Linear operations in Saks spaces (II)

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3.1. We shall prove now some general theorems concerning the Saks spaces [1]).

Let X denote the fundamental Banach space or an incomplete Banach space with the norm $\| \ \|$, and let $\| \ \|^*$ be a B- or F-norm, defining together with the former norm the Saks space X_s.

(A) *Let us denote by X_0 the space composed of the elements of X, with the norm*

$$\|x\|_0 = \|x\| + \|x\|^*; \tag{1}$$

then X_0 is a Fréchet space.

(A') *If we apply, in the definition of a Saks space, the norm $\|x\|_0$ as the "starred" norm, then the sphere $\|x\| \leqslant 1$ forms a Saks space $(X_0)_s$ satisfying conditions (Σ_1), (Σ_2) and (Σ_2').*

(A'') *If the set X^* is dense with respect to the norm $\| \ \|_0$ in the sphere $\|x\| \leqslant 1$, then it is dense in X_s.*

To prove (A'), let us suppose that $\|x_n\| \leqslant 1$, $\|x_n - x_m\|_0 \to 0$ as $n, m \to \infty$. The space X_s being complete, there exists an element $x_0 \in X_s$ such that $x_n \xrightarrow{l} x_0$; since (as we have noticed in 2.2, p. 266) the space X is such that $y_n \xrightarrow{l} y_0$ implies

$$\varliminf_{n\to\infty} \|y_n\| \geqslant \|y_0\|,$$

we see that $x_n - x_m \xrightarrow{l} x_n - x_0$ implies

$$\varliminf_{m\to\infty} \|x_n - x_m\| \geqslant \|x_n - x_0\|,$$

[1]) This is the second part of the paper [6] which will be denoted by the abbreviation LO(I). Subsequently some definitions and notations of LO(I) will be used. The results of this paper, as those of LO(I), were presented on 26th September 1948 to the VI Congress of Polish Mathematicians.

and since $\|x_n-x_m\|\to 0$ as $n,m\to\infty$, we see that $\|x_n-x_0\|\to 0$ and, since $\|x_n-x_0\|^*\to 0$, we get $\|x_n-x_0\|_0\to 0$. Thus the space $(X_0)_s$ is complete.

To prove that the space $(X_0)_s$ satisfies conditions $(\Sigma_1), (\Sigma_2)$, and (Σ_2'), it is sufficient in virtue of 1.32 to state that condition (Σ_1) is satisfied at every point of a dense set in X_s. For this purpose, let us consider the sphere $\|x-x_0\|_0\leqslant\varrho$, $\|x\|\leqslant 1$, $\|x_0\|\leqslant 1$. The element $y=(1-\alpha)x_0$, $0<\alpha<1$, lies in this sphere for sufficiently small α. Condition (Σ_1) is satisfied at the point y, because $\|x\|_0<\delta=1-\|y\|$ implies $y+x\,\epsilon\,X_0$.

Statement (A) immediately results from the completeness of the space $(X_0)_s$.

To prove (A″) let us notice that if $\|x_0\|<1$, then there exists $\bar{x}\,\epsilon\,X^*$ such that $\|\bar{x}-x_0\|_0<\varrho(1-\|x_0\|)$ for $0<\varrho<1$. Hence $\|\bar{x}\|<1$, $\|\bar{x}-x_0\|^*<\varrho$, and since the set of the elements x_0 satisfying $\|x_0\|<1$ is obviously dense in X_s, the set X^* is dense in X_s.

(B) *If $x_n\,\epsilon\,X$ and $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$, then X is a Banach space with respect to the norm $\|\ \|$, and conversely.*

Only completeness with respect to the norm $\|\ \|$ is to be proved. By hypothesis, $\|x_m-x_n\|\to 0$ as $m,n\to\infty$ implies $\|x_m-x_n\|^*\to 0$, whence also $\|x_m-x_n\|_0\to 0$ (where $\|\ \|_0$ is the norm defined by (1)). It suffices to notice that the space X is complete with respect to the norm $\|\ \|_0$ in virtue of (A).

Suppose now t X, with the norm $\|\ \|$, is a Banach space. From (A) it follows th is complete also with respect to the norm $\|\ \|_0$; moreover, $\|x_n\|_0\to 0$ plies $\|x_n\|\to 0$, whence, by a well-known theorem of Banach, $\|x_n\|\to 0$ implies $\|x_n\|_0\to 0$, which in turn implies $\|x_n\|^*\to 0$.

A-, *B*- or *F*-norm $\|\ \|_1$ is called *equivalent to the norm* $\|\ \|_2$ if $\|x_n\|_1\to 0$ implies $\|x_n\|_2\to 0$, and, conversely, $\|x_n\|_2\to 0$ implies $\|x_n\|_1\to 0$.

(C) *Each of the following conditions is necessary and sufficient in order that the norm $\|\ \|^*$ be equivalent to the B-norm $\|\ \|$:*

(a) *X is a Banach space with respect to the norm $\|\ \|$ and $\|x_n\|^*\to 0$ implies $\|x_n\|\to 0$,*

(b) *X is a Fréchet space with respect to the norm $\|\ \|^*$ and $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$,*

(c) *X is a Banach space with respect to the norm $\|\ \|$ and the set $X^*=\underset{x}{E}\{\|x\|<1\}$ is of the second category in X_s.*

Ad (a) and (b). If the norm $\|\ \|^*$ is equivalent to $\|\ \|$, then the norm $\|\ \|_0$ is equivalent to $\|\ \|$ and $\|\ \|^*$; by 3.1 (A) X is a Banach space with the norm $\|\ \|^*$, whence (a) and (b) are necessary.

If (a) is satisfied, then $\|x_n\|^*\to 0$ implies $\|x_n\|\to 0$ and in virtue of 3.1 (B) $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$.

If (b) is satisfied, then by 3.1 (B) X, provided with the norm $\|\ \|$, is a Banach space, and since this space is also an *F*-space with respect to the norm $\|\ \|^*$ and since $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$, therefore also $\|x_n\|^*\to 0$ implies $\|x_n\|\to 0$, by the theorem of Banach cited above.

