On the metric theory of inhomogeneous diophantine approximations

by

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Introduction. In this paper a problem of H. Steinhaus and some related questions are solved. We introduce the following notations:

Let K be the circle $\zeta_1^2+\zeta_2^2=1$ in the plane (ζ_1,ζ_2). If x is a real number, then by $\zeta=x'=\Phi(x)$ we denote the point in the plane (ζ_1,ζ_2) with the coordinates $(\cos 2\pi x, \sin 2\pi x)$. If g and h are real numbers, $g<h$, then the interval $I[g,h]$ is the set of points x' where $g\leqslant x\leqslant h$. Obviously an operation of addition may be defined in the set K under which this set becomes a group and the mapping Φ is a "natural" homomorphic mapping of the additive group of real numbers on the group K. Further the Lebesgue measure μ and a topology are defined on K in an obvious way. (We suppose that $\mu(I[g,h])=h-g$ if $1\geqslant h-g>0$ and specially that $\mu(K)=1$).

Let B be a non empty set the elements of which are sequences of real numbers $\{b_i\}$, $i=1,2,3,\ldots$, fulfilling the following conditions:

(1) $$b_i>0, \qquad i=1,2,\ldots,$$

(2) $$b_{i+1}\leqslant b_i, \qquad i=1,2,\ldots,$$

(3) $$\sum_{i=1}^{\infty} b_i=\infty.$$

We define the set $\alpha(B)$:

A real number x $(0\leqslant x<1)$ *belongs* to the set $\alpha(B)$ if the following condition is fulfilled:

For every sequence $\{b_i\}\epsilon B$ almost every point $\eta\epsilon K$ belongs to an infinite number of intervals $I[ix-b_i, ix+b_i]$, $i=1,2,3,\ldots$

This condition may be written in the equivalent form: if $\{b_i\}\epsilon B$ then

$$\mu\Big(\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i, ix+b_i]\Big)=1.$$

Let the set $\tilde{B}$ contain all sequences $\{b_i\}$ fulfilling conditions (1), (2), (3). The set $\alpha(\tilde{B})$ contains no rational number. H. Steinhaus has put forward the question whether all irrational numbers from the interval $(0,1)$ belong to the set $\alpha(\tilde{B})$.

We shall answer this question in the terms of the theory of diophantine approximations.

Let the function $\varphi(q)$ be defined for $q\geqslant 1$, non-negative and non increasing. A number x is said to *admit the approximation* $\varphi(q)$ if there is an infinite sequence of pairs of integers (p_n,q_n), $q_{n+1}>q_n$, fulfilling the inequality

$$|x-p_n/q_n|\leqslant\varphi(q_n).$$

It is known that every number x admits of the approximation $1/\sqrt{5}q^2$.

Let the number y belong to the set $Y_{\varphi(q)}$ if $0\leqslant y<1$ and if there is such a number $d_y\geqslant 1$ that the number y does not admit of the approximation $\varphi(d_y q)$.

The problem of H. Steinhaus is solved by the following

THEOREM 1. $\alpha(\tilde{B})=Y_{1/q^2}$.

It easily follows from this theorem that the set $\alpha(\tilde{B})$ is non empty and that its Lebesgue measure is zero. On the other hand we shall prove the following result:

Let the set B' contain only one sequence $\{b_i\}$ fulfilling the conditions (1), (2), (3). Then the Lebesgue measure of the set $\alpha(B')=\alpha(\{b_i\})$ is unity.

Further we give a generalization of theorem 1 by establishing the relation

$$\alpha(B_{\varphi(q)})=Y_{\varphi(q)}$$

where $B_{\varphi(q)}$ [1]) is a suitable subset of the set $\tilde{B}$ for a class of functions $\varphi(q)$.

Finally we generalize theorem 1 to the case of s linear forms with n variables.

1. The proof of theorem 1. We have to prove that the measure of the set

$$\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i, ix+b_i]$$

is unity for every sequence $\{b_i\}\epsilon\tilde{B}$ if and only if $x\epsilon Y_{1/q^2}$.

We start from the following

[1]) For the definition of the set $B_{\varphi(q)}$ see p. 94.

LEMMA 1. *If $x \bar{\epsilon} Y_{1/q^2}$, then there is such a sequence $\{b_i\} \epsilon \tilde{B}$ that the measure of the set*

$$\prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[ix - b_i, ix + b_i]$$

is zero.

Proof. Let us suppose that $x \bar{\epsilon} Y_{1/q^2}$. Then there is such a positive function $\varphi(q)$ that the function $q^2\varphi(q)$ steadily tends to zero as q increases to infinity and that the number x admits the approximation $\varphi(q)$.

Let (p_s, q_s) be a sequence of pairs of integers which fulfil the following inequalities:

$$(4) \qquad |x - p_s/q_s| \leqslant \varphi(q_s), \qquad s = 1, 2, 3, \ldots,$$

$$(5) \qquad q_{s+1} > q_s > 0, \qquad s = 1, 2, 3, \ldots,$$

$$(6) \qquad 1/q_s^2\varphi(q_s) > 2^{2s}, \qquad s = 1, 2, 3, \ldots,$$

$$(7) \qquad \varphi(q_{s+1})/\varphi(q_s) < 1/2, \qquad s = 1, 2, 3, \ldots$$

We estimate the Lebesgue measure of a certain set.

LEMMA 2. *Let us fix a positive integer n and a positive number b. Then the following inequality holds:*

$$\mu\Big(\sum_{i=1}^{\infty} I[ix - b, ix + b]\Big) \leqslant q_s \cdot 2\big(b + n\varphi(q_s)\big), \qquad s = 1, 2, 3, \ldots$$

From the obvious inequality

$$|x - p_s'/iq_s| < \varphi(q_s),$$

holding for suitable p_s', we obtain, on multiplying it by i,

$$|ix - p_s'/q_s| < i\varphi(q_s) \leqslant n\varphi(q_s) \quad \text{for} \quad i = 1, 2, \ldots, n,$$

whence it follows that all points $(ix)'$ for $i = 1, 2, \ldots, n$ are contained in the set

$$\sum_{j=1}^{q_s} I[j/q_s - n\varphi(q_s),\ j/q_s + n\varphi(q_s)].$$

Hence and from the inclusion

$$\sum_{i=1}^{n} I[ix - b, ix + b] \subset \sum_{j=1}^{q_s} I[j/q_s - b - n\varphi(q_s),\ j/q_s + b + n\varphi(q_s)]$$

we obtain the proof of lemma 2.

We return to the proof of lemma 1.

Let us denote by q_s^* the greatest integer less than or equal to $(1/\varphi(q_s))^{1/2}$. According to (6) and (7) we have $q_{s+1}^* > q_s^*$. We put

$$b_1 = b_2 = \ldots = b_{q_1^*} = (\varphi(q_1))^{1/2},$$

$$b_{q_s^*+1} = b_{q_s^*+2} = \ldots = b_{q_{s+1}^*} = (\varphi(q_{s+1}))^{1/2}, \qquad s = 1, 2, 3, \ldots$$

This sequence $\{b_i\}$ fulfils conditions (1), (2), (3) since we have

$$\sum_{s=1}^{\infty} b_s \geqslant \sum_{j=1}^{\infty} \sum_{s=q_j^*+1}^{q_{j+1}^*} b_s \geqslant \sum_{j=1}^{\infty} \left[\left(\frac{1}{\varphi(q_{j+1})}\right)^{1/2} - \left(\frac{1}{\varphi(q_j)}\right)^{1/2} - 1\right] \cdot (\varphi(q_{j+1}))^{1/2}$$

$$\geqslant \sum_{j=1}^{\infty} \left(1 - \left(\frac{\varphi(q_{j+1})}{\varphi(q_j)}\right)^{1/2} - (\varphi(q_{j+1}))^{1/2}\right) = \infty.$$

Now we shall prove that the series

$$\sum_{s=1}^{\infty} \mu\Big(\sum_{i=q_s^*+1}^{q_{s+1}^*} I[ix - b_i, ix + b_i]\Big)$$

converges. We use lemma 2 and have

$$\sum_{s=1}^{\infty} \mu\Big(\sum_{i=q_s^*+1}^{q_{s+1}^*} I[ix - b_i, ix + b_i]\Big)$$

$$\leqslant \sum_{s=1}^{\infty} \mu\Big(\sum_{i=1}^{q_{s+1}^*} I[ix - b_{q_{s+1}^*}, ix + b_{q_{s+1}^*}]\Big) \leqslant \sum_{s=1}^{\infty} q_{s+1} \cdot 2\big(b_{q_{s+1}^*} + q_{s+1}^*\varphi(q_{s+1})\big)$$

$$\leqslant \sum_{s=1}^{\infty} 2q_{s+1}\left(\sqrt{\varphi(q_{s+1})} + \frac{1}{\sqrt{\varphi(q_{s+1})}}\varphi(q_{s+1})\right)$$

$$= \sum_{s=1}^{\infty} 4q_{s+1}\sqrt{\varphi(q_{s+1})} < \sum_{s=1}^{\infty} 4 \cdot 2^{-s-1} = 2.$$

It easily follows that lemma 1 holds and specially that number x does not belong to the set $a(\tilde{B})$.

We proceed to the second part of theorem 1 and prove the inclusion $Y_{1/q^2} \subset a(\tilde{B})$.

First of all we shall prove the following

LEMMA 3. *If $x \epsilon Y_{1/q^2}$ and if $\{b_i\} \epsilon \tilde{B}$, then*

$$\mu\Big(\prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[ix - b_i, ix + b_i]\Big) > 0.$$

Let us fix a number $x \epsilon Y_{1/q^2}$. According to the definition of the set Y_{1/q^2} there is such a positive number c_x that the inequality

$$|x - p/q| > c_x/q^2 \tag{8}$$

holds for every pair of integers (p,q), $q>0$.

