I-T$ can be decomposed in one and only one way into two components,*

$$I-T=B_1+B_2,$$

where: 1° *B_1 is a linear transformation which maps all null-elements into* 0 *and B_2 maps all kernel-elements into* 0; 2° *B_1 coincides with $I-T$ for kernel-elements and B_2 coincides with $I-T$ for null-elements*; 3° *for any x, $B_1(x)$ is a kernel-element and $B_2(x)$ a null-element; B_1 and B_2 are orthogonal, i. e. $B_1B_2=B_2B_1=0$.*

Proof. Putting

$$B_1=T^{(0)}(I-T) \quad \text{and} \quad B_2=(I-T^{(0)})(I-T)$$

we obviously get $B_1+B_2=I-T$. By theorem 3 we have $T^{(0)}(I-T)=(I-T)T^{(0)}$ and $(I-T^{(0)})(I-T)=(I-T)(I-T^{(0)})$. Since $T^{(0)}$ maps every null-element into 0, B_1 has the same property. The transformations I and $T^{(0)}$ map the kernel-elements into themselves, hence B_2 maps every kernel-element into 0. Evidently, B_1 coincides with $I-T$ for kernel-elements and B_2 with $I-T$ for null-elements. Property 3° results from theorem 3.

THEOREM 5. *The transformation $T_1=I-B_1$ has an inverse, i. e. there exists a transformation T_1^{-1} defined on $\mathfrak{X}$ such that $T_1T_1^{-1}=T_1^{-1}T_1=I$. The equations $T^n(x)=0$ and $D^n(x)=0$, where $D=I-B_2$, have the same solutions. The equations $T^n(x)=y$ and $D^n(x)=y$ with the same right-hand sides either both have solutions or both have none.*

Proof. First of all we shall show that T_1 possesses a continuous inverse. By definition we have

$$T_1=I-B_1=I-(I-T)T^{(0)}.$$

Suppose that x is a solution of the equation $T_1(x)=0$. Then we have $(I-T^{(0)})x=-TT^{(0)}(x)$. Since, by theorem 3, $(I-T^{(0)})x$ is a null-element, we have $T^\nu(I-T^{(0)})x=0$, whence $T^{\nu+1}T^{(0)}(x)=0$, and, consequently, we infer that $T^{(0)}(x)$ is a null-element, which is possible if and only if

$T^{(0)}(x)=0$, since $T^{(0)}(x)$ is a kernel-element. Consequently, we have $x=T^{(0)}(x)-TT^{(0)}(x)=0$. Thus it is proved that the mapping T_1 is one-to-one.

We shall show that the range of T_1 is the whole of $\mathfrak{X}$. Let x be an arbitrary element of $\mathfrak{X}$. Since $T^{(0)}(x)$ is a kernel-element, there exists an element y satisfying the equation $T^{\nu+1}(y)=T^{(0)}(x)$.

Put $z=(I-T^{(0)})x+T^{\nu}(y)$. Then we have the equality

$$x=(I-T^{(0)})z+TT^{(0)}(z)=T_1(z).$$

In fact, $(I-T^{(0)})z=(I-T^{(0)})[(I-T^{(0)})x+T^{\nu}(y)]=(I-T^{(0)})^2x=(I-T^{(0)})x$.

On the other hand, by theorem 3, we get

$$TT^{(0)}(z)=TT^{(0)}[(I-T^{(0)})x+T^{\nu}(y)]=TT^{(0)}T^{\nu}(y)=TT^{\nu}(y)=T^{(0)}(x).$$

The assertion concerning D results from the following identities:

(1) $$T^n=T_1^nD^n=D^nT_1^n,$$

(2) $$D^n=(T_1^{-1})^nT^n.$$

Identity (1) follows from the identity

$$T=I-(B_1+B_2)=I-B_1-B_2+B_1B_2=(I-B_1)(I-B_2)=T_1D$$

by theorem 4.

Identity (2) is obtained by multiplying identity (1) by $(T_1^{-1})^n$.

THEOREM 6. *The transformation T can be represented as the sum of two linear transformations,*

$$T=H+K,$$

where H possesses a continuous inverse and the range of K is the space G_ν of all null-elements of T.

Proof. It follows immediately from theorems 3-5, if we set $H=T_1=I-T^{(0)}(I-T)$ and $K=-B_2=-(I-T^{(0)})(I-T)$. If $x\epsilon G_\nu$, *i. e.* $T^{\nu}x=0$, then

$$x=(I-T^{(0)})x=(I-T^{(0)})(I-T^{\nu})x=B_2(I+T+\ldots+T^{\nu-1})x.$$

Remark 1. If $T=I-U$, where U is completely continuous, then, as we know, G_ν is finite-dimensional, whence K is finite-dimensional and T may be represented as the sum of linear operators

$$I-U=H+K,$$

where transformation H possesses an inverse and K is finite-dimensional.

This representation theorem plays a fundamental role in the Riesz-Schauder theory.

Remark 2. In the case where $\mathfrak{X}$ is a locally convex linear topological space all the above theorems remain true, provided that the continuity of the transformations $T^{(0)}, B_1, B_2, D, H$ and K being, however, not guaranteed.

Notice that the "alternative of Fredholm" is also true for transformation T. This follows from. theorem 3.

Since the range of T is closed, it follows that transformation T is normally solvable, *i. e.* the equation $y=T(x)$ has a solution if and only if $f(y)=0$, for any linear functional f such that $\overline{T}(f)=0$.

Finally let us observe that the *linear equation* $\overline{T}(f)=f_0$ *possesses a solution if and only if*

(3) $$f_0(x)=0,$$

whenever x is a solution of the equation $T(x)=0$.

The necessity of the condition is obvious. If condition (3) is fulfilled, we can define a linear functional f as follows: if $y=T(x)$ we set $f(y)=f_0(x)$. By condition (3) the linear functional f is well defined on the range of T, but can be extended to the whole of $\mathfrak{X}$.

It remains to prove that the functional f defined above is linear on the range of T. But this follows from a theorem of Banach ([1], theorem 4, p. 40).

In fact, if $y_n\to y$ as $n\to\infty$, where $y=T(x)$, then there exists a sequence $\{x_n\}\subset\mathfrak{X}$ such that $x_n\to x$ as $n\to\infty$ and $y_n=T(x_n)$, hence $f(y_n)\to f(y)$ as $n\to\infty$.

Example. A very simple example which fulfils the above theory is given by an arbitrary projection P defined on X. In this case we have, evidently, $\mu=\bar{\mu}=\nu=1$. The representation theorem (Theorem 6) assumes here the very simple form

$$P=I-(I-P),$$

where $H=I$ and $K=-(I-P)$. It is clear that the range of $I-P$ coincides with the null-spaces of P.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (I)*, Stud. Math. 10 (1948), p. 184-208.

[3] F. Riesz, *Über lineare Funktionalgleichungen*, Acta Math. 41 (1918), p. 71-98.

[4] J. Schauder, *Über lineare, vollstetige Funktionaloperationen*, Stud. Math. 2 (1930), p. 183-196.

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