Ad (c). The necessity is obvious by (a). To prove the sufficiency, it is enough, in virtue of (B), to prove that $\|x_n\|^*\to 0$ implies $\|x_n\|\to 0$. Suppose that it is not so. Then there exist $x_n^0\,\epsilon\,X_s$ such that $\|x_n^0\|^*\to 0$, $\|x_n^0\|=1$. Write

$$X_n^*=\underset{x}{E}\{\|x\|\leqslant 1-1/n\}\qquad (n=2,3,\ldots),$$

and let $K(x_0,\varrho)$ be an arbitrary sphere in X_s. We shall prove that there are points y in $K(x_0,\varrho)$ such that $\|y\|=1$. If $\|x_0\|=1$, then $y=x_0$ gives the desired result; if $\|x_0\|<1$, let us choose a sequence ϑ_n of numbers such that $\|x_0+\vartheta_n x_n^0\|=1$ for $n=1,2,\ldots$ Then

$$\overline{\lim_{n\to\infty}}|\vartheta_n|<\infty$$

for $1=\|x_0+\vartheta_n x_n^0\|\geqslant\|\vartheta_n x_n^0\|-\|x_0\|=|\vartheta_n|-\|x_0\|$. It follows that $\|\vartheta_n x_n^0\|^*\to 0$, whence $y=x_0+\vartheta_n x_n^0\,\epsilon\,K(x_0,\varrho)$ if sufficiently large n is chosen. Since the set X_n^* is closed, it follows that it is non-dense, whence the set $X^*=\sum_{n=1}^{\infty}X_n^*$ is of the first category, in contradiction to the hypothesis.

(D) *Every (X_s,Y)-linear operation may be extended in a unique way to an operation $V(x)$ defined in the whole of X_0 and (X_0,Y)-linear.*

For $x\,\epsilon\,X$ let us set $V(x)=\varrho U(x/\varrho)$ where $|\varrho|\geqslant\|x\|$. Since

$$(\varrho'/\varrho)U(x/\varrho')=U(x/\varrho)\qquad\text{for}\qquad |\varrho|\geqslant|\varrho'|\geqslant\|x\|,$$

the definition of $V(x)$ does not depend on the choice of ϱ and since $\varrho U(x/\varrho)=U(x)$ if $\|x\|\leqslant 1$, $V(x)$ is an extension of $U(x)$. Let us choose $\varrho>\max(\|x\|,\|y\|,\|x+y\|)$, then

$$V(x+y)=\varrho U\big((x+y)/\varrho\big)=\varrho U(x/\varrho)+\varrho U(y/\varrho)=V(x)+V(y).$$

Let $\|x_n-x_0\|_0\to 0$; then there exists a $\varrho\geqslant\|x_n\|$ for $n=0,1,\ldots$, whence $\|x_n/\varrho-x_0/\varrho\|^*\to 0$,

$$V(x_n)=\frac{1}{\varrho}\,U(x_n/\varrho)\to\frac{1}{\varrho}\,U(x_0/\varrho)=V(x_0).$$

Finally, it is obvious that the requirement that the extended operation be (X_0,Y)-linear determines uniquely the operation $V(x)$.

3.11. Sometimes, if the norm $\|\ \|^*$ is equal to the norm $\|\ \|$, we shall say that *X_s is a Banach space.*

3.2. In the sequel $U_n(x)$ will stand for (X_s, Y)-linear operations into a Banach or Fréchet space Y. We shall denote by $\omega(x)$ the *oscillation of the sequence* $\{U_n(x)\}$ at x, *i. e.* the number defined by the formula

$$\omega(x)=\lim_{k\to\infty}\sup_{m,n\geqslant k}\|U_n(x)-U_m(x)\|.$$

We shall denote by R_σ and S_σ the sets of the points x at which $\omega(x)\geqslant\sigma$ and $\omega(x)<\sigma$ respectively (the number σ may be equal to ∞). Then S_σ is F_σ-set, since the sets

$$\underset{x}{E}\{\sup_{m,n\geqslant k}\|U_n(x)-U_m(x)\|\leqslant a\}$$

are closed for each finite a.

By C we shall denote the set of those points of X_s for which $\omega(x)=0$, *i. e.* the sets of points of convergency of the sequence $\{U_n(x)\}$; D will stand for the set of divergence of this sequence, *i. e.* $D=\underset{x}{E}\{\omega(x)\neq 0\}$.

3.21. The functional $\omega(x)$ is, in general, of Baire's second class and therefore may be discontinuous everywhere; this functional has the following properties:

(a) *$U_n(x)$ converges at x_0 if and only if $\omega(x_0)=0$; $\omega(0)=0$.*

(b) *$|\omega(x)-\omega(y)|\leqslant 2\omega((x-y)/2)$ if $\omega(x)$ and $\omega(y)$ are finite.*

(c) *If $x_1, x_2, x_1+x_2\,\epsilon\, X_s$, then $\omega(x_1+x_2)\leqslant\omega(x_1)+\omega(x_2)$.*

(d) *The sequence $\{U_n(x)\}$ is bounded at x if and only if*

$$\lim_{\vartheta\to 0}\omega(\vartheta x)=0.$$

If Y is a Banach space, this condition is equivalent to $\omega(x)<\infty$. In general Fréchet spaces the inequality $\omega(x)<\infty$ is only necessary for the boundedness of the sequence.

(e) *If the oscillation $\omega(x)$ is continuous at 0, then the sequence $\{U_n(x)\}$ is bounded everywhere in X_s.*

(f) *If the oscillation $\omega(x)$ is continuous at x_0, $\|x_0\|<1$ and $\omega(x_0)=0$, then $\omega(x)$ is continuous everywhere.*

(g) *Let the space X_s satisfy condition (Σ_1). If $\omega(x)$ is continuous at a point x_0 belonging to the closure of the set C, then it is continuous in the whole of X_s.*

(h) *Suppose that the space X_s satisfies condition (Σ_1) and $\omega(x)$ vanishes in a set dense in X_s. Then either $\omega(x)\equiv 0$ and hence it is continuous in the entire space, or $\omega(x)$ is discontinuous at any point, whence it is of Baire's second class.*

The proofs of (a) and (c) are trivial; (b) follows from the inequality

$$\big|\|U_n(x)-U_m(x)\|-\|U_n(y)-U_m(y)\|\big|\leqslant\|2[U_n(x/2)-U_m(x/2)]-$$
$$-2[U_n(y/2)-U_m(y/2)]\|\leqslant 2\left\|U_n\left(\frac{x-y}{2}\right)-U_m\left(\frac{x-y}{2}\right)\right\|.$$

To prove the sufficiency of (d), choose an $\varepsilon_1>0$ such that $\|y\|<\varepsilon_1$, $|\vartheta|\leqslant 1$ imply $\|\vartheta y\|<\varepsilon$. This is possible by 1.22. There exists a δ_1 such that $0<\delta_1<1$ and such that $\omega(\vartheta x)<\varepsilon_1$ if $|\vartheta|\leqslant\delta_1$. Thus we can find a k such that $\|\delta_1[U_n(x)-U_m(x)]\|=\|U_n(\delta_1 x)-U_m(\delta_1 x)\|<\varepsilon_1$ for $m,n\geqslant k$; hence we get $\|U_n(\vartheta x)-U_m(\vartheta x)\|<\varepsilon$ if $|\vartheta|\leqslant\delta_1$, $m,n\geqslant k$. The boundedness of the sequence $\{U_n(x)\}$ follows from the inequality

$$\|\vartheta U_n(x)\|\leqslant\|U_k(\vartheta x)\|+\|U_k(\vartheta x)-U_n(\vartheta x)\|<2\varepsilon$$

valid for sufficiently small ϑ and $n\geqslant k$.