Lemma 3 is an easy consequence of the following

LEMMA 4. *If* $\{b_i\} \epsilon \tilde{B}$, *then the inequality*

$$\mu\Big(\sum_{i=k}^{\infty} I[ix - b_i, ix + b_i]\Big) > c_x/7, \qquad k=1,2,3,\ldots, \tag{9}$$

holds (it will be remembered that $c_x \leqslant 1/\sqrt{5}$*).*

Proof. We put

$$b_i' = \min(c_x/3i, b_i), \qquad i = 1,2,3,\ldots,$$

and

$$b_i'' = b_{7^j}' \quad \text{for} \quad i = 7^{j-1}+1, 7^{j-1}+2, \ldots, 7^j, \qquad j = 1,2,3,\ldots$$

Apparently we have

$$\sum_{i=1}^{\infty} b_i' = \infty, \qquad \sum_{i=1}^{\infty} b_i'' \geqslant \tfrac{1}{7}\sum_{i=8}^{\infty} b_i' = \infty.$$

Let us suppose that there is such an index k_0 that inequality (9) is false; consequently we have

$$\mu\Big(\sum_{i=k_0}^{7^j} I[ix - b_i'', ix + b_i'']\Big) \leqslant c_x/7, \qquad j = j_0, j_0+1, \ldots, \qquad 7^{j_0} > k_0+1. \tag{10}$$

The set

$$\sum_{i=k_0}^{7^j} I[ix - b_i'', ix + b_i'']$$

is the sum of a finite system of disjoint intervals. On replacing b_i by $2b_i$ each of the above-mentioned intervals will be lengthened at most two-fold, *i. e.* we shall have

$$\mu \sum_{i=k_0}^{7^j} I[ix - 2b_i'', ix + 2b_i'') \leqslant 2c_x/7.$$

Let L_{j+1} be the set of indices i fulfilling the conditions

$$7^j < i \leqslant 7^{j+1}, \qquad (ix)' \bar{\epsilon} \sum_{i=k_0}^{7^j} I[ix - 2b_i'', ix + 2b_i''].$$

It follows from (8) that the number of the points $(ix)'$, $i=1,2,\ldots,7^{j+1}$, which are contained in an interval $I[y-d, y+d]$ is not greater than $2d \cdot 7^{j+1}/c_x + 1$ [2]).

As the set

$$\sum_{i=k_0}^{7^j} I[ix - 2b_i'', ix + 2b_i'']$$

again is the sum of disjoint intervals the number of which does not exceed 7^j and as

$$\mu\Big(\sum_{i=k_0}^{7^j} I[ix - 2b_i'', ix + 2b_i'']\Big) \leqslant 2c_x/7,$$

therefore the number of the indices i which fulfil the conditions

$$1 \leqslant i \leqslant 7^{j+1}, \qquad i \bar{\epsilon} L_{j+1},$$

is not greater than

$$\frac{2c_x}{7} \cdot \frac{1}{c_x} \cdot 7^{j+1} + 7^j = 3 \cdot 7^j.$$

Since the number of all indices i satisfying $7^j < i \leqslant 7^{j+1}$ is $6 \cdot 7^j$, therefore the set L_{j+1} has at least $3 \cdot 7^j$ elements. Further we have

$$\sum_{i=k_0}^{7^{j+1}} I[ix - b_i'', ix + b_i''] - \sum_{i=k_0}^{7^j} I[ix - b_i'', ix + b_i''] \supset \sum_{i \epsilon L_{j+1}} I[ix - b_i'', ix + b_i''].$$

Each two of the intervals appearing on the right side are disjoint. For if any two intervals $I[ix - b_i'', ix + b_i'']$ and $I[lx - b_l'', lx + b_l'']$, $i, l \epsilon L_{j+1}$, $i \neq l$, were not disjoint, there would exist a number y such that the interval

$$I[y - c_x/3 \cdot 7^{j+1}, y + c_x/3 \cdot 7^{j+1}]$$

would contain both points, $(ix)'$ and $(lx)'$. We have

$$b_l'' = b_i'' = b_{7^{j+1}}' \leqslant c_x/3 \cdot 7^{j+1}.$$

But in virtue of footnote [2]) the interval

$$I[y - c_x/3 \cdot 7^{j+1}, y + c_x/3 \cdot 7^{j+1}]$$

[2]) Let us denote by σ the number of the points $(ix)'$, $i=1,2,\ldots,7^{j+1}$, which are contained in the interval $I[y-d, y+d]$. It follows that there are two indices i_1, i_2, $0 < i_1 < i_2 \leqslant 7^{j+1}$ such that the points $(i_1 x)'$, $(i_2 x)'$ belong to the interval $I[t, t+2d/(\sigma-1)]$. The point $((i_2 - i_1)x)'$ belongs to the interval $I[t - i_1 x, t - i_1 x + 2d/(\sigma-1)] \subset I[-2d/(\sigma-1), 2d/(\sigma-1)]$. That means that there is such an integer s that $|(i_2-i_1)x - s| \leqslant 2d/(\sigma-1)$, $|x - s/(i_2-i_1)| \leqslant 2d/(\sigma-1)(i_2-i_1)$. According to (8) we get

$$2d/(\sigma-1)(i_2-i_1) \geqslant c_x/(i_2-i1)^2, \qquad \sigma - 1 \leqslant (2d/c_x)(i_2-i_1) \leqslant (2d/c_x)7^{j+1}.$$

may contain at most one point $(kx)'$, $k=1,2,\ldots,7^{j+1}$. Thus we get

$$\mu\Big(\sum_{i=k_0}^{7^{j+1}} I[ix-b_i'', ix+b_i'']\Big)$$
$$\geqslant \mu\Big(\sum_{i=k_0}^{7^j} I[ix-b_i'', ix+b_i'']\Big) + \sum_{i\in L_{j+1}} \mu\big(I[ix-b_i'', ix+b_i'']\big)$$
$$\geqslant \mu\Big(\sum_{i=k_0}^{7^j} I[ix-b_i'', ix+b_i'']\Big) + \sum_{i\in L_{j+1}} 2b_i''.$$

Obviously we have

$$\sum_{i\in L_{j+1}} 2b_i'' = \sum_{i\in L_{j+1}} 2b'_{7^{j+1}} \geqslant 3\cdot 7^j\cdot 2b'_{7^{j+1}} = 7^{j+1} b'_{7^{j+1}} = \sum_{i=7^j+1}^{7^{j+1}} b_i''.$$

Therefore

$$\mu\left(\sum_{i=k_0}^{7^j} I[ix-b_i'', ix+b_i'']\right) + \sum_{i\in L_{j+1}} 2b_i''$$
$$\geqslant \mu\left(\sum_{i=k_0}^{7^j} I[ix-b_i'', ix+b_i'']\right) + \sum_{i=7^j+1}^{7^{j+1}} b_i''.$$

As the last inequality holds for $j=j_0, j_0+1, j_0+2, \ldots$ and as the series $\sum_{i=1}^{\infty} b_i''$ diverges, we find that

$$\mu\left(\sum_{i=k_0}^{7^{j+1}} I[ix-b_i'',\ ix+b_i'']\right) > c_x/7$$

for large j and the contradiction of inequality (10) completes the proof of lemma 4.

In order to complete the proof of the theorem 1 we shall need the following

LEMMA 5. *Let us suppose that U and V are subsets of K, that the measure μ of U is positive and that V is dense in K. Let the set $W=U\oplus V$ contain all points $w=u+v$ where $u\in U$, $v\in V$. Then $\mu(W)=1$.*

Let U_1 and V_1 be sets of real numbers. Let the Lebesgue measure of the set U_1 be positive and let the set V_1 be dense (in E_1). Then the set $U_1\oplus V_1$ contains almost all real numbers. This follows from the known fact that almost all points of the set U_1 are points of outer density for this set (Saks [4], Chapter IV, § 10) and we easily find by means of the mapping Φ that lemma 5 holds.

Now we are ready to complete the proof of theorem 1 by proving the following

LEMMA 6. *If $x\in Y_{1/q^2}$ and if $\{b_i\}\in\tilde{B}$, then*

$$\mu\left(\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i,\ ix+b_i]\right)=1.$$

Proof. Let us fix a sequence $\{b_i\}\in\tilde{B}$. From the condition $b_{n+1}\leqslant b_n$ it can be seen that if

$$\xi\in\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i,\ ix+b_i],$$

then also

$$\xi-(sx)'\in\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i, ix+b_i] \quad \text{for} \quad s=0,1,2,\ldots$$

Applying lemma 5 for

$$U=\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i, ix+b_i], \qquad V=\{(-sx)'\}, \qquad s=0,1,2,\ldots$$

(which can be done on account of lemma 3 and the fact that V, defined above, is a set dense for irrational x), we immediately obtain

$$\mu\left(\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[ix-b_i,\ ix+b_i]\right)=1.$$

Remark. From a general theorem due to Khintchine [2] it follows at once that the Lebesgue measure of the set Y_{1/q^2} is zero. Jarnik [1] (or Kurzweil [3], § 6) investigated this set by means of the Hausdorff measure and found that the Hausdorff dimension of the set Y_{1/q^2} is unity.

2. In this section we pass to the following

THEOREM 2. *Let us suppose that $\{b_i\}\in\tilde{B}$. Then the Lebesgue measure of the set $\alpha(\{b_i\})$ is one.*

Let us fix a sequence $\{b_i\}\in\tilde{B}$. It is easy to find a sequence $\{b_i'\}\in\tilde{B}$ fulfilling the conditions

(11) $$\lim (b_i'/b_i)=0,$$

(12) $$b_i'<1/8i, \quad i=1,2,3,\ldots$$

Let us denote by S the square $0\leqslant x<1$, $0\leqslant y<1$ in the plane (x,y), and let us put

$$U_{n,m}=\underset{(x,y)}{E}\,[|y-nx-m|\leqslant b_n], \qquad m,n \text{ whole, } n \text{ positive},$$

$$V_n=\sum_{m=-\infty}^{\infty} U_{n,m}, \qquad n=1,2,3,\ldots, \qquad W=\prod_{s=1}^{\infty}\sum_{n=s}^{\infty} V_n,$$

$$U'_{n,m} = \underset{(x,y)}{E}\,[|y-nx-m| \leqslant b'_n, \quad m, n \text{ whole}, \ n \text{ positive},$$

$$V'_n = \sum_{m=-\infty}^{\infty} U'_{n,m}, \qquad W' = \prod_{s=1}^{\infty} \sum_{n=s}^{\infty} V'_n.$$

We start from the following

LEMMA 7. *To every positive integer N_0 there is such a positive integer N that*

$$\mu_2\Big(S \frown \Big(\sum_{n=N_0}^{N} V'_n\Big)\Big) \geqslant 1/8\ ^3).$$

Consequently $\mu_2(S \frown W') \geqslant 1/8$.

