On the identity of Morrey-Calkin and Schauder-Sobolev spaces

by

P. SZEPTYCKI (Warszawa)

**§ 1. Introduction.** In direct methods of the calculus of variations for quadratic functionals, and in applications of Hilbert space theory to the boundary value problems for second order elliptic equations (Friedrichs' method) a fundamental part is played by a class of $W_a^1$-spaces, which were first defined by J. Schauder. The study of those spaces was next developed by Sobolev [4] and his school.

On the other hand a class of $\mathfrak{P}_a$-spaces was discovered by Ch. B. Morrey jr. and Calkin [3]. This class was next applied by Morrey [2] in his fundamental research in connection with the calculus of variations and second order quasilinear elliptic equations.

Several months ago K. Maurin stated the hypothesis that the spaces $\mathfrak{P}_2$ and $W_2^1$ are the same. The paper which I am going to present goes still farther. It proves the identity of $W_a^1$ and $\mathfrak{P}_a$ for $a>1$.

The theories of spaces $\mathfrak{P}_a$ and $W_a^1$ have been developed independently.

There are theorems whose proofs are trivial in the theory of $W_a^1$-spaces (*e. g.* those of completeness) and rather difficult in $\mathfrak{P}_a$, but there are more theorems that are trivial in $\mathfrak{P}_a$ but not in $W_a^1$ (we think this to be a consequence of the fact that $\mathfrak{P}_a$-spaces are better known than $W_a^1$). The theorem which will be proved in this paper gives us a possibility of simplifying some proofs and enriches the theory of Schauder-Sobolev spaces by new theorems.

**§ 2. Notation and definitions.** In our considerations we shall denote by $G$ a fixed and bounded domain of $N$-dimensional arithmetical space. The points of this domain will be denoted by small letters of latin alphabet. Consequently we have $x=(x_1,\dots,x_N)$, $y=(y_1,\dots,y_N)$, and for $x_1,\dots,x_{k-1},x_{k+1},\dots,x_N$ we shall write $x_k'$. We use the symbol $[ab]$ to denote the $N$-dimensional *cell*, *i. e.* a set of points $x$ for which $a_i\leqslant x_i\leqslant b_i$ $(i=1,2,\dots,N)$. Similarly, $[a_k',b_k']$ is a set of points $x$ for which $a_i\leqslant x_i\leqslant b_i$ $(i=1,2,\dots,k-1,k+1,\dots,N)$, and $(x_k',a_k)$ is the point $(x_1,\dots,x_{k-1},a_k,x_{k+1},\dots,x_N)$.

Definition 1. A function $f(x)$ which is defined on the region $G$ is of *class* $\mathfrak{P}(G)$ if

1° $f$ is summable on each cell $[ab]\subset G$,

2° there exist functions $v_1,\ldots,v_N$, satisfying 1°, such that

$$\int_a^b v_k(x)\,dx = \int_{a'_k}^{b'} \big(f(x'_k, b_k) - f(x'_k, a_k)\big)\,dx'_k$$

holds for almost all interior cells of $G$ (*i. e.* for all cells $[ab]$ for which the point $(ab)$ does not belong to the set of $2N$-dimensional measure zero).

Definition 2. The generalized partial derivative $D_{x_k}f$ (in the sense of Morrey-Calkin) with respect to $x_k$ of the function $f$ of class $\mathfrak{P}(G)$ is a *Lebesgue derivative* of the set function $\int_e v_k(x)\,dx$.

It is clear that the equality $D_{x_k}f = v_k$ holds almost everywhere.

Definition 3. A function $f$ is of *class* $\mathfrak{P}_a(G)$ if

1° $f$ is of class $\mathfrak{P}(G)$,

2° $f$ and $D_{x_k}f$ $(k=1,2,\ldots,N)$ are of class $L^a(G)$.

In consequence of condition 2° of definition 3 we can define in $\mathfrak{P}_a(G)$ a norm

$$\|f\|^a = \int_G \Big\{|f(x)|^a + \Big[\sum_{k=1}^N \big(D_{x_k} f(x)\big)^2\Big]^{a/2}\Big\}\,dx.$$

It has been proved by Morrey and Calkin that with this norm $\mathfrak{P}_a(G)$ is a Banach space (the addition and the multiplication by numbers being defined in the ordinary way). In the case of $a=2$ it is a Hilbert space.