Suppose now that the sequence $\{U_n(x)\}$ is bounded. Given an $\varepsilon>0$ there exists a $\vartheta_0>0$ such that $|\vartheta|\leqslant\vartheta_0$, $n=1,2,\ldots$, implies $\|\vartheta U_n(x)\|<\varepsilon/2$. Hence $\|U_n(\vartheta x)-U_m(\vartheta x)\|<\varepsilon$ for $|\vartheta|\leqslant\vartheta_0$, $n,m=1,2,\ldots$; it follows that $\omega(\vartheta x)\leqslant\varepsilon$.

(e) follows immediately from (d).

We prove (f) only for $x_0=0$. By (e) and (d) we get $\omega(x)<\infty$ everywhere; then we apply (b).

Now we prove (g). By the continuity of $\omega(x)$ at x_0 we get $\omega(x_0)=0$. Given any $\varepsilon>0$ we have $\omega(x)<\varepsilon/2$ in a sphere $K(x_0,\varrho)$. By (Σ_1) there exists a $\delta>0$ such that $\|x\|^*<\delta$ implies $x=x_1-x_2$, $x_1\,\epsilon\, K(x_0,\varrho)$, $x_2\,\epsilon\, K(x_0,\varrho)$, whence $\omega(x)=\omega(x_1)+\omega(x_2)<\varepsilon$. Thus $\omega(x)$ is continuous at $x=0$ and it is sufficient to apply (f).

(h) follows directly from (g).

3.3. For a more detailed study of the convergence and divergence of the sequence $\{U_n(x)\}$ we shall introduce some numerical constants connected with this sequence.

The supremum of the numbers λ for which the set $R_\lambda\neq 0$ will be called the *index of oscillation* or shortly *λ_0-index* of the sequence $\{U_n(x)\}$. Thus

$$\lambda_0=\sup_{x\epsilon X_s}\omega(x).$$

The greatest number σ for which the set R_σ is residual will be called the *index of residual oscillation* or shortly *σ_0-index* of the sequence $\{U_n(x)\}$. Its existence follows from the fact that $\sigma_n\to\sigma$, $\sigma_n<\sigma$, implies

$$R_\sigma=\prod_{n=1}^{\infty}R_{\sigma_n}.$$

The number
$$\omega_0 = \lim_{\varrho\to 0} \sup_{\|x\|^*\leqslant\varrho} \omega(x)$$
will be called the *zero-oscillation* or shortly the *ω_0-oscillation* of the sequence $\{U_n(x)\}$.

The number
$$\mu_0 = \lim_{\varrho\to 0} \sup_{(n),\|x\|^*\leqslant\varrho} \|U_n(x)\|$$
will be called the *zero-modulus of continuity* or shortly *μ_0-modulus* of the sequence $\{U_n(x)\}$.

These constants may assume infinite values.

3.4. THEOREM 1. *If the space X_s satisfies condition (Σ_1), then*

(i) $\omega_0 \leqslant 2\sigma_0,$

(i') $\mu_0 \leqslant 2\sigma_0.$

Proof. Let $\sigma_0<\infty$, $\sigma_0<\sigma$. Then S_σ is an F_σ-set of the second category, whence S_σ contains a sphere $K(x_0,\varrho)$. By (Σ_1) there exists a $\delta>0$ such that $\|x\|^*<\delta$ implies $x=x_1-x_2$ with $x_1,x_2\in K(x_0,\varrho)$. Hence
$$\omega(x)\leqslant\omega(x_1)+\omega(x_2)<2\sigma,$$
and this implies (i).

Now let us prove (i'). It is sufficient to consider only the case $\sigma_0<\infty$. Let $\varepsilon>0$ be chosen freely, then let $\sigma_0<\sigma<\sigma_0+\varepsilon$ and set
$$S_\sigma^k = \mathop{E}_{x}\{\sup_{n,m\geqslant k}\|U_n(x)-U_m(x)\|\leqslant\sigma-1/k\};$$
these sets are closed, $S_\sigma=\sum_{k=1}^{\infty}S_\sigma^k$, and since S_σ is of the second category, one of the sets S_σ^k, say S_σ^l, contains a sphere. By (Σ_1) we get for $\|x\|^*<\delta$
$$\sup_{m,n\geqslant l}\|U_n(x)-U_m(x)\|\leqslant 2\sigma-2/l<2\sigma_0+2\varepsilon,$$
$$\sup_{m\geqslant l}\|U_m(x)\|\leqslant 2\sigma_0+2\varepsilon+\|U_l(x)\|,$$
whence for sufficiently small $\|x\|^*$
$$\mu_0-\varepsilon\leqslant\sup_{m\geqslant l}\|U_m(x)\|\leqslant 2\sigma_0+3\varepsilon,$$
and therefore $\mu_0\leqslant 2\sigma_0$.

THEOREM 2. *In every space X_s the following inequalities are satisfied:*

(j) $\omega_0\geqslant\sigma_0,$ (j') $\omega_0\leqslant 2\mu_0.$

If X_s satisfies condition (Σ_1) or (Σ_2), then the inequalities

(j'') $\mu_0\leqslant 2\omega_0$ *or* (j''') $\mu_0\leqslant 4\omega_0$

are satisfied respectively.

Proof. Since $\omega(x)\geqslant\sigma_0$ is a residual set, we get
$$\sup_{\|x\|^*\leqslant\varrho}\omega(x)\geqslant\sigma_0$$
and hence $\omega_0\geqslant\sigma_0$. The inequalities
$$\sup_{m,n\geqslant k}\|U_n(x)-U_m(x)\|\leqslant 2\sup_{(n)}\|U_n(x)\|$$
imply (j'), without any additional hypotheses about X_s.

If X_s satisfies condition (Σ_1), condition (j'') directly results from (i') and (j).

