Invariant subspaces of completely continuous operators in locally convex linear topological spaces

by

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Recently N. Aronszajn and K. T. Smith [2] have proved the following

THEOREM: Let $X$ be a Banach space and $U$ a linear completely continuous operator in $X$. There exist proper invariant subspaces of $U$.

The purpose of this note is to prove that any linear completely continuous operator in a locally convex linear topological space possesses the same property.

The proof of this statement is based on the above theorem and on a method developed in the paper [1].

Let $X$ be a locally convex linear topological space, i.e., a linear space on which a topology is imposed in such a fashion that the postulated operations of addition and multiplication by real numbers are continuous in the topology. The local convexity means that for every neighbourhood $U_i$ of the element $x \in X$ there exists a convex neighbourhood $\mathfrak{H}_i$ such that $\mathfrak{H}_i \subseteq U_i$.

A linear operator $U$ having its domain and range in $X$ is said to be completely continuous if there exists a neighbourhood $\mathcal{U}$ of 0 such that the image $U(\mathcal{U})$ is compact in the sense that every infinite subset has a cluster point (i.e., non-isolated point).

It is known (for references see [1]) that the space $X$ is isomorphic to a certain $\mathfrak{F}$-space in which a class of pseudonorms $p|_i$, where $\varnothing \subseteq \Phi$, is defined ($\Phi$ is an abstract set and $\mathcal{F} = \mathcal{K}$). The system of neighbourhoods of zero consists of the sets of all elements $x$ such that $|x|_i < \varepsilon (i = 1, 2, \ldots, k)$, $\varepsilon > 0$.

LEMMA. Let $U$ be a linear completely continuous operator. Then there exists a pseudonorm $|x|_i = \sup |x|_i$ ($i = 1, 2, \ldots, m$) such that for every $\varnothing \subseteq \Phi$ there is a number $M_\varnothing$ satisfying the inequality $|U(x)|_\varnothing \leq M_\varnothing |x|$.

1) For the proof see [2], p. 198.

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A linear completely continuous operator is already continuous; this follows immediately from the above lemma.

A closed linear subspace \( N \subseteq X \) is said to be an invariant subspace of \( U \) if \( U(N) \subseteq N \). \( N \) is a proper invariant subspace if \( (0) \neq N \neq X \).

**Theorem.** Let \( U \) be a linear completely continuous operator which maps the locally convex linear topological space \( X \) into itself. There exist proper invariant subspaces of \( U \).

**Proof.** On the basis of the above lemma we construct an auxiliary Banach space as follows.

Let \( \| \cdot \| \) be the pseudonorm chosen above. We divide the space \( X \) into classes and we say that \( x_1 \) and \( x_2 \) belong to the same class \( X \) if \( \| x_1 - x_2 \| = 0 \).

The set of all elements \( x \in X \) such that \( \| x \| = 0 \) constitutes the zero class. We denote by \( X^* \) the quotient space obtained, which is a linear normed space with the norm \( \| x \| = \| x \| \). The transformation \( U \) defines a transformation \( y = U(x) \) in the space \( X^* \), where \( x \in X \). \( y = U(x) \) if it follows from the lemma that \( U \) is a linear completely continuous transformation.

We denote by \( X' \) the completion of the space \( X^* \). \( X' \) is a Banach space. The transformation \( U \) can be extended to the whole space \( X' \). It can easily be verified that the range of this extension is contained in \( X' \).

By the theorem of N. Aronszajn and K.T. Smith [2] there exists a proper invariant subspace \( N \) of \( U \). Denote by \( X \) the intersection of the sets \( X \) and \( X' \). Since \( U(X') \subseteq X \) and \( U(X) \subseteq \text{R} \), we have \( U(X) \subseteq \text{R} \). Hence \( U(X) \subseteq \text{R} \). Denote by \( X \) the union of all elements of all classes belonging to \( X \), i.e. \( x \in X \), if there exists a class \( x \in X \) such that \( x \in X \). \( X \) is evidently a linear set. It can immediately be verified that \( U(X) \subseteq X \). Since the operator \( U \) is continuous we need only to take (in \( X \)) the closure \( X \), and we find that \( X = X \) is a proper invariant subspace of \( U \). This completes the proof.

**References**


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**On linear functional equations in \((B_k)\)-spaces**

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The well known classical Riesz-Schauder [3, 4] theory deals with a very important class of linear transformations in Banach spaces. The basic class of transformations considered in this theory consists of transformations of the form \( H + U \), where \( H \) is an isomorphism onto and \( U \) is completely continuous. One of the principal properties of such a transformation is that it may be represented as the sum of an isomorphism onto and a finite-dimensional linear transformation. Hence every transformation of this class has a finite nullity (i.e., the space of all characteristic elements is finite-dimensional).

The following natural question arises:

Is the powerful algebraic method of F. Riesz strongly connected exclusively with the above class of transformations?

Is it possible to extend this method to a wider class of linear transformations, including, for instance, also some transformations with infinite nullity?

In the present paper an attempt is made to generalize the Riesz-Schauder theory in the above mentioned direction. The generalization given here is extended to a class of linear transformations which includes the subclass of all projections.

Let \( X \) be a fixed \((B_k)\)-space (i.e., a linear complete metric space with a topology determined by a sequence of pseudo-norms see [2]).

We shall say that a linear transformation \( T \) has the domain and range in \( X \) possesses the property \((B_k)\) if there exists a non-negative integer \( \mu \) such that \( T^{n+1}(x) = 0 \) implies \( T^n(x) = 0 \).

Notice that if a linear transformation \( T \) possesses property \((B_k)\), then \( T^n(x) = 0 \) implies \( T^n(x) = 0 \) for \( n > \mu \). We shall say that a linear transformation \( T \) is of finite order if it possesses property \((B_k)\) for some \( \mu \), the least non-negative integer \( \mu \) is called the order of the transformation \( T \). If \( T \) is of order \( \mu \), then \( T^n(x) = 0 \) implies \( T^n(x) = 0 \) for \( n > \mu \), and for \( n < \mu \) there exists an element \( x \in X \) such that \( T^{n+1}(x) = 0 \), but \( T^n(x) \neq 0 \).