Definition 4. A function $f$ is of *class* $C_c^\infty(G)$ if

1° $h$ is of class $C^\infty(G)$,

2° a support of $h$ is compact in $G$ (*i. e.* a closure of the set of points $x$ for which $h(x)\neq 0$ is compact in $G$).

Definition 5. The space $W'_a(G)$, $a>1$, is defined as the *completion* (*i. e.* the space of classes of all fundamental and equivalent sequences) of $C^\infty(G)$ in the sense of the norm

$$\|h\|^a = \int_G \left\{|h(x)|^a + \left[\sum_{k=1}^N \left(\frac{\partial h(x)}{\partial x_k}\right)^2\right]^{a/2}\right\}dx.$$

Definition 6. A function $f$ of the class $L^a(G)$ is *differentiable with respect to $x_k$ in the sense of Sobolev* if there exists a function $g$ of the class $L^a(G)$ such that the equality

$$\int_G f(x)\frac{\partial h(x)}{\partial x_k}\,dx = -\int_G g(x)\,h(x)\,dx$$

holds for all functions $h\in C_c^\infty(G)$.



Therefore $g \underset{\text{def}}{=} \partial[f(x)]/\partial x_k$ is a partial derivative of $f$ with respect to $x_k$ in the sense of Sobolev.

**§ 3. Fundamental theorem.** In this section we shall give a proof of the following theorem.

THEOREM. *The spaces $\mathfrak{P}_a(G)$ and $W_a^1(G)$ are equivalent in the following sense: if $f\in\mathfrak{P}_a(G)$, then $f\in W_a^1(G)$ and inversely. Moreover, the existence of derivatives of $f$ in the Morrey-Calkin sense is equivalent to the existence of corresponding derivatives in the sense of Sobolev and the equality $\partial[f]/\partial x_k = D_{x_k}f$ holds almost everywhere in $G$.*

Proof. In proving this theorem we shall use the following well-known results:

THEOREM 1 (Morrey-Calkin (v. [2] and [3])). *The necessary and sufficient condition that $f$ be of class $\mathfrak{P}(G)$ is that it satisfy* 1° *of definition* 1 *and that for each cell $[ab]\subset G$ there exist summable functions $v_k(x)$ and a sequence $\{f_p(x)\}$, each $f_p$ satisfying a uniform Lipschitz condition on $[ab]$, such that*

$$\lim_{p\to\infty}\int_a^b\Big\{|f_p(x)-f(x)| + \sum_{k=1}^N\Big|\frac{\partial f_p(x)}{\partial x_k} - v_k\Big|\Big\}\,dx = 0.$$

THEOREM 2 (Sobolev)[1]. *$W_a^1(G)$ is the space of all functions of class $L^a(G)$ which are differentiable in the sense of Sobolev with respect to $x_k$ for $k=1,2,\ldots,N$.*

From theorem 1 it immediately follows that $W_a^1(G)\subset\mathfrak{P}_a(G)$. Indeed, let $f\in W_a^1(G)$ and $\{f_p\}$ be a sequence of $f_p\in C^\infty(G)$ such that

$$\lim_{p\to\infty}\|f_p - f\|_{W_a^1} = 0.$$

This sequence satisfies of course the conditions of theorem 1. Let in the space $L^a$

$$v_k = \lim_{p\to\infty}\frac{\partial f_p}{\partial x_k}.$$

From the inequality of Hölder we get

$$\int_a^b |f(x)-f_p(x)|\,dx \leqslant \sqrt[a]{\int_a^b |f(x)-f_p(x)|\,dx}\cdot\sqrt[\beta]{|ab|} \leqslant \|f-f_p\|_{W_a^1}\sqrt[\beta]{|ab|}.$$

[1]) See [4]; this follows also from Friedrichs' theorem [1].

We have also

$$\int_a^b \left| v_k(x) - \frac{\partial f_p(x)}{\partial x_k} \right| dx \leqslant \sqrt[\alpha]{\int_a^b \left| v_k(x) - \frac{\partial f_p(x)}{\partial x_k} \right| dx} \cdot \sqrt[\beta]{|ab|} \leqslant \|f - f_p\|_{W_a^1} \sqrt[\beta]{|ab|},$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

and this, by the definition of functions $v_k$, completes the proof of our inclusion.