Now let us consider the case where X_s satisfies condition (Σ_2). Choose a sequence $x_n\xrightarrow{l}0$ and a sequence $\{l_n\}$ of indices so that
$$\lim_{n\to\infty}\|U_{l_n}(x_n)\|=\mu_0.$$

By (Σ_2) there exists a sequence $\{\hat{x}_{k_n}\}$ satisfying conditions (Σ_2): (i)-(iii) of 1.31 and condition 2° of LO(I), p. 270, and such that
$$\lim_{n\to\infty}\|U_{m_n}(\hat{x}_{k_n})\|=\mu_0$$
where $m_n=l_{k_n}$. Similarly to 2.4 (proof of Theorem 2), we can prove that $|a_n|\leqslant 1$ implies
$$\sum_{n=1}^{\infty}a_n\hat{x}_{k_n}/2\in X_s\ ^{2}).$$

For any $a=\{a_n\}$ belonging to the space (I) (cf. LO(I), p. 243) consider the sequence of operations $\{V_n(a)\}$ defined by the formula
$$V_i(a)=\sum_{n=1}^{\infty}a_nU_i(\hat{x}_{k_n}/2).$$
Denoting the zero-modulus of continuity and the zero-oscillation of this sequence by μ_0^a and ω_0^a respectively, we get $2\mu_0^a\geqslant\mu_0$, $\omega_0^a\leqslant\omega_0$. Hence, space (I) satisfying condition (Σ_1), we get $\mu_0^a\leqslant 2\omega_0^a$, $\mu_0\leqslant 4\omega_0$.

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[2]) In LO(I), p. 270, line 12, 14 and 17, $\hat{x}_{k_n}$ is misprinted for $\hat{x}_{k_n}/2$.

3.41. We need the following

Lemma. *Let $K(x_0,\varrho)$*[3] *be a fixed sphere, and denote by γ the infimum of the numbers σ for which the set S_σ is dense in $K(x_0,\varrho)$. Then, given any $x\in K(x_0,\varrho)$,*

$$\text{(k)} \qquad \omega(x)\leqslant 2\omega_0+\gamma.$$

Suppose that $\omega_0<\infty$; then $\omega(x)<\infty$ everywhere. If $x\in K(x_0,\varrho)$ and $\bar{x}\in S_\sigma\cdot K(x_0,\varrho)$, $\sigma>\gamma$ and the distance $d(\bar{x},x)$ is sufficiently small, we get by 3.21(b), $\varepsilon>0$ being arbitrary,

$$\omega(x)-\omega(\bar{x})\leqslant 2\omega((x-\bar{x})/2)\leqslant 2\omega_0+\varepsilon, \qquad \omega(x)\leqslant 2\omega_0+\sigma+\varepsilon.$$

3.42. Theorem 3. *Suppose that the sequence $\{U_n(x)\}$ converges in a set dense in X_s. Then*

$$\text{(l)}\quad \lambda_0\geqslant\omega_0, \qquad \text{(l')}\quad \lambda_0\geqslant\sigma_0, \qquad \text{(l'')}\quad 2\omega_0\geqslant\lambda_0.$$

Proof. (l) and (l') trivially result from the definitions. To prove (l'') we apply lemma 3.41 (in this case $\gamma=0$).

3.43. Theorem 4. *If the space X_s satisfies condition (Σ_1) or (Σ_2), then the following conditions are equivalent:*

$$\text{(m)}\quad \omega_0=0, \qquad \text{(m')}\quad \mu_0=0,$$

(m'') *$\omega(x)$ is continuous everywhere.*

Proof. The equivalence of (m) and (m') results from Theorem 1 and 2; $\omega_0=0$ is equivalent to the continuity of $\omega(x)$ at 0, and this by 3.21 (f), implies the continuity of $\omega(x)$ in the whole of X_s.

Remark. By Theorem 2, (m) and (m'') follow from (m') without any additional assumptions on X_s.

Theorem 5. *If the space X_s satisfies condition (Σ_1), then each of the conditions* (m), (m'), (m'') *is equivalent to the following:*

$$\text{(n)} \qquad \sigma_0=0.$$

Proof. This is an immediate consequence of theorems 1,2 and 4.

We complete these theorems by the following remark. If X_s and Y are Banach spaces, then the μ_0-modulus may assume the values 0 or ∞ only; $\mu_0=0$ if and only if $\overline{\lim\limits_{n\to\infty}}\|U_n\|<\infty$, where

$$\|U_n\|=\sup_{x\in X_s}\|U_n(x)\|.$$

[3]) ϱ may be infinite here. In this case $K(x_0,\infty)=X_s$.

Indeed, if $\mu_0<\infty$, then there exists a $\varrho>0$ such that $\|U_n(\varrho x)\|\leqslant\mu_0+1$ for $n=1,2,\ldots$ and $\|x\|=\|x\|^*=1$, whence

$$\|U_n(x)\|\leqslant\frac{\mu_0+1}{\varrho}\|x\| \quad \text{for} \quad x\in X_s;$$

it follows hence that the μ_0-modulus of the sequence $\{U_n(x)\}$ is equal to 0. The last inequality implies also $\|U_n\|\leqslant 1/\varrho$. If

$$\overline{\lim_{n\to\infty}}\|U_n\|<\infty,$$

then from the inequalities $\|U_n(x)\|\leqslant\|U_n\|\,\|x\|$ it is apparent that $\mu_0=0$.

Let us notice that every Saks space has the following property:

3.44. *If $K(x_0,\varrho)\subset C$, then $C=X_s$.*

For the proof let us remark first that we may assume $\|x_0\|<1$. Let $\varrho'=\min(1-\|x_0\|,\varrho)$. Then every element x satisfying the inequalities $\|x-x_0\|\leqslant\varrho'$, $\|x-x_0\|^*\leqslant\varrho'$ is in $K(x_0,\varrho)$. Therefore $x_0+\varrho''z\in K(x_0,\varrho)$ for every $z\in X_s$ and sufficiently small ϱ'', whence $x_0+\varrho''z\in C$, and $x_0\in C$ implies $z\in C$.

3.45. Theorem 6. 1° *Let $\mu_0=0$; then for every Saks space X_s the set C of the points of convergence is either non-dense or identical with the entire space.*

2° *If $\mu_0>0$, the set C is of the first category, whence D is residual if X_s satisfies condition (Σ_1); if X_s satisfies condition (Σ_2), the set D is dense in X_s.*

Proof. *Ad* 1°. If $\mu_0=0$ and X_s is an arbitrary Saks space, then by the remark which follows Theorem 4, the set C is closed. It suffices to apply 3.44.

Ad 2°. If X_s satisfies condition (Σ_1), it suffices to apply Theorem 1; if X_s satisfies condition (Σ_2), Theorems 2, 3 and 3.44 lead to the desired result.