Proof. We shall use the estimation

(13) $$\mu_2\Big(S \frown \sum_{n=N_0}^{N} V'_n\Big) \geqslant \sum_{n=N_0}^{N} \mu_2(S \frown V'_n) - \sum_{N_0 \leqslant j < k \leqslant N} \mu_2(S \frown V'_j \frown V'_k).$$

Let us denote by K the set $K = \underset{(x,y)}{E}\,[0 \leqslant x < 1]$. Apparently

$$\mu_2(S \frown V'_n) = \mu_2(K \frown U'_{n,0}) = 2b'_n, \qquad n = 1, 2, 3, \ldots$$

We shall prove that

$$\mu_2(S \frown V'_j \frown V'_k) \leqslant 8 \cdot b'_j \cdot b'_k, \qquad j \neq k, \ j = 1, 2, \ldots, \ k = 1, 2, \ldots$$

We start from the observation that

$$\mu_2(S \frown V'_j \frown V'_k) = (K \frown U'_{k,0}) \frown V'_j, \qquad j < k,$$

which may easily be verified. The set $K \frown U'_{k,0} \frown V'_j$ consists of $k-j$ parallelograms. The measure od each of these parallelograms is equal to $4b'_j b'_k/(k-j)$. We obtain hence

$$\mu_2(S \frown V'_j \frown V'_k) = (k-j)\, 4\, b'_j b'_k/(k-j) = 4b'_j b'_k.$$

Now we rewrite inequality (13) in the form

$$\mu_2\Big(S \frown \sum_{n=N_0}^{N} V'_n\Big) \geqslant 2 \sum_{n=N_1}^{N} b'_n - 4 \sum_{N_0 \leqslant j < k \leqslant N} b'_j b'_k \geqslant 2 \sum_{n=N_0}^{N} b'_n - \Big(2 \sum_{n=N_0}^{N} b'_n\Big)^2.$$

As $b'_n \leqslant 1/8$, we can choose the index N in such a way that

$$3/8 \leqslant 2 \sum_{n=N_0}^{N} b'_n < 5/8.$$

It follows that

$$\mu_2\Big(S \frown \sum_{n=N_0}^{N} V'_n\Big) \geqslant 3/8 - (3/8)^2 > 1/8,$$

and the proof of lemma 7 is complete.

[3]) By μ_2 we denote the Lebesgue measure in the plane.

Next we prove

LEMMA 8. $\mu_2(S \frown W) = 1$.

This proposition will be proved in the following way: if $(x, y) \epsilon W'$, then there is a sequence of pairs of integers q_n, p_n fulfilling the conditions

$$1 < q_n < q_{n+1}, \qquad |y - q_n x - p_n| \leqslant b'_{q_n}.$$

Let us choose a positive integer s. According to (11) the inequalities

$$|y - q_n x - p_n| \leqslant b_{q_n}/s, \qquad |sy - q_n sx - sp_n| \leqslant b_{q_n}$$

hold for all large n and that means that the point (sx, sy) belongs to the set W.

Let us denote by $C[x_0, y_0, d]$ $(d > 0)$ the square $x_0 - d \leqslant x \leqslant x_0 + d$, $y_0 - d \leqslant y \leqslant y_0 + d$ in the plane (x, y). Let us choose a number δ, $0 < \delta < 1$.

As the measure of the set W' is positive, there is such a square $C[x_0, y_0, 1/2s]$ (s being a positive integer) that

$$\mu_2(W' \frown C[x_0, y_0, 1/2s]) > \delta/S^2.$$

Then we have

$$\mu_2(W \frown C[sx_0, sy_0, 1/2]) > \delta,$$

and, as the point $(x+u, x+v)$ belongs to the set W if the point (x, y) belongs to the set W and if u and v are integers, it follows that $\mu_2(W \frown S) > \delta$. As δ $(0 < \delta < 1)$ is arbitrary, we have $\mu_2(W \frown S) = 1$.

Theorem 2 is an easy consequence of lemma 8. For $0 \leqslant x < 1$ let us put

$$A_x = \underset{y}{E}\,[0 \leqslant y < 1, \ (x, y) \epsilon W].$$

Then we have the following equivalence: $x \epsilon \alpha(\{b_n\})$ if and only if $\mu_1(A_x) = 1$[4]) and the proof of theorem 2 is complete.

3. The purpose of this section is to establish the relation $\alpha(B_{\varphi(q)}) = Y_{\varphi(q)}$.

We shall need some facts about continuous fractions. It is known that every irrational number x, $0 < x < 1$, can be developed into a regular continuous fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_s + \cdots}}}$$

[4]) μ_1 is the linear Lebesgue measure.

(a_i are positive integers, $i=1,2,3,\ldots$) in a unique way. If

$$\frac{p_i}{q_i}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_i}}}$$

(p_i, q_i relatively prime integers, q_i positive, $i=1,2,3,\ldots$), then the partial denominators q_i fulfil the following relations:

(14) $$q_{i+1}=a_{i+1}q_i+q_{i-1}, \qquad i=0,1,2,\ldots$$

(we put $q_{-1}=0$, $q_0=1$); and the following inequality holds:

(15) $$1/q_i(q_i+q_{i+1})<|x-p_i/q_i|<1/q_i\cdot q_{i+1}, \qquad i=1,2,3,\ldots$$

Finally it is well known that all rational numbers which approximate well a given irrational number x are contained in the sequence of the partial fractions of number x. More exactly

(16) if $|x-r/s|<1/2s^2$ where r,s $(s>0)$ are integers, then $r/s=p_i/q_i$ for a suitable i.

We suppose through this section that the function $\varphi(q)$ is defined for $q\geqslant 1$ and fulfils the following conditions:

(17) the function $q\cdot\varphi(q)$ does not increase,

(18) $0<q^2\cdot\varphi(q)\leqslant 1$ for $q\geqslant 1$.

Let us define the set $B_{\varphi(q)}$. *A sequence $\{b_i\}\epsilon\tilde{B}$ belongs to the set $B_{\varphi(q)}$* if and only if there is a function $\delta(q)$ defined for $q\geqslant 1$, $\delta(q)\geqslant 1$, $\delta(q)\to\infty$ steadily as $q\to\infty$ and a sequence of positive integers $t_1<t_2<t_3<\ldots$,

$$t_{i+1}>1/t_i\varphi(t_i\delta(t_i)), \qquad i=1,2,3,\ldots,$$

such that the series

(19) $$\sum_{i=1}^{\infty} t_i b_{[1/t_i\varphi(t_i\delta(t_i))]}$$

diverges. Here $[q]$ is the greatest integer less than or equal to q.

We state

THEOREM 3. $\alpha(B_{\varphi(q)})=Y_{\varphi(q)}$.

We start from the following

LEMMA 9. *If $\{b_i\}\epsilon B_{\varphi(q)}$ and $y\epsilon Y_{\varphi(q)}$, then*

$$\mu\Big(\prod_{k=1}^{\infty}\sum_{i=k}^{\infty} I[iy-b_i, iy+b_i]\Big)>0.$$

Proof. Let us fix a sequence $\{b_i\}\epsilon B_{\varphi(q)}$ and a number $y\epsilon Y_{\varphi(q)}$. Let the function $\delta(q)$ and the sequence $\{t_i\}$ correspond to the sequence $\{b_i\}$ according to the definition of the set $B_{\varphi(q)}$.

We put $\delta'(q)=(\delta(q))^{1/2}$ and

$$b'_1=b'_2=\ldots=b'_{[1/t_1\varphi(t_1\delta(t_1))]}=b_{[1/t_1\varphi(t_1\delta(t_1))]},$$

$$b'_i=b_{[1/t_s\varphi(t_s\delta(t_s))]} \quad \text{for } [1/t_{s-1}\varphi(t_{s-1}\delta(t_{s-1}))]<i\leqslant[1/t_s\varphi(t_s\delta(t_s))], \quad s=2,3,4,\ldots,$$

and using the conditions (18) and (17) we get

$$\frac{1}{t_i\varphi(t_i\delta(t_i))}=\frac{t_i\delta^2(t_i)}{(t_i\delta(t_i))^2\varphi(t_i\delta(t_i))}\geqslant t_i\delta^2(t_i), \qquad i=1,2,\ldots,$$

$$\frac{1}{q\delta(q)\varphi(q\delta(q))}\geqslant\frac{1}{q\delta'(q)\varphi(q\delta'(q))}, \qquad q\geqslant 1.$$

We multiply the last inequality by $\delta(q)$ and get

(20) $$\frac{1}{q\varphi(q\delta(q))}\geqslant\delta'(q)\frac{1}{q\varphi(q\delta'(q))}, \qquad q\geqslant 1.$$

Lemma 9 will be proved if we prove the inequality

(21) $$\mu\Big(\prod_{k=i}^{\infty}\sum_{i=k}^{\infty} I[iy-b_i, iy+b_i]\Big)\geqslant 1/32.$$

Let p_i/q_i $(i=1,2,3,\ldots)$ be the partial fractions of number y. According to the definition of the set $Y_{\varphi(q)}$ and according to inequality (15) there is such a number $c_y\geqslant 1$ that

(22) $$q_{i+1}<1/q_i\varphi(c_y\cdot q_i) \quad \text{for} \quad i=1,2,3,\ldots$$

Let us fix an integer s_0 fulfilling the conditions

(23) $$\delta'(t_{s_0})>c_y, \qquad t_{s_0}>q_1, \qquad t_{s_0}>64, \qquad \delta'(t_{s_0})>32.$$

Let us define a sequence of indices $\{i_s\}$, $s=s_0, s_0+1,\ldots$, by means of the condition

(24.1) $$q_{i_s-1}<t_s\leqslant q_{i_s}, \qquad s=s_0,s_0+1,\ldots$$

We have $i_s>1$, $s=s_0,s_0+1,\ldots$, and according to (17), (22) and (23) we get

(24.2) $$q_{i_s}\leqslant\frac{1}{q_{i_s-1}\varphi(c_y\cdot q_{i_s-1})}\leqslant\frac{1}{t_s\varphi(c_y\cdot t_s)}\leqslant\frac{c_y}{\delta'(t_s)t_s\varphi(t_s\delta'(t_s))}\leqslant\frac{1}{t_s\varphi(t_s\delta'(t_s))}.$$

Let us suppose that inequality (21) is false. Then there is such a $k_0 \geqslant 2$ that

$$(25.1)\qquad \mu\Big(\sum_{i=k_0}^{k} I[iy-b_i', iy+b_i']\Big)<1/32 \quad \text{for} \quad k=k_0, k_0+1, \ldots$$

Let us estimate the measure of the set

$$D_s=\sum_{i=k_0}^{[q_{i_s+1}/4]} I[iy-b_i', iy+b_i'] - \sum_{i=k_0}^{[1/t_s\varphi(t_s\delta'(t_s))]} I[iy-b_i', iy+b_i']$$

for $s=s_1, s_1+1, \ldots$, where the integer s_1 fulfills the conditions $s_1 \geqslant s_0$, $t_{s_1} > 4k_0$.