We shall now prove the inverse inclusion. For this purpose we shall prove the following lemmas:

LEMMA 1. *If $f \in L^\alpha(G)$ is differentiable with respect to $x_k$ in the sense of Lebesgue (i. e. if there exists a function $f_{x_k}$ such that $f(x) = \int f_{x_k}(x)\,dx_k + C$), and the derivative in the sense of Lebesgue of $f$ with respect to $x_k$ is of class $L^\alpha(G)$, then $f$ is differentiable with respect to $x_k$ in the sense of Sobolev and the respective derivatives are equal almost everywhere in $G$.*

Proof. Let $h \in C_c^\infty(G)$ and $D$ be the support of $h$ in $G$. We shall restrict ourselves to the case where $D$ is a normal domain with respect to the hyperplane $x_k = 0$. In other cases we divide $D$ into normal domains.

We use the Lebesgue theorem on integrating by parts. We have

$$\int_G f(x) \frac{\partial h(x)}{\partial x_k} dx = \int_D f(x) \frac{\partial h(x)}{\partial x_k} dx = \int_{D_k} \int_{a(x_k')}^{b(x_k')} f(x_k', x_k) \frac{\partial h(x_k', x_k)}{\partial x_k} dx_k dx_k'$$

$$= \int_{D_k'} [f(x_k', x_k) h(x_k', x_k)]_{a(x_k)}^{b(x_k)} dx_k' - \int_G f_{x_k}(x) h(x)\, dx,$$

where $a(x_k')$ and $b(x_k')$ are the points at which the line $(x_k', t)$ intersects the boundary of domain $D$, and $D_k$ is the projection of $D$ on the hyperplane $x_k = 0$; we denote by $f_{x_k}$ the Lebesgue derivative of $f$ with respect to $x_k$.

The condition $h(x) = 0$ on the boundary of domain $D$ gives us the assertion

$$\int_G f(x) \frac{\partial h(x)}{\partial x_k} dx = -\int_G h(x) g(x)\, dx, \quad \text{where} \quad g = \frac{\partial f}{\partial x_k}.$$

LEMMA 2. *If $f \in \mathfrak{P}_\alpha(G)$, then $f$ is differentiable in the sense of Lebesgue with respect to $x_k$ $(k = 1, 2, \ldots, N)$ and the derivatives of Lebesgue are equal almost everywhere to those of Morrey-Calkin.*

Proof. From the condition 2° of definition 1 and from a theorem of Fubini it follows that there exists such a number $a_k$ that the equality

$$\int_{a_k'}^{b_k'} \int_{a_k}^{x_k} v(x)\, dx = \int_{a_k'}^{b_k'} \big(f(x_k', x_k) - f(x_k', a_k)\big)\, dx_k'$$



holds for almost all $x_k$ and for almost all $(N-1)$ dimensional cells $[a_k' b_k']$ for which $[ab] \subset G$. Hence it follows that

$$\int_{a_k}^{x_k} v(x)\, dx_k = f(x_k', x_k) + C, \quad \text{q. e. d.}$$

From the condition 2° of definition 3 we have $D_{x_k} f \in L^\alpha(G)$ and by lemma 1 and theorem 2 the proof of inclusion $\mathfrak{P}_\alpha(G) \subset W_\alpha^1(G)$ is completed.

The proof presented here gives also the equality almost everywhere of the derivatives of Morrey-Calkin and of Sobolev. This completes the proof of the theorem.

**§ 4. Some applications of the proved theorem.** The fundamental theorem makes it possible to transfer the known properties of Morrey-Calkin spaces to the theory of Schauder-Sobolev and inversely. The proofs of some theorems are simplified in that way.

1. From the definition of $W_\alpha^1$-space we have drawn a not very obvious conclusion about the completeness of the $\mathfrak{P}_\alpha$-space.

2. We think that it would be very interesting to transfer the so-called "substitution theorem" to the theory of Schauder-Sobolev.