The following theorem is now immediately obtained:

Theorem 6'. *Let X_s satisfy condition (Σ_1); then the set C of convergence of the sequence $\{U_n(x)\}$ is either of the first category, or identical with X_s.*

We complete Theorem 6 by the following remark. *Let X_s satisfy condition (Σ_1) or (Σ_2) and let the sequence $\{U_n(x)\}$ converge everywhere; then this sequence converges uniformly on every compact set.*

Suppose that the set X^* is compact and that the sequence $\{U_n(x)\}$ does not converge uniformly on X^*. Then there exists an $\varepsilon>0$ and elements

$x_{n_i} \epsilon X^*$ such that the sequence $\{x_{n_i}\}$ converges to an element x_0 of X^* and

$$\|U_{n_i}\tfrac{1}{2}(x_{n_i}-x_0)\| = \|\tfrac{1}{2}[U_{n_i}(x_{n_i})-U_{n_i}(x_0)]\| \geqslant \varepsilon$$

for $i=1,2,\ldots$ This, however, is impossible for $\mu_0=0$.

THEOREM 7. *Let $U_n(x)\to U(x)$ in the whole of X_s. If X_s satisfies condition (Σ_1) or (Σ_2), then $U(x)$ is a (X_s, Υ)-linear operation*[4]).

Proof. The additivity of $U(x)$ is obvious. By Theorem 6, $\mu_0=0$, whence it immediately follows that $U(x)$ is continuous at 0. Now it suffices to apply 2.2 (A).

3.46. Theorem 6 and 2.2 immediately imply the following one:

Let X_s satisfy condition (Σ_1) or (Σ_2) and let Υ^* be a family of linear functionals in X_s such that every sequence $\eta_i \epsilon \Upsilon^*$ contains a subsequence convergent everywhere in X_s; then the functionals $\eta \epsilon \Upsilon^*$ are equicontinuous.

3.5. By B we shall henceforth denote the set of those $x \epsilon X_s$ for which the sequence $\{U_n(x)\}$ is bounded, by U the set $X_s - B$, *i. e.* the set of those points at which the sequence is unbounded.

THEOREM 8. *Let X_s be an arbitrary Saks space. Then $B=X_s$ if and only if the following condition is satisfied:*

(*) *given any $\varepsilon>0$ there is a $\vartheta_0>0$ such that*

(2) $$|\vartheta| \leqslant \vartheta_0,\ \|x\|^* \leqslant \vartheta_0 \text{ implies } \|\vartheta U_n(x)\| < \varepsilon \text{ for } n=1,2,\ldots$$

Proof. Let $V_n(x)$ denote the extension of $U_n(x)$ as in 3.1(D). If $B=X_s$, then the sequence $\{V_n(x)\}$ is bounded for every $x \epsilon X_0$, as can easily be seen. By a known theorem (Mazur und Orlicz [4]) there exists a $\delta>0$ such that

(3) $$\|V_n(x)\| < \varepsilon \quad \text{for} \quad n=1,2,\ldots,\ \|x\|+\|x\|^* < \delta.$$

Let us choose δ' so that $\|x\|^*<\delta'$ imply $\|\vartheta x\|^*<\delta/2$ for $|\vartheta|\leqslant 1$. Let $\vartheta_0=\min(1,\delta',\delta/2)$, $\|x\|\leqslant 1$. Since $|\vartheta|\leqslant\vartheta_0$, $\|x\|^*<\vartheta_0$ implies $\vartheta x \epsilon X_s$ and $\|\vartheta x\|+\|\vartheta x\|^*<\delta$, it follows from $U_n(x)=V_n(x)$ and (3) that (2) is satisfied.

To prove the sufficiency let us notice that condition (*) implies $\|\vartheta_n U_n(x)\| = \|U_n(\vartheta_n x)\| < \varepsilon$ for almost all n's if $\vartheta_n\to 0$.

3.51. (A) *The inequality $\|x\|^*\leqslant\vartheta_0$ in condition (2) may be replaced by $x \epsilon X_s$ if one of the following conditions is satisfied:*

(a) *the fundamental space X is complete with respect to the norm $\|\ \|$,*

(b) *$\|\ \|^*$ is a B-norm.*

[4]) Concerning this theorem in the case when the condition (Σ_1) is satisfied cf. Alexiewicz [1].

The proofs result from the fact that in case (a) $\|x_n\|\to 0$ implies $\|x_n\|^*\to 0$ and in case (b) the norm $\|\ \|^*$ is homogeneous.

Let us denote by $\mu_0(\vartheta)$ the zero-modulus of continuity of the sequence $\{\vartheta U_n(x)\}$. Then it is easily seen that

(B) *Condition (*) is equivalent to the relation*

(4) $$\lim_{\vartheta\to 0} \mu_0(\vartheta) = 0.$$

For $|\vartheta|\leqslant 1$ the inequality

(5) $$\mu_0 \geqslant \mu_0(\vartheta)$$

is always satisfied.

THEOREM 9. 1° *For arbitrary X_s, if $\lim\limits_{\vartheta\to 0}\mu_0(\vartheta)=0$, the set B of points of boundedness is identical with the whole space.*

2° *For every X_s, if*

$$\overline{\lim_{\vartheta\to 0}}\, \mu_0(\vartheta) > 0,$$

the set of unboundednes U is dense in X_s. If X_s satisfies condition (Σ_1), B is of the first category, whence the set U is residual.

Proof. *Ad* 1°. It suffices to apply Theorem 8 and 3.51 (B).

Ad 2°. As in 3.44, we can prove that if B contains a sphere, then $B=X_s$. If

$$v_0 = \overline{\lim_{\vartheta\to 0}}\, \mu_0(\vartheta) > 0,$$

and X_s is an arbitrary Saks space, then the set U is non-empty whence, by the foregoing remark, it is dense in X_s. Suppose that X_s satisfies condition (Σ_1). There exist a sequence $\vartheta_k\to 0$, a sequence of elements $\{x_k\}$ and a sequence $\{n_k\}$ of indices such that $\|\vartheta_k U_{n_k}(x_k)\|\geqslant v_0/2$ and $\|x_k\|^*\to 0$. Then the zero-modulus of the sequence of operations $\{\vartheta_k U_{n_k}(x)\}$ exceeds $v_0/2$, whence by Theorem 6 this sequence diverges in a residual set $\bar{D}$. Obviously $\bar{D}\subset U$.