We have

$$y=p_{i_s+1}/q_{i_s+1}+r, \qquad |r|<1/(q_{i_s+1})^2.$$

Hence

$$(25.2)\qquad iy \equiv p_i'/q_{i_s+1}+r_i \pmod 1,$$

$$0 \leqslant p_i' < q_{i_s+1}, \qquad |r_i|<1/4q_{i_s+1}, \qquad i=1,2,\ldots,[q_{i_s+1}/4],$$

$p_i' \neq p_j'$ for $i \neq j$, as the numbers p_{i_s+1}, q_{i_s+1} are relatively prime, and the number of the points $(iy)'$, $i=k_0, k_0+1, \ldots, [q_{i_s+1}/4]$ which are contained in the interval $I[u-d, u+d]$ does not exceed

$$(25.3)\qquad 2q_{i_s+1}\cdot 2d+1\ ^{5)}.$$

The set

$$\sum_{i=k_0}^{[1/t_s\varphi(t_s\delta'(t_s))]} I[iy-b_i', iy+b_i']$$

is the sum of a finite system of disjoint intervals. If we replace b_i' by $2b_i'$, then each of these intervals grows at most twice and consequently

$$(25.4)\qquad \mu\Big(\sum_{i=k_0}^{[1/t_s\varphi(t_s\delta'(t_s))]} I[iy-2b_i', iy+2b_i']\Big)<1/16.$$

Let L_{s+1}, $s=s_1, s_1+1, s_1+2, \ldots$, be the set of indices i fulfilling the conditions

$$k_0 \leqslant i \leqslant [q_{i_s+1}/4], \qquad (iy)' \,\bar{\epsilon} \sum_{i=k_0}^{[1/t_s\varphi(t_s\delta'(t_s))]} I[iy-2b_i', iy+2b_i'].$$

The set

$$\sum_{i=k_0}^{[1/t_s\varphi(t_s\delta'(t_s))]} I[iy-2b_i', iy+2b_i']$$

[5]) This easily follows by means of the Schubfachprinzip from (25.2) and from the fact that $(iy)' \epsilon I[u-d, u+d]$ is equivalent to

$$u-d \leqslant p_i'/q_{i_s+1}+n_i \leqslant u+d$$

if $0 \leqslant u-d$, $u+d \leqslant 1$. The case that $u-d<0$ or $u+d>1$ is also very simple.

consists of disjoint intervals the number of which does not exceed

$$(25.5)\qquad [1/t_s\varphi(t_s\delta'(t_s))]-k_0+1.$$

Using (25.3), (25.4) and (25.5) we find that the number of the elements of the set L_{s+1} is not less than

$$q_{i_s+1}/4-k_0-2q_{i_s+1}/16-\big(1/t_s\varphi(t_s\delta'(t_s))-k_0+1\big)=q_{i_s+1}/8-1/t_s\varphi(t_s\delta'(t_s))-1.$$

According to (20) we have

$$\frac{1}{t_s\varphi(t_s\delta'(t_s))} \leqslant \frac{1}{\delta'(t_s)}\cdot\frac{1}{t_s\varphi(t_s\delta(t_s))}.$$

Using the last inequality (23) we get

$$\frac{1}{t_s\varphi(t_s\delta'(t_s))} \leqslant \frac{1}{32}\cdot\frac{1}{t_s\varphi(t_s\delta(t_s))},$$

and according to the definition of the sequence t_s and according to (24.1) we get

$$(25.6)\qquad 1/t_s\varphi(t_s\delta'(t_s)) \leqslant t_{s+1}/32 \leqslant q_{i_s+1}/32.$$

It follows that the number of the elements of the set L_{s+1} is not less than

$$q_{i_s+1}/8-q_{i_s+1}/32-1=q_{i_s+1}/16+(q_{i_s+1}/32-1),$$

and finally it follows from the third inequality (23) and from (25.6) that the number of the elements of the set L_{s+1} is greater than $q_{i_s+1}/16$.

Let us now distinguish two cases:

1° $b'_{q_{i_s+1}} \geqslant 1/4q_{i_s+1}$ for a suitable $s \geqslant s_1$.

Then we have

$$I[jy-1/4q_{i_s+1}, jy+1/4q_{i_s+1}] < D_s, \qquad j \epsilon L_{s+1},$$

and these intervals are mutually disjoint. It follows that

$$\mu(D_s) \geqslant \frac{1}{16}\, q_{i_s+1}\cdot\frac{1}{2q_{i_s+1}}=\frac{1}{32},$$

which contradicts inequality (25.1).

2° $b'_{q_{i_s+1}} < 1/4q_{i_s+1}$ for $s=s_1, s_1+1, \ldots$

In this case the inclusion

$$I[jy-b_j', jy+b_j'] \subset D_s, \qquad j \epsilon L_{s+1}, \qquad s_1=s_1, s+1, \ldots,$$

holds and these intervals are again mutually disjoint. As $q_{i_{s+1}} \geqslant t_{s+1}$, it follows that

$$\mu(D_s) \geqslant t_{s+1} b'_{q_{i_{s+1}}}/16 = t_{s+1} b_{[1/t_{s+1}\varphi(t_{s+1}\delta(t_{s+1}))]}/16$$

according to (24.2); as the series (19) diverges and as the sets D_{s_2} and D_{s_3} are disjoint for $s_2 \neq s_3$, we again get a contradiction of inequality (25.1). Thus lemma 9 is proved.

The inclusion $Y_{\varphi(q)} \subset \alpha(B_{\varphi(q)})$ is an easy consequence of the following

LEMMA 10. *If* $\{b_i\} \epsilon B_{\varphi(q)}$ *and if* $y \epsilon Y_{\varphi(q)}$ *then*

$$\mu\Big(\prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[iy - b_i, iy + b_i]\Big) = 1.$$

Proof. Let us choose a

$$\xi \epsilon \prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[iy - b_i,\ iy + b_i].$$

As the sequence $\{b_i\}$ is non-increasing, we easily verify that

$$\xi - (jy)' \epsilon \prod_{k=j+1}^{\infty} \sum_{i=k}^{\infty} I[(i-j)y - b_i, (i-j)y + b_i]$$

$$\subset \prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[iy - b_i, iy + b_i], \quad j = 1, 2, 3, \ldots$$

Number y is irrational as it does not admit the approximation $\varphi(d_y \cdot q)$ for a suitable $d_y \geqslant 1$. Consequently the set J of all points $(-jy)'$, $j = 1, 2, 3, \ldots$, is dense in K. We apply lemma 5 with

$$V = J, \quad U = \prod_{k=1}^{\infty} \sum_{i=k}^{\infty} I[iy - b_i, iy + b_i] = W,$$

and the proof of lemma 10 is complete.

The converse inclusion $\alpha(B_{\varphi(q)}) \subset Y_{\varphi(q)}$ is a consequence of the following

LEMMA 11. *If* $x \bar{\epsilon} Y_{\varphi(q)}$, *then there is a sequence* $\{b_i\}$ *fulfilling the conditions*

(27) $$\{b_i\} \epsilon B_{\varphi(q)},$$

(27) $$x \bar{\epsilon} \alpha(\{b_i\}).$$

Proof. As $x \bar{\epsilon} Y_{\varphi(q)}$, there is such a function $\lambda(q)$ defined for $q \geqslant 1$, $\lambda(q) \to \infty$ steadily as $q \to \infty$, that the number x admits the approximation $\varphi(q\lambda(q))$.

Let us choose a function $\beta'(q)$ which is defined for $q \geqslant 1$ and fulfils the conditions

(29) $$\lambda(q) \geqslant \beta'(q) \geqslant 1, \quad q \geqslant \beta'(q), \quad q \geqslant 1,$$

(30) $$\beta'(q) \to \infty \quad \text{steadily as} \quad q \to \infty,$$

(31) $$\lambda(q)/\beta'(q) \to \infty \quad \text{steadily as} \quad q \to \infty,$$

(32) $$q/\beta'(q) \to \infty \quad \text{with} \quad q \to \infty.$$

Such a function $\beta'(q)$ will be found in the following manner.

We find such a number s_1 that $\big(\lambda(2)/\lambda(1)\big)^{2s_1} = 2$ if $\lambda(2)/\lambda(1) > 2$; otherwise we put $s_1 = 1/2$ and define

$$\beta'(q) = \big(\lambda(q)/\lambda(1)\big)^{s_1}, \quad 1 \leqslant q \leqslant 2.$$

Having defined $\beta'(2^n)$ we find such a number s_{n+1} that

$$\big(\lambda(2^{n+1})/\lambda(2^n)\big)^{2s_{n+1}} = 2$$

if $\lambda(2^{n+1})/\lambda(2^n) > 2$; otherwise we put $s_{n+1} = 1/2$ and define

$$\beta'(q) = \beta'(2^n)\big(\lambda(q)/\lambda(2^n)\big)^{s_{n+1}}, \quad 2^n < q \leqslant 2^{n+1},$$

We easily verify that the function $\beta'(q)$ fulfils conditions (29)-(32). Further we put

$$\delta_1(q) = \frac{\alpha\big(q/\beta'(q)\big)}{\beta'\big(q/\beta'(q)\big)}, \quad q \geqslant 1,$$

$$\delta(q) = \inf_{q \leqslant s < \infty} \delta_1(s), \quad q \geqslant 1.$$

As $\delta_1(q) \geqslant 1$ and $\delta_1(q) \to \infty$ with $q \to \infty$, we get $\delta(q) \geqslant 1$ and $\delta(q) \to \infty$ steadily with $q \to \infty$.

Finally we have

$$\delta\big(q \cdot \beta'(q)\big) \leqslant \delta_1\big(q \cdot \beta'(q)\big) = \frac{\alpha\left(\dfrac{q \cdot \beta'(q)}{\beta'\big(q \cdot \beta'(q)\big)}\right)}{\beta'\left(\dfrac{q \cdot \beta'(q)}{\beta'\big(q \cdot \beta'(q)\big)}\right)} \leqslant \frac{\lambda(q)}{\beta'(q)}.$$

We put $\beta(q) = (\beta'(q))^{1/2}$ and get

(33) $$\lambda(q) \geqslant \beta^2(q) \cdot \delta\big(q \cdot \beta(q)\big).$$

Let p'_i/q'_i $(i = 1, 2, 3, \ldots)$ be the partial fractions of the number x. According to inequality (15), using the assumption that number x admits the approximation $\varphi\big(q\alpha(q)\big)$, and using (16) and (18), we find such a sequence of indices $i_1 < i_2 < i_3 < \dots$ that the following conditions are fulfilled:

$$\beta(q_{i_1}) > 2, \qquad \sum_{n=1}^{\infty} \frac{1}{\beta(q_{i_n})} < \infty,$$

$$q_{i_n+1} > \frac{1}{2q_{i_n}\varphi\big(q_{i_n}\lambda(q_{i_n})\big)}, \qquad n = 1,2,3,\dots$$

Let us define:

$$t_n = [q_{i_n}\beta(q_{i_n})], \qquad n = 1,2,3,\dots,$$

$$b_j = 1/q_{i_1}\beta(q_{i_1}), \qquad j = 1,2,\dots, \quad [1/t_1\varphi\big(t_1\delta(t_1)\big)],$$

$$b_j = 1/q_{i_n}\cdot\beta(q_{i_n}) \quad \text{for} \quad [1/t_{n-1}\varphi\big(t_{n-1}\delta(t_{n-1})\big)] < j \leqslant [1/t_n\varphi\big(t_n\delta(t_n)\big)] \quad (n = 2,3,\dots).$$