Let $f$ be of class $W_\alpha^1(G)$, $D \subset G$ and $g \in W_\alpha^1(D)$ where $g = f$ on the boundary of $D$. Then the function $h$ defined as

$$h = \begin{cases} f(x) & \text{if} \quad x \in G - D, \\ g(x) & \text{if} \quad x \in D, \end{cases}$$

is of class $W_\alpha^1(G)$ and

$$\frac{\partial [h]}{\partial x_k} = \begin{cases} \dfrac{\partial [f(x)]}{\partial x_k} & \text{if} \quad x \in G - D, \\ \dfrac{\partial [g(x)]}{\partial x_k} & \text{if} \quad x \in D, \end{cases} \qquad k = 1, 2, \ldots, N.$$

3. In the theory of Morrey-Calkin a remarkable part is played by theorems, introduced by Rademacher, on the change of variables by so-called "transformations of class $K$". We transfer these theorems to the theory of Schauder-Sobolev:

THEOREM (Generalization of a theorem of Evans). *If $f \in W_\alpha'(G)$ and $g(y) = f(x(y))$ where $x(y)$ is a regular transformation of domain $H$ on domain $G$, then $G \in W_\alpha^1(H)$.*

THEOREM. *Weak convergence is preserved by transformations of class $K$, i. e. if $f_n \rightharpoonup f$ on $G$ in $W_\alpha^1(G)$ and $x = x(y)$ is the transformation of class $K$, of the domain $H$ on $G$, and if $g_n(y) = f_n(x(y))$ and $g(y) = f(x(y))$, then $g_n \rightharpoonup g$ on $H$ in $W_\alpha^1(H)$.*

References

[1] O. K. Friedrichs, *On the identity of weak and strong extensions of differential operators*, Transactions of the American Math. Soc. 55 (1944), p. 132-152.

[2] Ch. B. Morrey, *Multiple integral problem and principle of variation in related topics*, Princeton 1943.

[3] — and J. W. Calkin, *Functions of several variables and absolute continuity*, Duke Math. Journal 6 (1940), p. 170-215.

[4] С. Л. Соболев, *Некоторые применения функционального анализа в математической физике*, Москва 1948.

*Reçu par la Rédaction le 24. 9. 1954*

# Invariant subspaces of completely continuous operators in locally convex linear topological spaces

by

M. ALTMAN (Warszawa)

Recently N. Aronszajn and K. T. Smith [2] have proved the following

THEOREM. *Let $X$ be a Banach space and $U$ a linear completely continuous operator in $X$. There exist proper invariant subspaces of $U$.*

The purpose of this note is to prove that any linear completely continuous operator in a locally convex linear topological space possesses the same property.

The proof of this statement is based on the above theorem and on a method developed in the paper [1].

Let $X$ be a *locally convex linear topological space*, *i. e.* a linear space on which a topology is imposed in such a fashion that the postulated operations of addition and multiplication by real numbers are continuous in the topology. The *local convexity* means that for every neighbourhood $\mathfrak{U}_x$ of the element $x \in X$ there exists a convex neighbourhood $\mathfrak{V}_x$ such that $\mathfrak{V}_x \subset \mathfrak{U}_x$.

A linear operator $U$ having its domain and range in $X$ is said to be *completely continuous* if there exists a neighbourhood $\mathfrak{V}$ of 0 such that the image $U(\mathfrak{V})$ is compact in the sense that every infinite subset has a *cluster point* (*i. e.* non-isolated point).

It is known (for references see [1]) that the space $X$ is isomorphic to a certain $(B_a)$-space in which a class of pseudonorms $|x|_\vartheta$, where $\vartheta \in \Phi$, is defined ($\Phi$ is an abstract set and $\overline{\overline{\Phi}} = \aleph_a$). The system of neighbourhoods of zero consists of the sets of all elements $x$ such that $|x|_{\vartheta_i} < \varepsilon$ $(i=1,2,\ldots,k)$, $\varepsilon > 0$.

LEMMA. *Let $U$ be a linear completely continuous operator. Then there exists a pseudonorm $|x| = \sup_i |x|_{\vartheta_i}$ $(i=1,2,\ldots,n)$ such that for every $\vartheta \in \Phi$ there is a number $M_\vartheta$ satisfying the inequality $|U(x)|_\vartheta \leqslant M_\vartheta |x|$* [1]).

[1]) For the proof see [2], p. 196.