Suppose that the set C is dense in X_s and $B=X_s$. If X_s is a Banach space, then by 3.51 (A) and (2)

$$\|\vartheta_0 U_n(x)\| \leqslant \varepsilon \quad \text{for} \quad n=1,2,\ldots,\ x \epsilon X_s,$$

whence the inequality

$$\|U_n(x)\| = \|\vartheta_0 U_n(x/\vartheta_0)\| < \varepsilon, \qquad n=1,2,\ldots,$$

is satisfied for $\|x\|=\|x\|^*\leqslant\vartheta_0$.

It follows that $\mu_0=0$ and, by Theorem 6,1°, $C=X_s$ in this case. Let us notice that for arbitrary Saks spaces, even those which satisfy condition (Σ_1) or (Σ_2), the condition $B=X_s$ does not imply $\mu_0=0$; the set C may in that case be dense in X_s without, however, being identical with the whole space X_s. On the other hand, by (5), $\mu_0=0$ implies $B=X_s$.

3.6. *Let y_{in} be elements of a Fréchet space Y satisfying the following conditions:*

(o_1) *the set y_{in}, $i,n=1,2,\ldots$, is bounded,*

(o_2) $\sup\limits_{(i)}\|y_{in}\|=\eta_n>\eta>0$ *for* $n=1,2,\ldots,$

(o_3) *the series* $\sum\limits_{n=1}^{\infty}\|\vartheta y_{in}\|$, $i=1,2,\ldots$, *are uniformly convergent in the interval* $0\leqslant\vartheta\leqslant 1$,

(o_4) $\lim\limits_{i\to\infty} y_{in}=y_n$ *for* $n=1,2,\ldots,$

(o_5) $\lim\limits_{n\to\infty} y_n=0$.

Under these hypotheses there exists a nought-or-one sequence $\{\lambda_n\}$ such that the series

$$z_i=\sum_{n=1}^{\infty}\lambda_n y_{in} \tag{6}$$

converges for $i=1,2,\ldots$, and the sequence $\{z_i\}$ is bounded and divergent.

Proof. Suppose, for a moment, that $y_n=0$ for $n=1,2,\ldots$; this, together with the hypotheses (o_1)-(o_4), implies the possibility of defining two increasing sequences of indices $\{i_k\}$ and $\{n_k\}$ such that

1° $$\sum_{n=n_{k+1}}^{\infty}\|\vartheta y_{in}\|<\frac{1}{2^k}\quad\text{for}\quad i=1,2,\ldots,i_k,\ 0\leqslant\vartheta\leqslant 1,$$

2° $$\sum_{n=1}^{n_{k+1}}\|\vartheta y_{in}\|<\frac{1}{2^k}\quad\text{for}\quad i=i_{k+1},i_{k+1}+1,\ldots,\ 0\leqslant\vartheta\leqslant 1$$

(we set $i_0=1$). For $i_{k-1}\leqslant i\leqslant i_k$, $k=1,2,\ldots$, $0\leqslant\vartheta\leqslant 1$, the following inequality is satisfied:

$$\sum_{l=1}^{\infty}\|\vartheta y_{in_l}\|\leqslant\sum_{l=1}^{k-1}\|\vartheta y_{in_l}\|+\|\vartheta y_{in_k}\|+\sum_{l=k+1}^{\infty}\|\vartheta y_{in_l}\|<1/2^{k-2}+\|\vartheta y_{in_k}\|+1/2^{k+1}. \tag{7}$$

By (o_2) there exists an index j_n $(n=1,2,\ldots)$ such that

$$\|y_{j_n n}\|\geqslant\frac{3}{4}\eta.$$

Let us write

$$\lambda_n=\begin{cases}\dfrac{1+(-1)^k}{2} & \text{for}\quad n=n_k,\ k=1,2,\ldots,\\ 0 & \text{elsewhere;}\end{cases}$$

then we define the elements z_i by formula (6).

The sequence $\{z_i\}$ is bounded by (7) and (o_1). Now

$$z_i=\sum_{n=1}^{n_{k-1}}\lambda_n y_{in}+\lambda_{n_k}y_{in_k}+\sum_{n=n_{k+1}}^{\infty}\lambda_n y_{in},$$

whence if we set $j_{n_k}=i'_k$, there follows

$$\|z_{i'_{2k-1}}-z_{i'_{2k}}\|\geqslant\|y_{i'_{2k}n_{2k}}\|-2\frac{1}{2^k}\geqslant\frac{3}{4}\eta-\frac{1}{2^{k-1}}\to\frac{3}{4}\eta$$

as $k\to\infty$; thus the sequence $\{z_i\}$ diverges, for (o_3) implies $i'_k\to\infty$.

Now we remove the hypothesis that all $y_n=0$. Choose elements y_{l_n} so that the series $\sum\limits_{n=1}^{\infty}\|\vartheta y_{l_n}\|$ be uniformly convergent for $0\leqslant\vartheta\leqslant 1$ and set $y_{in}^0=y_{il_n}-y_{l_n}$. The elements y_{in}^0 satisfy conditions (o_1)-(o_4); moreover $\lim\limits_{i\to\infty} y_{in}^0=0$.

By what has just been proved there exists a nought-or-one sequence $\{\lambda_n^0\}$ such that the sequence

$$z_i^0=\sum_{n=1}^{\infty}\lambda_n^0 y_{in}^0$$

is bounded and divergent. If we set

$$\lambda_i=\begin{cases}\lambda_n^0 & \text{for}\quad i=l_n,\ n=1,2,\ldots,\\ 0 & \text{elsewhere,}\end{cases}$$

we may easily verify that sequence (6) is bounded and divergent.

Theorem 10. *Let Y be a Fréchet space in which some neighbourhood of the element 0 is bounded. Let the sequence of operations $\{U_n(x)\}$ have the following properties:*

(p_1) *the sequence $\{U_n(x)\}$ converges in a set W dense in X_s,*

(p_2) *if $w_i\in W$, $w_i\xrightarrow{l}0$ and $U_n(w_i)\to U(w_i)$, then $U(w_i)\to 0$,*

(p_3) *the set U of points of unboundedness of the sequence $\{U_n(x)\}$ is non-empty.*

Under these hypotheses there exists an element x_0 such that the sequence $\{U_n(x_0)\}$ is bounded but divergent.

Proof. Let us remark first that if $x\in B$, then $\sup\limits_{(n)}\|\vartheta U_n(x)\|$ is a continuous function of ϑ; this immediately follows from the inequality

$$\big|\|\vartheta U_n(x)\|-\|\vartheta' U_n(x)\|\big|\leqslant\|(\vartheta-\vartheta')U_n(x)\|$$

and from LO(I), 1.2. By (p_3) and Theorem 9 there is an $\varepsilon_0>0$ and for every $n=1,2,\ldots$ there exist elements x_n and numbers $0<\vartheta_n<1/n$ such that

1° $\sup\limits_{(i)} \|\vartheta_n U_i(x_n)\| \geqslant \varepsilon_0$,

2° $\|x_n\|^* < 1/n$,

3° the sphere $\|y\| \leqslant \varepsilon_0$ in Y is bounded.