It is apparent that $\{b_j\} \epsilon B_{\varphi(q)}$ and we have to prove that

(34) $$x \bar{\epsilon}\, \alpha\{b_j\}.$$

We easily find that

(35) $$\frac{\varphi\big(q_{i_n}\lambda(q_{i_n})\big)}{t_n\varphi\big(t_n\delta(t_n)\big)} \leqslant \frac{1}{q_{i_n}\beta^2(q_{i_n})},$$

as according to (17) and (33) we have

$$\frac{\varphi\big(q_{i_n}\lambda(q_{i_n})\big)}{t_n\varphi\big(t_n\delta(t_n)\big)} \leqslant \frac{\delta\big(q_{i_n}\beta(q_{i_n})\big)}{q_{i_n}\lambda(q_{i_n})} \cdot \frac{q_{i_n}\lambda(q_{i_n})\varphi\big(q_{i_n}\lambda(q_{i_n})\big)}{q_{i_n}\beta(q_{i_n})\delta\big(q_{i_n}\beta(q_{i_n})\big)\varphi\big(q_{i_n}\delta\big(q_{i_n}\beta(q_{i_n})\big)\big)} \leqslant \frac{1}{q_{i_n}\beta^2(q_{i_n})}\cdot 1.$$

Relation (34) will be proved if we prove that the series

$$\sum_{n=2}^{\infty} \mu\Big(\sum_{j=1+[1/t_{n-1}\varphi(t_{n-1}\delta(t_{n-1}))]}^{[1/t_n\varphi(t_n\delta(t_n))]} I[jx - b_j, jx + b_j]\Big)$$

converges.

We use lemma 2 and have

$$\sum_{n=2}^{\infty} \mu\Big(\sum_{j=1+[t_{n-1}\varphi(t_{n-1}\delta(t_{n-1}))]}^{[1/t_n\varphi(t_n\delta(t_n))]} I[jx - b_j, jx + b_j]\Big)$$

$$\leqslant \sum_{n=2}^{\infty} q_{i_n}\cdot 2\left(\frac{1}{q_{i_n}\beta(q_{i_n})} + \frac{1}{t_n\varphi\big(t_n\delta(t_n)\big)}\varphi\big(q_{i_n}\lambda(q_{i_n})\big)\right) \leqslant 4\sum_{n=2}^{\infty}\frac{1}{\beta(q_{i_n})} < \infty$$

according to (35). Thus the proof of lemma 11 is finished and theorem 3 is completely proved.

Now we define the set $Y'_{\varphi(q)}$. Let the function $\varphi(q)$ fulfil conditions (17) and (18). Number y belongs to the set $Y'_{\varphi(q)}$ if and only of there is

such a positive number $c_y \leqslant 1$ that number y does not admit the approximation $c_y\cdot\varphi(q)$. As

$$c_y\varphi(q) = \frac{c_y}{q}q\varphi(q) \geqslant \frac{c_y}{q}\cdot\frac{q}{c_y}\varphi\left(\frac{q}{c_y}\right) = \varphi\left(\frac{q}{c_y}\right)$$

for $q \geqslant 1$, we have $Y'_{\varphi(q)} \subset Y_{\varphi(q)}$.

We state

THEOREM 4. *Let the function $\varphi(q)$ fulfil conditions* (17) *and* (18) *and let us suppose that there is such a number $l > 0$ that the inequality*

(36) $$\varphi(2q) > l\varphi(q)$$

holds for $q \geqslant 1$. Then $Y'_{\varphi(q)} = Y_{\varphi(q)}$ and according to theorem 3 *we have $\alpha(B_{\varphi(q)}) = Y'_{\varphi(q)}$.*

Proof. We have to prove that $Y'_{\varphi(q)} \supset Y_{\varphi(q)}$.

Let us suppose that $x \bar{\epsilon}\, Y'_{\varphi(q)}$. Then there is such a function $\gamma(q)$, $1 \geqslant \gamma(q) > 0$, $\gamma(q) \to 0$ steadily with $q \to \infty$, that number x-admits the approximation $\gamma(q)\varphi(q)$.

By means of condition (36) we easily find such a function $\lambda(q)$, $\lambda(q) \geqslant 1$, $\lambda(q) \to \infty$ steadily with $q \to \infty$, that the inequality

(37) $$\varphi\big(q\lambda(q)\big) \geqslant \gamma(q)\varphi(q)$$

holds for $q \geqslant 1$.

Apparently we have $l < 1$. We put $v_0 = 1$ and find such a sequence of real numbers $v_0 < v_1 < v_2 < \dots$ that $\gamma(v_n) \leqslant l^n$, $n = 0,1,2,\dots$ Let $\lambda(q)$ be a non-decreasing function fulfilling conditions $\lambda(1) = 1$, $\lambda(v_n) = 2^{n-1}$, $n = 1,2,3,\dots$ Then this function $\lambda(q)$ fulfils inequality (37), as $v_n \to \infty$ with $n \to \infty$, and if $v_n \leqslant q \leqslant v_{n+1}$, then

$$\varphi\big(q\lambda(q)\big) \geqslant \varphi(q2^n) \geqslant l^n\varphi(q) \geqslant \gamma(v_n)\varphi(q) \geqslant \gamma(q)\varphi(q).$$

It follows that number x admits the approximation $\varphi\big(q\lambda(q)\big)$. Number x does not belong to the set $Y_{\varphi(q)}$ and theorem 4 is proved.

Finally, we show that theorem 1 is a special case of theorem 3.

We have to show that

(38) $$B_{1/q^2} = \tilde{B}.$$

Let us fix a sequence $\{b_i\} \epsilon \tilde{B}$. We shall find a function $\delta(q)$, $\delta(q) \geqslant 1$ for $q \geqslant 1$, $\delta(q) \to \infty$ steadily with $q \to \infty$, and a sequence of positive integers $t_1 < t_2 < t_3 < \dots$ such that $t_{n+1} > t_n\cdot\delta^2(t_n)$ and that

$$\sum_{n=1}^{\infty} t_n b_{[t_n\delta^2(t_n)]} = \infty.$$

As the series

$$\sum_{n=1}^{\infty} s^n b_{s^{n+1}} \Big(\geqslant s^{-2} \sum_{i=s^2}^{\infty} b_i \Big), \qquad s=2,3,\ldots,$$

diverges, there is such a sequence of indices $2 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \ldots$, $s_i \to \infty$ with $i \to \infty$, that the series

$$\sum_{n=1}^{\infty} (s_n)^n b_{(s_n)^{n+1}}$$

diverges.

We put $t_n = (s_n)^n$ and find such a function $\delta(q)$, $\delta(q) \geqslant 1$ for $q \geqslant 1$, $\delta(q) \to \infty$ steadily with $q \to \infty$, that $\delta(t_n) < (s_n)^{1/2}$ $(n=1,2,3,\ldots)$. It follows that

$$\sum_{n=1}^{\infty} t_n b_{[t_n \delta^2(t_n)]} = \infty, \qquad \{b_i\} \epsilon B_{1/q^2},$$

and finally $\tilde{B} \subset B_{1/q^2}$.

4. In this section we shall generalise theorem 1 to the case of s linear forms with n variables. We shall introduce the following notation.

Let E_1 be the additive group of real numbers, let K_1 be the one-dimensional toroidal group (which means that K_1 is the factor group E_1/W_1, where W_1 is the additive group of whole numbers). Let Φ_1 be the natural homomorphism of the group E_1 on the group K_1. If $x_1 \epsilon E_1$, we put $x_1' = \Phi_1(x_1)_1 \epsilon K_1$. If $\xi_1 \epsilon K_1$, let us denote by $\psi_1(\xi_1)$ such a real number that $\Phi_1(\psi_1(\xi_1)) = \xi_1$ and that $-1/2 \leqslant \psi_1(\xi_1) < 1/2$. We introduce the notion of the norm $\|\xi_1\|$ of an element $\xi_1 \epsilon K_1$: we put $\|\xi_1\| = |\psi_1(\xi_1)|$. Let

$$E_s = E_1 \times E_1 \times \ldots \times E_1 \qquad (s \text{ times}),$$

$$K_s = K_1 \times K_1 \times \ldots \times K_1 \qquad (s \text{ times}).$$

If $\xi = (\xi_1, \xi_2, \ldots, \xi_s) \epsilon K_s$ $(\xi_i \epsilon K_1, i=1,2,\ldots s)$, we define the norm of the element ξ

$$\|\xi\| = \max_{i=1,2,\ldots,s} \|\xi_i\|.$$

If $\xi, \eta \epsilon K_s$ and if n is an integer, we define $\xi + \eta$ and $n\xi$ in an obvious way. If $\eta \epsilon K_s$, $0 < d \leqslant 1/2$, we define the "cube" $C(\eta, d)$

$$C(\eta, d) = \mathop{E}_{\xi} \xi [\xi \epsilon K_s, \|\xi - \eta\| \leqslant d].$$

We denote by μ the invariant complete measure on K_s and we suppose that this measure is normed by the condition $\mu(K_s) = 1$.

Let (y_{ij}), $i=1,2,\ldots,s$, $j=1,2,\ldots,n$, be a matrix with s rows and n columns (y_{ij} are real numbers). We form the system of linear forms

$$y_{i1} q_1 + y_{i2} q_2 + \ldots + y_{in} q_n - p_i, \qquad i=1,2,\ldots,s. \tag{39}$$

Let the function $\varphi(t)$ be defined for $t \geqslant 1$, non-negative and non-increasing. We say that *system* (39) *admits the approximation* $\varphi(t)$, if for every number Q there is a point with $n+s$ integral coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_s)$ such that the inequalities

$$(40) \qquad \begin{aligned} &|y_{i1} q_1 + y_{i2} q_2 + \ldots + y_{in} q_n - p_i| \leqslant \varphi\big(\max_{j=1,2,\ldots,n} |q_j|\big), \\ &\max_{j=1,2,\ldots,n} |q_j| > Q, \qquad i = 1,2,\ldots,s, \end{aligned}$$

hold. Let

$$\eta^{(j)} = \big(\Phi_1(y_{1,j}), \Phi_2(y_{2,j}), \ldots, \Phi_1(y_{s,j})\big) \epsilon K_s, \qquad j=1,2,\ldots,n.$$

Then the system of inequalities (40) is equivalent to the inequality

$$\|q_1 \eta^{(1)} + q_2 \eta^{(2)} + \ldots + q_n \eta^{(n)}\| \leqslant \varphi\big(\max_{j=1,2,\ldots,n} |q_j|\big), \tag{41}$$

and we say that the *system of points* $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ *admits the approximation* $\varphi(q)$, if for every number Q there is such a point with n integral coordinates $(q_1, q_2, \ldots, q_n)$, $\max_{j=1,2,\ldots,n} |q_j| > Q$ that the inequality (41) holds.

