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Invariant subspaces of completely continuous operators in locally convex linear topological spaces

by

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Recently N. Aronszajn and K. T. Smith [2] have proved the following

THEOREM. *Let X be a Banach space and U a linear completely continuous operator in X. There exist proper invariant subspaces of U.*

The purpose of this note is to prove that any linear completely continuous operator in a locally convex linear topological space possesses the same property.

The proof of this statement is based on the above theorem and on a method developed in the paper [1].

Let X be a *locally convex linear topological space*, *i. e.* a linear space on which a topology is imposed in such a fashion that the postulated operations of addition and multiplication by real numbers are continuous in the topology. The *local convexity* means that for every neighbourhood $\mathfrak{U}_x$ of the element $x \in X$ there exists a convex neighbourhood $\mathfrak{V}_x$ such that $\mathfrak{V}_x \subset \mathfrak{U}_x$.

A linear operator U having its domain and range in X is said to be *completely continuous* if there exists a neighbourhood $\mathfrak{V}$ of 0 such that the image $U(\mathfrak{V})$ is compact in the sense that every infinite subset has a *cluster point* (*i. e.* non-isolated point).

It is known (for references see [1]) that the space X is isomorphic to a certain (B_a)-space in which a class of pseudonorms $|x|_\vartheta$, where $\vartheta \in \Phi$, is defined (Φ is an abstract set and $\overline{\overline{\Phi}} = \aleph_a$). The system of neighbourhoods of zero consists of the sets of all elements x such that $|x|_{\vartheta_i} < \varepsilon$ $(i=1,2,\ldots,k)$, $\varepsilon > 0$.

LEMMA. *Let U be a linear completely continuous operator. Then there exists a pseudonorm $|x| = \sup_i |x|_{\vartheta_i}$ $(i=1,2,\ldots,n)$ such that for every $\vartheta \in \Phi$ there is a number M_ϑ satisfying the inequality $|U(x)|_\vartheta \leqslant M_\vartheta |x|$* [1]).

[1]) For the proof see [2], p. 196.

A linear completely continuous operator is already continuous; this follows immediately from the above lemma.

A closed linear subspace $N\subset X$ is said to be an *invariant subspace* of U if $U(N)\subset N$. N is a proper invariant subspace if $(0)\neq N\neq X$.

THEOREM. *Let U be a linear completely continuous operator which maps the locally convex linear topological space X into itself. There exist proper invariant subspaces of U.*

Proof. On the basis of the above lemma we construct an auxiliary Banach space as follows.

Let $|x|$ be the pseudonorm chosen above. We divide the space X into classes and we say that x_1 *and* x_2 *belong to the same class* X if $|x_1-x_2|=0$. The set of all elements $x\epsilon X$ such that $|x|=0$ constitutes the zero class. We denote by X^* the quotient space obtained, which is a linear normed space with the norm $|\mathfrak{x}|=|x|$. The transformation U defines a transformation $\mathfrak{y}=\mathfrak{U}(\mathfrak{x})$ in the space X^*, where $x\epsilon\mathfrak{x}$, $y\epsilon\mathfrak{y}$ and $y=U(x)$. It follows from the lemma that $\mathfrak{U}$ is a linear completely continuous transformation.

We denote by X' the completion of the space X^*. X' is a Banach space. The transformation $\mathfrak{U}$ can be extended to the whole space X'. It can easily be verified that the range of this extension is contained in X^*.

By the theorem of N. Aronszajn and K. T. Smith [2] there exists a proper invariant subspace $\mathfrak{N}$ of $\mathfrak{U}$. Denote by $\mathfrak{N}_1$ the intersection of the sets $\mathfrak{N}$ and X^*. $\mathfrak{N}_1=\mathfrak{N}X^*$ is a linear set. Since $\mathfrak{U}(\mathfrak{N})\subset X^*$ and $\mathfrak{U}(\mathfrak{N})\subset\mathfrak{N}$, we have $\mathfrak{U}(\mathfrak{N})\subset\mathfrak{N}_1$, hence $\mathfrak{U}(\mathfrak{N}_1)\subset\mathfrak{N}_1$. Denote by N_1 the union of all elements of all classes belonging to $\mathfrak{N}_1$, *i. e.* $x\epsilon N_1$ if there exists a class $\mathfrak{x}\epsilon\mathfrak{N}_1$ such that $x\epsilon\mathfrak{x}$. N_1 is evidently a linear set. It can immediately be verified that $U(N_1)\subset N_1$. Since the operator U is continuous we need only to take (in X) the closure N of N_1 and we find that $N=\overline{N}_1$ is a proper invariant subspace of U. This completes the proof.

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On linear functional equations in (B_0)-spaces

by

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The well known classical Riesz-Schauder [3, 4] theory deals with a very important class of linear transformations in Banach spaces. The basic class of transformations considered in this theory consists of transformations of the form $H+U$, where H is an isomorphism onto and U is completely continuous. One of the principal properties of such a transformation is that it may be represented as the sum of an isomorphism onto and a finite-dimensional linear transformation. Hence every transformation of this class has a finite nullity (*i. e.* the space of all characteristic elements is finite-dimensional).

The following natural question arises:

Is the powerful algebraic method of F. Riesz strongly connected exclusively with the above class of transformations?

Is it possible to extend this method to a wider class of linear transformations, including, for instance, also some transformations with infinite nullity?

In the present paper an attempt is made to generalize the Riesz-Schauder theory in the above mentioned direction. The generalization given here is extended to a class of linear transformations which includes the subclass of all projections.

Let $\mathfrak{X}$ be a fixed (B_0)-space (*i. e.* a linear complete metric space with a topology determined by a sequence of pseudo-norms (see [2]).

We shall say that a *linear transformation T having its domain and range in $\mathfrak{X}$ possesses the property* (R_μ) if there exists a non-negative integer μ such that $T^{\mu+1}(x)=0$ implies $T^{\mu}(x)=0$.

Notice that if a linear transformation T possesses property (R_μ), then $T^n(x)=0$ implies $T^\mu(x)=0$ for $n>\mu$. We shall say that a linear transformation T is of *finite order* if it possesses property (R_μ) for some μ. The least non-negative integer μ is called the *order of the transformation* T. If T is of order μ then $T^n(x)=0$ implies $T^\mu(x)=0$ for $n>\mu$, and for $n<\mu$ there exists an element $x\epsilon\mathfrak{X}$ such that $T^{n+1}(x)=0$, but $T^n(x)\neq 0$.