The set W being dense in X_s, it is possible to choose elements $w_n^0 \epsilon W$ and numbers ϑ_n^0 such that $0 < \vartheta_n^0 < 1/n$ and

1′° $\sup\limits_{(i)} \|\vartheta_n^0 U_i(w_n^0)\| = \varepsilon_0$,

2′° $\|w_n^0\|^* < 1/n$.

Choose an increasing sequence of indices $\{i_n\}$ such that the series

$$\sum_{n=1}^{\infty} \|\vartheta U_i(\vartheta_{i_n}^0 w_{i_n}^0)\| \quad \text{for} \quad i = 1, 2, \ldots$$

be uniformly convergent in the interval $0 \leqslant \vartheta \leqslant 1$, and such that

$$\sum_{n=1}^{\infty} \vartheta_{i_n}^0 \leqslant 1, \qquad \sum_{n=1}^{\infty} \|\vartheta_{i_n}^0 \text{‹}$$

this is possible, for $\vartheta_i^0 w_i^0 \xrightarrow{l} 0$. Let us set

$$y_{in} = U_i(\vartheta_{i_n}^0 w_{i_n}^0) \quad \text{for} \quad i, n = 1, 2, \ldots$$

By the hypotheses and by 1′°, 2′° it follows that the hypotheses of Lemma 3.6 are satisfied whence there exists a bounded and divergent sequence of elements of the form

$$z_i = \sum_{n=1}^{\infty} \lambda_n y_{in} = \sum_{n=1}^{\infty} \lambda_n U_i(\vartheta_{i_n}^0 w_{i_n}^0) = U_i\Big(\sum_{n=1}^{\infty} \lambda_n \vartheta_{i_n}^0 w_{i_n}^0\Big);$$

now it is sufficient to put

$$x_0 = \sum_{n=1}^{\infty} \lambda_n \vartheta_{i_n}^0 w_{i_n}^0.$$

THEOREM 10′. *Theorem 10 remains true if we replace in its hypotheses X_s by a Fréchet space X, the l-convergence in* (p_2) *by the convergence in X, and if the continuity of $U_n(x)$ is understood as (X, Y)-continuity.*

The proof is quite analogous.

3.7. In this section Y will stand for a Banach space, and Υ for its conjugate space. $\Upsilon_0 \subset \Upsilon$ will denote a fundamental set of functionals in Y (for the definition see 2.3). A sequence $\{y_n\}$ of elements of Y will be called Υ_0-*weakly convergent to* $y_0 \epsilon Y$ if $\eta(y_n) \to \eta(y_0)$ for every $\eta \epsilon \Upsilon_0$. If $\Upsilon_0 = \Upsilon$, then according to the usual terminology Υ_0-weakly convergent sequences will be called simply *weakly convergent*.

Let us notice that for every fundamental set the Υ_0-weak limit — if it exists — is uniquely determined.

Given a sequence $\{U_n(x)\}$ we shall denote by $C_w(\Upsilon_0)$ the set of those $x \epsilon X_s$ for which this sequence is Υ_0-weakly convergent to an element of Y.

THEOREM 11. *Let Y be a separable space and let X_s satisfy the condition* (Σ_1); *then the set $C_w(\Upsilon_0)$ is either of the first category or identical with the entire space.*

Proof. Suppose that the set $C_w(\Upsilon_0)$ is of the second category. By Theorems 6′ and 7 for every $\eta \epsilon \Upsilon_0$ the functionals $\eta(U_n(x))$ are convergent in the whole space X_s to a linear functional $\xi_\eta(x)$; moreover, there exists $v(x) \epsilon Y$ such that $\xi_\eta(x) = \eta(v(x))$ for $x \epsilon C_w(\Upsilon_0)$. Let $\eta_i \epsilon \Upsilon_0$, since the norms $\|\eta_i\|$ are bounded, the separability of Y implies that the sequence $\xi_{\eta_i}(x)$ contains a partial sequence, convergent for every $x \epsilon C_w(\Upsilon_0)$, whence, by Theorem 6′, in the whole of X_s. The functionals $\xi_\eta(x)$ $(\eta \epsilon \Upsilon_0)$ are thus, in virtue of 3.46, equicontinuous everywhere. This implies

$$\|v(x_i) - v(x_j)\| \to 0 \tag{8}$$

when $x_i, x_j \epsilon C_w(\Upsilon_0)$, $x_i \xrightarrow{l} x_0$, $x_j \xrightarrow{l} x_0$.

Indeed, it is sufficient to choose $\eta^{ij} \epsilon \Upsilon_0$ so that

$$\left|\xi_{\eta^{ij}}\big(\tfrac{1}{2}(x_i - x_j)\big)\right| = \left|\eta^{ij}\Big(\tfrac{1}{2}\big(v(x_i) - v(x_j)\big)\Big)\right| \geqslant c\|v(x_i) - v(x_j)\|/2,$$

and then apply equicontinuity at 0 of the functionals $\xi_{\eta^{ij}}$ (in the formula above, c denotes the constant in condition 2.3 (f_1)). Now we define, for $x \epsilon \overline{C_w(\Upsilon_0)}$, the operation

$$V(x) = \lim_{i \to \infty} v(x_i)$$

where $x_i \epsilon C_w(\Upsilon_0)$, $x_i \xrightarrow{l} x$. By (8), this kind of definition is justified and the defined operation is uniquely determined; moreover $V(x) = v(x)$ for $x \epsilon C_w(\Upsilon_0)$ and $V(x)$ is a (X_s, Y)-continuous operation in $\overline{C_w(\Upsilon_0)}$. Since $x_i \xrightarrow{l} x_0$, $x_i \epsilon C_w(\Upsilon_0)$, $\eta \epsilon \Upsilon_0$ implies

$$\xi_\eta(x_i) = \eta(v(x_i)) \to \xi_\eta(x_0),$$
$$\eta(v(x_i)) \to \eta(V(x_0)),$$

we get $\xi_\eta(x_0) = \eta(V(x_0))$ in $\overline{C_w(\Upsilon_0)}$. There exists a sphere K contained in $\overline{C_w(\Upsilon_0)}$; therefore

$$\xi_\eta(x) = \eta(V(x)) \tag{9}$$

for every $x \epsilon K$. Hence, by (Σ_1), we see that the operation $V(x)$ may be extended over the whole space X_s so that (9) be satisfied.