It is known that each system of points $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ admits the approximation $c_{n,s} t^{-n/s}$ where the positive number $c_{n,s}$ depends on n and s only.

Let us denote by $Y_{n,s}$ the set whose elements are systems of points $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ that do not admit the approximation $cq^{-n/s}$ where c is a suitable positive number depending on the system $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$.

We denote by q the point $(q_1, q_2, \ldots, q_n)$ where the numbers q_i $(i=1,2,\ldots,n)$ are whole and put

$$\|q\| = \max_{i=1,2,\ldots,n} |q_i|.$$

Let the set $B_{n,s}$ contain all generalized sequences $\{b_{q_1,q_2,\ldots,q_n}\}$, $q_1 = \ldots -1, 0, 1, \ldots$, $q_2 = \ldots -1, 0, 1, \ldots$, $q_n = \ldots, -1, 0, 1, \ldots$, which fulfil the conditions

$$b_{q_1,q_2,\ldots,q_n} \geqslant 0, \tag{42}$$

$$b_{q_1,q_2,\ldots,q_n} \leqslant b_{q_1',q_2',\ldots,q_n'} \quad \text{if} \quad \|q\| \geqslant \|q'\|, \; q' = (q_1', q_2', \ldots, q_n'), \tag{43}$$

$$\sum (b_{q_1,q_2,\ldots,q_n})^s = \infty, \tag{44}$$

where the sum runs over all points q with n integral coordinates $q_1, q_2, \ldots, q_n$, $\infty < q_1 < \infty$, $-\infty < q_2 < \infty, \ldots, -\infty < q_n < \infty$.

According to this definition we have $b_{q_1,q_2,\ldots,q_n} = b_{q_1',q_2',\ldots,q_n'}$ if $\|q\| = \|q'\|$.

A *system* $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ *belongs to the set* $\alpha(B_{n,s})$ if for every sequence $\{_{q_1,q_2,\ldots,q_n}\} \epsilon B_{n,s}$ almost every point $\xi \epsilon K_s$ belongs to an infinite number of cubes

$$C[q_1 \eta^{(1)} + q_2 \eta^{(2)} + \ldots + q_n \eta^{(n)}, b_{q_1,q_2,\ldots,q_n}].$$

This is equivalent to the condition that

$$\mu\Big(\prod_{k=1}^{\infty}\sum_{\|q\|\geqslant k} C[q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)}, b_{q_1,q_2,\ldots,q_n}]\Big)=1$$

for every sequence $\{b_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$.

Now we are ready to state

THEOREM 5. $\alpha(B_{n,s})=Y_{n,s}$.

We shall prove the first part of theorem 5 if we prove the following

LEMMA 12. *If the system of points* $(\xi^{(1)},\xi^{(2)},\ldots,\xi^{(n)})$ *does not belong to the set* $Y_{n,s}$, *then there is such a sequence* $\{b_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$ *that the measure* μ *of the set*

$$\prod_{k=1}^{\infty}\sum_{\|q\|\geqslant k} C[q_1\xi^{(1)}+q_2\xi^{(2)}+\ldots+q_n\xi^{n}, b_{q_1,q_2,\ldots,q_n}]$$

is zero.

Proof. As the system $(\xi^{(1)},\xi^{(2)},\ldots,\xi^{(n)})$ does not belong to the set $Y_{n,s}$, there is such a sequence of points with n integral coordinates

$$q^{(j)}=(q_1^{(j)},q_2^{(j)},\ldots,q_n^{(j)}) \tag{45}$$

that

$$\|q_1^{(j)}\xi^{(1)}+q_2^{(j)}\xi^{(2)}+\ldots+q_n^{(j)}\xi^{(n)}\|<c_j\|q^{(j)}\|^{-n/s},\qquad \|q^{(j)}\|<\|q^{(j+1)}\|,\qquad j=1,2,3,\ldots,$$

where $c_j>0$, $c_j\to 0$ with $j\to\infty$.

Let us choose an integer $N>\|q^{(1)}\|$ and a number $b>0$ and let us estimate the measure of the set

$$H(N,b)=\sum_{\|q\|\leqslant N} C[q_1\xi^{(1)}+q_2\xi^{(2)}+\ldots+q_n\xi^{(n)}, b].$$

Let us choose such an index j that $\|q^{(j)}\|<N$. Without loss of generality we may suppose that $\|q^{(j)}\|=q_1^{(j)}$. Then we have the inclusion [6])

$$H(N,b)\subset\sum C[q_1\xi^{(1)}+q_2\xi^{(2)}+\ldots+q_n\xi^{(n)}, b+N\|q^{(j)}\|^{-1}c_j\|q^{(j)}\|^{-n/s}] \tag{46.1}$$

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[6]) The inclusion (46.1) is a consequence of the fact that all points $q_1'\xi^{(1)}+q_2'\xi^{(2)}+\ldots+q_n'\xi^{(n)}$, $(q_1',q_2',\ldots,q_n')=q'$, $\|q'\|\leqslant N$, belong to the set

$$\sum C[q_1\xi^{(1)}+q_2\xi^{(2)}+\ldots+q_n\xi^{(n)}, N\|q^{(j)}\|^{-1}c_j\|q^{(j)}\|^{-n/s}] \tag{46.2}$$

where the sum runs over the same points q as in the inclusion (46.1).

Let us fix a point $q_1'\xi^{(1)}+q_2'\xi^{(2)}+\ldots+q_n'\xi^{(n)}$, $\|q'\|\leqslant N$, and let us suppose that $|q_1'|\geqslant\|q^{(j)}\|-1$. We find such an integer m that $|q_1'+mq_1^{(j)}|\leqslant\|q^{(j)}\|-1$. As we have assumed that $|q_1^{(j)}|=\|q^{(j)}\|$, it follows that $|m|\leqslant N/\|q^{(j)}\|$. The norm of the difference of the points $q_1'\xi^{(1)}+q_2'\xi^{(2)}+\ldots+q_n'\xi^{(n)}$ and $(q_1'+mq_1^{(j)})\xi^{(1)}+(q_2'+mq_2^{(j)})\xi^{(2)}+\ldots+(q_n'+mq_n^{(j)})\xi^{(n)}$ does not exceed $|m|c_j\|q^{(j)}\|^{-n/s}\leqslant N\|q^{(j)}\|^{-1}c_i\|q^{(j)}\|^{-n/s}$, and the inequalities $|q_1'+mq_1^{(j)}|\leqslant\|q^{(i)}\|-1$, $|q_i'+mq_i^{(j)}|\leqslant|q_i'|+|m|\|q^{(j)}\|\leqslant 2N$, $i=1,2,\ldots,n$, hold. That means that the point $q_1'\xi^{(1)}+q_2'\xi^{(2)}+\ldots+q_n'\xi^{(n)}$ belongs to the set (46.2).

where the sum runs over all points q with n integral coordinates $q_1,q_2,\ldots,q_n$ which fulfil the conditions

$$-\|q^{(j)}\|\leqslant q_1\leqslant\|q^{(j)}\|-1,\qquad -2N\leqslant q_2\leqslant 2N,\ \ldots,\qquad -2N\leqslant q_n\leqslant 2N.$$

From inclusion (46.1) we get

$$\mu(H(N,b))\leqslant 2\|q^{(j)}\|(6N)^{n-1}(b+N\|q^{(i)}\|^{-1}c_j\|q^{(j)}\|^{-n/s})^s. \tag{46.3}$$

Let us suppose that

$$1>c_j>2^{(n+s)/s}\cdot c_{j+1},\qquad j=1,2,\ldots \tag{47}$$

(If this condition is not fulfilled, we choose a suitable subsequence from sequence (45)). We shall further suppose without loss of generality that the numbers $c_j^{-s/(n+s)}$, $j=1,2,3,\ldots$, are integers.

Let us define $b_{q_1,q_2,\ldots,q_n}=1$ if $\|q\|\leqslant\|q^{(1)}\|\cdot c_1^{-s/(n+s)}$

$$b_{q_1,q_2,\ldots,q_n}=\|q_i^{(j)}\|_i^{-n/s}c_j^{n/(n+s)}$$

if

$$N_{j-1}=\|q^{(j-1)}\|\cdot c_{j-1}^{-s/(n+s)}<\|q\|\leqslant\|q^{(j)}\|\cdot c_j^{-s/(n+s)}=N_j,\qquad j=1,2,3,\ldots$$

First of all we verify that the series $\sum_q(b_{q_1,\ldots,q_n})^s$ diverges (the sum runs over all points q with n integral coordinates). We have

$$\begin{aligned}\sum_q(b_{q_1,q_2,\ldots,q_n})^s&\geqslant\sum_{j=2}^{\infty}\sum_{N_{j-1}<\|q\|\leqslant N_j}(b_{q_1,q_2,\ldots,q_n})^s\\&\geqslant\sum_{j=2}^{\infty}\|q^{(j)}\|^{-n}c_j^{ns/(n+s)}[(2\|q^{(j)}\|c_j^{-s/(n+s)}+1)^n-(2\|q^{(j-1)}\|c_{j-1}^{-s/(n+s)}+1)^n]\\&\geqslant\sum_{j=2}^{\infty}\|q^{(j)}\|^{-n}c_j^{ns/(n+s)}[\|q^{(j)}\|^n c_j^{-ns/(n+s)}-\|q^{(j-1)}\|^n c_{j-1}^{-n/(n+s)}]\\&\geqslant\sum_{j=2}^{\infty}[1-(c_j/c_{j-1})^{ns/(n+s)}]\geqslant\sum_{j=2}^{\infty}[1-2^{-n}]=\infty.\end{aligned}$$

We have proved that the sequence $\{b_{q_1,q_2,\ldots,q_n}\}$ fulfils condition (44). It is obvious from the definition of the sequence $\{b_{q_1,q_2,\ldots,q_n}\}$ that conditions (42) and (43) are also satified. Consequently the sequence $\{b_{q_1,q_2,\ldots,q_n}\}$ belongs to the set $B_{n,s}$. According to (46.3) and (47) we have

$$\begin{aligned}&\sum_{j=2}^{\infty}\mu\Big(\sum_{N_{j-1}<\|q\|\leqslant N_j}C[q_1\xi^{(1)}+q_2\xi^{(2)}+\ldots+q_n\xi^{(n)}, b_{q_1,q_2,\ldots,q_n}]\Big)\\&\leqslant\sum_{j=2}^{\infty}\mu\big(H(N_j,\|q^{(j)}\|^{-n/s}\cdot c_j^{n/(n+s)})\big)\\&\leqslant\sum_{i=2}^{\infty}2\|q^{(j)}\|\cdot(6\|q^{(j)}\|\cdot c_j^{-s/(n+s)})^{n-1}(\|q^{(j)}\|^{-n/s}\cdot c_j^{n/(n+s}+c_j^{-s/(n-s)}\cdot c_j\|q^{(j)}\|^{-n/s})^s\\&\leqslant\sum_{j=2}^{\infty}6^{n-1}\cdot 2^{s+1}\cdot c_j^{s/(n+s)}<\infty,\quad\text{q. e. d.}\end{aligned}$$

In order to prove the second part of theorem 5 we shall need some lemmas. First of all we state the following

DENSITY THEOREM. *Almost all points of a measurable set $A\subset K_s$ are points of outer density for A and almost all points of the complement of the set A are points of dispersion of the set A.*

This theorem holds in E_s (Saks [4], Chapter IV, § 10) and it is apparent that it also holds in K_s.

From the Density Theorem we easily deduce the following

LEMMA 13. *Let $U\subset K_s$, $V\subset K_s$, $W=U\oplus V$ (which means that the set W contains all points $\zeta=\xi+\eta$ where $\xi\in U,\eta\in V$).*

If $\mu(U)>0$ and if the set V is dense in K_s, then $\mu(W)=1$.

As another consequence of the Density Theorem we shall prove the following

LEMMA 14. *If $(\eta^{(1)},\eta^{(2)},\ldots,\eta^{(n)})\in Y_{n,s}$, then the set*

$$q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)} \tag{48}$$

where $q=(q_1,q_2,\ldots,q_n)$ runs over all points with n integral coordinates $-\infty<q_1<\infty$, $-\infty<q_2<\infty$, ..., $-\infty<q_n<\infty$ is dense in K_s.

Proof. Let us fix a system $(\eta^{(1)},\eta^{(2)},\ldots,\eta^{(n)})\in Y_{n,s}$ and let us consider the points

$$q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)} \tag{49}$$

where

$$\max_{j=1,2,\ldots,n}|q_j|\leqslant 2^{ks}$$

and k is a positive integer which we shall choose later.

According to the definition of the set $Y_{n,s}$ there is such a positive constant c that the norm of the difference of any two different points of set (49) is greater than

$$c/2^{nk}>1/2^{nk+c_1}$$

where c_1 is a suitable positive integer. Let

$$\zeta^{(m)}_{t_1,t_2,\ldots,t_s}=\big(\Phi_1(t_1/2^m),\Phi_1(t_2/2^m),\ldots,\Phi_1(t_s/2^m)\big)$$

where $t_1,t_2,\ldots,t_s$ and m are integers and let us consider the set of cubes

$$C\big(\zeta^{(nk+c_1)}_{t_1,t_2,\ldots,t_s},1/2^{(nk+c_1+1)}\big) \tag{50.1}$$

where $t_1,t_2,\ldots,t_s$ run over all integers

$$0\leqslant t_1<2^{kn+c_1},\quad 0\leqslant t_2<2^{kn+c_1},\quad \ldots,\quad 0\leqslant t_s<2^{kn+c_1}.$$

Set (50.1) contains 2^{kns+c_1s} cubes which cover K_s, and as every cube of this set contains at most one point of set (49), it follows that the number of the cubes of (50,1) which contain a point of set (49) exceeds 2^{kns}.

If lemma 14 is false, then there is a cube $C[\xi,1/2^{r+1}]$ that contains no point of the set (48). (We may suppose that $\xi=\xi^{(r)}_{t_1,t_2,\ldots,t_s}$ where $r,t_1,t_2,\ldots,t_s$ are positive integers).

If $\zeta\in K_s$ and if m is a positive integer, we denote by $\zeta/2^m$ one of the points $\zeta'\in K_s$ which fulfil the equation $2^m\zeta'=\zeta$.

If $A\subset K_s$ we denote by $A/2^m$ the set of all points ζ' fulfilling the relation $2^m\xi'\in A$.

We easily verify that

$$2^{-m}C[\xi,1/2^{r+1}]=\sum C[2^{-m}\xi+\zeta^{(m)}_{t_1,t_2,\ldots,t_s},1/2^{r+m+1}]$$

where the sum runs over all systems of integers $t_1,t_2,\ldots,t_s$, $0\leqslant t_1<2^m$, $0\leqslant t_2<2^m,\ldots,$ $0\leqslant t_s<2^m$.

As the points of set (48) form a group, the set

$$2^{-m}C[\xi,1/2^{n+1}]$$

contains no point of this set. Apparently each cube

$$C[\eta,1/2^{m-1}],\qquad m\geqslant 2,$$

contains a cube

$$C[2^{-m}\xi+\zeta^{(m)}_{t_1,t_2,\ldots,t_s},1/2^{m+r-1}].$$

It follows that no point of K_s is a point of dispersion for the set

$$\sum_{m=1}^{\infty}2^{-m}C[\xi,1/2^{r+1}]$$

and that

$$\mu\Big(\sum_{m=1}^{\infty}2^{-m}C[\xi,1/2^{r+1}]\Big)=1.$$

Consequently there is such a positive integer k' that

$$\mu\Big(\sum_{m=1}^{k'}2^{-m}C[\xi,1/2^{r+1}]\Big)>1-2^{-c_1s} \tag{50.2}$$

and that $k'+r+1=nk+c_1+1$ (where the last equation defines the number k occurring in (49) and (50)).

It follows that the set

$$\sum_{m=1}^{k'}2^{-m}C[\xi,1/2^{r+1}]$$

is a sum of the cubes of set (50.1).

Let us denote by N the number of these cubes. According to (50.2) we have the inequalities

$$N\cdot 2^{-(kn+c_1)s}\geqslant 1-2^{-c_1 s}$$

and

$$N\geqslant 2^{(kn+c_1)s}-2^{kns}.$$

As the set

$$\sum_{m=1}^{k'} 2^{-m}C[\xi,1/2^{r+1}]$$

contains no point of set (48) we get a contradiction of the statement that at least 2^{kns} cubes of set (50.1) contain a point of set (49).

Further we shall need the following

LEMMA 15. *Let us have*

$$D=C[\xi^{(1)},b_1]+C[\xi^{(2)},b_2]+\ldots+C[\xi^{(k)},b_k]\subset K_s$$

where $\xi^{(1)},\xi^{(2)},\ldots,\xi^{(k)}\in K_s$, $1/4\geqslant b_1\geqslant b_2\geqslant\ldots\geqslant b_k>0$.

Let

$$F=C[\xi^{(1)},2b_1]+C[\xi^{(2)},2b_2]+\ldots+C[\xi^{(k)},2b_k].$$

Then we have $\mu(F)\leqslant 2^s\mu(D)$.

Proof. We shall prove this lemma by means of induction. Lemma 15 is obviously true for $k=1$. Let this lemma hold for $k-1$ and let us define the transformation T

$$T(\xi)=2(\xi-\xi^{(k)}),\qquad \xi\in K_s.$$

If G is a subset of K_s then by $T(G)$ we mean the set of all points $T(\xi)$ where $\xi\in G$. Let

(50.3) $$G=C[\xi^{(k)},b_k]-\sum_{j=1}^{k-1}C[\xi^{(j)},b_i].$$

Apparently

(50.4) $$\mu(T(G))=2^s\mu(G).$$

We easily find that

$$F=\sum_{j=1}^{k-1}C[\xi^{(j)},2b_j]+T(G).$$

It follows that

$$\mu(F)\leqslant\mu\Big(\sum_{j=1}^{k-1}C[\xi^{(j)},2b_j]\Big)+\mu(T(G)).$$

According to the assumption that lemma 15 holds for $k-1$ and according to (50.4) we get

$$\mu(F)\leqslant 2^s\mu\Big(\sum_{j=1}^{k-1}C[\xi^{(j)},b_j]\Big)+2^s\mu(G).$$

Using (50.3) we get

$$\mu(F)\leqslant 2^s\mu\Big(\sum_{j=1}^{k-1}C[\xi^{(j)},b_j]+G\Big)=2^s\mu(D)$$

and the proof of lemma 15 is complete.

After these preliminaries we return to the proof of the second part of theorem 5. We shall prove the following

LEMMA 16. *If* $\{b_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$ *and if* $(\eta^{(1)},\eta^{(2)},\ldots,\eta^{(n)})\in Y_{n,s}$ *then*

$$\mu\Big(\prod_{k=1}^{\infty}\sum_{\|q\|\geqslant k}C[q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)},b_{q_1,q_2,\ldots,q_n}]\Big)>0.$$

Proof. Let us fix a system $(\eta^{(1)},\eta^{(2)},\ldots,\eta^{(n)})\in Y_{n,s}$. According to our assumptions this system does not admit the approximation $ct^{-n/s}$ where $0<c\leqslant 1$. Let us choose a sequence $\{b_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$. Let k be a positive integer which fulfils the inequality

(51) $$k>2^{2n+2}.$$

Let

$$b''_{q_1,q_2,\ldots,q_n}=\min(b_{q_1,q_2,\ldots,q_n},1/4,(1/2)(2\|q\|)^{-n/s}),$$

$$b'_{q_1,q_2,\ldots,q_n}=b''_{k^j,0,\ldots,0}\quad\text{if}\quad k^{j-1}<\|q\|\leqslant k^j\qquad (j=2,3,4,\ldots),$$

$$b'_{q_1,q_2,\ldots,q_n}=b''_{k,0,\ldots,0}\quad\text{if}\quad \|q\|\leqslant k.$$

It is easy to verify that $\{b''_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$ and that $\{b'_{q_1,q_2,\ldots,q_n}\}\in B_{n,s}$ as

$$\sum_{q}(b'_{q_1,q_2,\ldots,q_n})^s\geqslant k^{-n}\sum_{\|q\|>k}(b''_{q_1,q_2,\ldots,q_n})^s\ ^{7)}.$$

Lemma 16 will be proved if we prove that for every $t\geqslant 1$

(52) $$\mu\Big(\sum_{\|q\|>t}C[2_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)},b'_{q_1,q_2,\ldots,q_n}]\Big)\geqslant 2^{-5s}c^s.$$

Let us suppose that there is such an integer $t_0=k^{j_0}$, $j_0\geqslant 1$ that inequality (52) is false if $t=t_0$.

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[7]) $$\frac{(2k^{j+1}+9)^n-(2k^j+9)^n}{(2k^{j+2}+9)^n-(2k^{j+1}+9)^n}\geqslant\frac{9}{k^n}$$

as $k^n(2k^{j+1}+9)-(2k^{j+2}+9)^n\geqslant k^n(2k^j+9)^n-(2k^{j+1}+9)^n$, $j=0,1,2,\ldots$

Let

$$G_j = \sum_{k^j \geqslant \|q\| > t_0} C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, b_{q_1,q_2,\ldots,q_n}]$$

$$H_j = \sum_{k^j \geqslant \|q\| > t_0} C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, 2b_{q_1,q_2,\ldots,q_n}] \quad (j = j_0+1, j_0+2, \ldots).$$

We shall estimate the measure of the set $G_{j+1} - G_j$, $j = j_0+1, j_0+2, \ldots$ Let us denote by L_{j+1} the set of points $q = (q_1, q_2, \ldots, q_n)$ (q_j integers) fulfilling the conditions

$$k^{j+1} \geqslant \|q\| > k^j, \quad q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)} \bar{\epsilon} H_j.$$

Let us denote by J_{j+1} the set of points $q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}$ where $k^j < \|q\| \leqslant k^{j+1}$, $j = j_0+1, j_0+2, \ldots$

If $\zeta, \zeta' \epsilon J_{j+1}$, then

$$\|\zeta - \zeta'\| \geqslant c(2k^{j+1})^{-n/s}.$$

It follows that the set $C[\xi, b] \frown S_{j+1}$ does not contain more than

$$(2bc^{-1}(2k^{j+1})^{n/s} + 1)^s$$

elements.

Further let

$$H_j^{(1)} = \sum C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, 2b'_{q_1,q_2,\ldots,q_n}]$$

where the sum runs over all points $q = (q_1, q_2, \ldots, q_n)$ (q_j integers) fulfilling the conditions

$$k^j \geqslant \|q\| > t_0, \quad 2b'_{q_1,q_2,\ldots,q_n} \geqslant c(2k^{j+1})^{-n/s},$$

and

$$H_j^{(2)} = \sum C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, 2b'_{q_1,q_2,\ldots,q_n}]$$

where the sum runs over all points $q = (q_1, q_2, \ldots, q_n)$ (q_j integers) fulfilling the conditions:

$$k^j \geqslant \|q\| > t_0, \quad 2b'_{q_1,q_2,\ldots,q_n} < c(2k^{j+1})^{-n/s}.$$

According to our assumption that inequality (52) is false we have $\mu(G_j) < 2^{-5s}c^s$. Using lemma 15 we get $\mu(H_j^{(1)}) < 2^s \cdot 2^{-5s} \cdot c^s = 2^{-4s} \cdot c^s$ and $\mu(H_j^{(1)}) < 2^{-4s} \cdot c^s$.

The set $H_j^{(1)}$ has the following property: if $\xi \epsilon H_j^{(1)}$ then there is such a point $\tau(\xi) \epsilon K_s$ that we have the relation

$$\xi \epsilon C[\tau(\xi), (1/8)c(2k^{j+1})^{-n/s}] \subset H_j^{(1)}.$$

If $\xi^{(1)}, \xi^{(2)} \epsilon J_{j+1} \frown H_j^{(1)}$, then the cubes

$$c[\tau(\xi^{(1)}, (1/8)c(2k^{j+1})^{-n/s}], \quad C[\tau(\xi^{(2)})(1/8)c(2k^{j+1})^{-n/s}]$$

are necessarily disjoint. Consequently the number of the elements of the set $J_{j+1} \frown H_j^{(1)}$ does not exceed

$$2^{-4s}c^s(1/4)c(2k^{j+1})^{-n/s})^{-s} = 2^{n-2s} \cdot k^{(j+1)n}.$$

The set $J_{j+1} \frown H_j^{(2)}$ apparently does not contain more than $(2k^j+1)^n$ elements. According to the inclusion

$$J_{j+1} \frown H_j \subset J_{j+1} H_j^{(1)} + J_{j+1} \frown H_j^{(2)}$$

we find that the number of the points of the set $J_{j+1} \frown H_j$ does not exceed

$$2^{n-2s}k^{(j+1)n} + (2k^{j+1})^n.$$

It follows that the number of the elements of the set L_{j+1} is greater than

$$(2k^{j-1})^n - 2(2k^j+1)^n - 2^{n-2s}k^{(j+1)n} \geqslant k^{(j+1)n}(2^n - 2(3/k)^n - 2^{n-2s}) \geqslant k^{(j+1)n}$$

(according to (51)).

As the inclusion

$$G_{j+1} - G_j \supset \sum_{q \epsilon L_{j+1}} C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, b'_{q_1,q_2,\ldots,q_n}]$$

holds and as the inequality $b'_{q_1,q_2,\ldots,q_n} \leqslant c(2\|q\|)^{-n/s}/2$ implies that the cubes that occur on the right side of this inclusion are mutually disjoint, we get

$$\mu(G_{j+1} - G_j) \geqslant \sum_{q \epsilon L_{j+1}} (b'_{q_1,q_2,\ldots,q_n})^s$$

$$\geqslant k^{(j+1)n}(b''_{k^{j+k},0,\ldots,0})^s \geqslant 2^{-2n} \sum_{k^j < \|q\| \leqslant k^{j+1}} (b''_{k^{j+1},0,\ldots,0})^s$$

$$= 2^{-2n} \sum_{k^j < \|q\| \leqslant k^{j+1}} (b'_{q_1,q_2,\ldots,q_n})^s.$$

As the series $\sum_q (b'_{q_1,q_2,\ldots,q_n})^s$ diverges, we get a contradiction, which proves inequality (52) and lemma 16.

Let us now finish the proof of the second part of theorem 5. We shall prove the following

LEMMA 17. *If* $(\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)}) \epsilon Y_{n,s}$ *and if* $\{b_{q_1,q_2,\ldots,q_n}\} \epsilon B_{n,s}$, *then*

$$\mu\Big(\prod_{k=1}^{\infty} \sum_{\|q\| \geqslant k} C[q_1\eta^{(1)} + q_2\eta^{(2)} + \ldots + q_n\eta^{(n)}, b_{q_1,q_2,\ldots,q_n}]\Big) = 1.$$

Proof. We fix a sequence $\{b_{q_1,q_2,\ldots,q_n}\} \epsilon B_{n,s}$. We find such a sequence $\{b^{(1)}_{q_1,q_2,\ldots,q_n}\} \epsilon B_{n,s}$ that

$$\frac{b^{(1)}_{q_1,q_2,\ldots,q_n}}{b_{q_1,q_2,\ldots,q_n}} \to 0 \quad \text{with} \quad \|q\| \to \infty.$$

As the series $\sum_q b^{(1)}_{sq_1,sq_2,\ldots,sq_n}$ diverges for $s=1,2,3,\ldots$, there is such a sequence of positive integers $s_0\leqslant s_1\leqslant s_2\leqslant\ldots$, $s_i\to\infty$ with $i\to\infty$, that the series

$$\sum_q b^{(1)}_{s_{\|q\|}\cdot q_1,s_{\|q\|}\cdot q_2,\ldots,s_{\|q\|}q_n}$$

diverges. Let

$$b^{(2)}_{q_1,q_2,\ldots,q_n}=b^{(1)}_{s_{\|q\|}\cdot q_1,s_{\|q\|}\cdot q_2,\ldots,s_{\|q\|}\cdot q_n}.$$

It is apparent that $\{b^{(2)}_{q_1,q_2,\ldots,q_n}\}\epsilon B_{n,s}$ and that

$$\frac{b^{(2)}_{q_1,q_2,\ldots,q_n}}{b_{q_1+q'_1,q_2+q'_2,\ldots,q_n+q'_n}}\to 0$$

if q' is a fixed point and $\|q\|\to\infty$. Consequently we have

$$\mu\Big(\prod_{k=1}^{\infty}\sum_{\|q\|\leqslant k}C[q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)},b^{(2)}_{q_1,q_2,\ldots,q_n}]\Big)>0$$

according to lemma 16.

Further if

$$\xi\,\epsilon\prod_{k=1}^{\infty}\sum_{\|q\|\geqslant k}C[q_1\eta^{(1)}+q_2\eta^{(2)}+\ldots+q_n\eta^{(n)},b^{(2)}_{q_1,q_2,\ldots,q_n}],$$

then

$$\xi+q'_1\eta^{(1)}+\ldots+q'_n\eta^{(n)}\,\epsilon\prod_{k=1}^{\infty}\sum_{\|q\|\geqslant k}C[q_1\eta^{(1)}+\ldots+q_n\eta^{(n)},b_{q_1,q_2,\ldots,q_n}].$$

According to lemma 14 the set $q'_1\eta^{(1)}+q'_2\eta^{(2)}+\ldots+q'_n\eta^{(n)}$ where q' runs over all points with n integral coordinates is dense in K_s, and we finish the proof of lemma 17 applying lemma 13. As lemma 17 is equivalent to the inclusion $Y_{n,s}\subset\alpha\{B_{n,s}\}$ theorem 5 is completely proved.

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Sur le mouvement plan d'un liquide visqueux compressible

par

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1. Je présente dans ce mémoire les formules qui expriment les composantes de la force exercée sur un courbe S par un liquide visqueux, compressible, entourant cette courbe. Ces formules sont une généralisation des résultats de W. Wolibner [2] concernant le cas du liquide visqueux incompressible[1]).

Soit S une courbe plane, simple, fermée qui sans se déformer se déplace parallèlement à une droite avec une vitesse constante, égale à U dans un liquide visqueux, compressible, remplissant tout le plan à l'extérieur de S. J'admets pour simplifier que la courbe S possède partout une tangente continue. Soit XY un système de coordonnées liées à la courbe S, ayant l'axe X parallèle à la vitesse de S; soient u,v les composantes de la vitesse du liquide par rapport au système immobile $\bar{X}\bar{Y}$, parallèle au système XY; soient p,ϱ,μ,ν la pression, la densité et les coefficients de viscosité du liquide. J'admets que les forces extérieures n'existent pas.

Les équations du mouvement et l'équation de continuité ont la forme

$$\varrho\left(\frac{\partial u}{\partial t}+(u-U)\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\nu\frac{\partial\Theta}{\partial x}+\mu\Delta u,$$

(1)
$$\varrho\left(\frac{\partial v}{\partial t}+(u-U)\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\nu\frac{\partial\Theta}{\partial y}+\mu\Delta v,$$

$$\frac{\partial\varrho}{\partial t}+\frac{\partial}{\partial x}[\varrho(u-U)]+\frac{\partial}{\partial y}(\varrho v)=0,$$

où
$$\Theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y},\qquad \Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

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[1]) Cf [1]. P. Udeschini a démontré qu'il n'existe pas de mouvement permanent d'un liquide visqueux, compressible, entourant un corps solide et y adhérant, qui serait régulier à l'intérieur du liquide et tel qu'à l'infini serait satisfaite la condition $\lim_{r\to\infty}r^2(v-v_\infty)=0$.