THEOREM 11′. *The assertion of Theorem 11 remains unaltered if we replace the hypothesis of separability of Y by that of separability of X_s and suppose, moreover, that $\Upsilon_0 = \Upsilon$.*

Proof. Let Y_0 denote the closed linear span of the elements $\{U_n(x)\}$ where $x \in X_s$, $n=1,2,\ldots$ The space X_s being separable, (X_s, Y)-continuity of $U_n(x)$ implies that the set Y_0 is separable. Since, for every $x \in C_w(\Upsilon)$ and $\eta \in \Upsilon$, $\eta(U_n(x)) \to \eta(v(x))$ where $v(x) \in Y$, it follows by a well-known theorem that the elements $v(x)$ are in Y_0, and it is sufficient to apply Theorem 11 with Y replaced by Y_0.

THEOREM 12. *Let the sequence $\{U_n(x)\}$ be Υ_0-weakly convergent to $U(x)$ for every $x \in X_s$. Each of the following conditions is sufficient in order that the weak limit $U(x)$ be an (X_s, Y)-linear operation*:

(r_1) *Y is separable; X_s satisfies condition (Σ_1);*

(r_2) *Y is an arbitrary Banach space, X_s is separable and satisfies condition (Σ_1), and $\Upsilon_0 = \Upsilon$;*

(r_3) *Y is an arbitrary Banach space satisfying condition* (Z) (see LO(I), p. 271), *X_s satisfies condition (Σ_1);*

(r_4) *Y is an arbitrary Banach space, X_s satisfies condition (Σ_2), and $\Upsilon_0 = \Upsilon$.*

Proof. Let us remark first that in each case under consideration the uniqueness of the weak limit implies the additivity of the operation $U(x)$. Moreover, by Theorem 7, $\eta(U(x))$ is a linear functional in X_s for every $\eta \in \Upsilon_0$. In cases (r_1), (r_2), and (r_3) the (X_s, Y)-continuity follows by the application of Theorems 1, 1′, and 3′ of LO(I) respectively. In case (r_4) the proof is analogous to that of Theorem 1 of LO(I). Only, it is to be noticed that, actually, in the proof on p. 268 in LO(I) we do not use the separability of the whole space Y but only of the closed linear span Y_0 of the elements $U(\hat{x}_\lambda)$. The hypothesis $\Upsilon_0 = \Upsilon$ ensures the separability of Y_0.

3.8. Let Λ denote the space of nought-or-one sequences $\lambda = \{\lambda_n\}$, Y — a Fréchet space. The series

$$(*) \qquad \sum_{n=1}^{\infty} y_n \quad \text{where} \quad y_n \in Y$$

is called *perfectly bounded* if for arbitrary $\lambda \in \Lambda$ the partial sums of the series $\sum_{n=1}^{\infty} \lambda_n y_n$ compose a bounded set; the series is called *perfectly convergent* if for every $\lambda \in \Lambda$ the series $\sum_{n=1}^{\infty} \lambda_n y_n$ is convergent. It is easily seen that every perfectly convergent series is perfectly bounded. In some spaces perfect boundedness of a series implies its perfect convergency; according to the terminology adopted in LO(I) such spaces are said *to have property* (Z). If Y is a Banach space, then the series $(*)$ is perfectly bounded if and only if there exists a $K > 0$ such that

$$\Big\|\sum_{n=1}^{i} \lambda_n y_n\Big\| \leqslant K \quad \text{for every} \quad \{\lambda_n\} \in \Lambda \quad \text{and} \quad i=1,2,\ldots$$

The set Z_p (the set Z_p^b) of those sequences $z = \{y_n\}$ for which the series $(*)$ is perfectly convergent (perfectly bounded) forms a linear space under the usual definitions of addition and multiplication by scalars. If we define the norm of the element $z \in Z_p$ by the formula

$$\|z\|_p = \sup_{(k)\lambda \in \Lambda} \Big\|\sum_{n=1}^{k} \lambda_n y_n\Big\|,$$

then Z_p becomes a Fréchet or Banach space according to the space Y. The same norm may be defined in the space Z_p^b. The series $(*)$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} \|y_n\|$ converges. Let Z_a denote the set of those sequences $z = \{y_n\}$ for which the series converges absolutely.

3.81. *The set Z_a is linear under the usual definitions of addition and multiplication by scalars if and only if the following condition is satisfied:*

(o) *There exists $\alpha_0 > 0$ such that $|\alpha| \leqslant \alpha_0$ implies $\|\alpha y\| \leqslant K(\alpha)\|y\|$ for $\|y\| \leqslant \varrho$, where $K(\alpha) < \infty$, and ϱ may depend on α.*

If (o) is satisfied, then it is easy to prove that, given real α, β, there exists a $K < \infty$ such that for sufficiently small $\|y'\|, \|y''\|$

$$\|\alpha y' + \beta y''\| \leqslant K(\|y'\| + \|y''\|).$$

Hence $\sum_{n=1}^{\infty} \|y_n'\| < \infty$ and $\sum_{n=1}^{\infty} \|y_n''\| < \infty$, implies $\sum_{n=1}^{\infty} \|\alpha y_n' + \beta y_n''\| < \infty$.

Suppose now that the set Z_a is linear and condition (o) is not satisfied. Then there exist an $\alpha > 0$ and a sequence $\{y_n\}$ such that

$$\|\alpha y_n\| \geqslant k_n \|y_n\|, \quad \|y_n\| \leqslant 1/n^2, \quad k_n \geqslant n \quad \text{for} \quad n=1,2,\ldots$$

Let us choose positive integers r_n so that $1/n^2 \leqslant r_n \|y_n\| \leqslant 2/n^2$, set $r_0 = 0$ and let

$$y_i^* = y_n \quad \text{for} \quad r_0 + r_1 + \ldots + r_{n-1} < i \leqslant r_0 + r_1 + \ldots + r_n.$$

Obviously

$$\sum_{i=1}^{\infty} \|y_i^*\| \leqslant \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty,$$

on the other hand, however,

$$\sum_{i=1}^{\infty} \|\alpha y_i^*\| \geqslant \sum_{n=1}^{\infty} \frac{k_n}{n^2} = \infty,$$

which is contradictory.

3.82. In Z_a we define the norm as

$$\|z\|_a = \sum_{n=1}^{\infty} \|y_n\|. \tag{10}$$

The set Z_a with the norm defined by (10) *is a Fréchet space if and only if condition* (o) *is satisfied.*

The necessity being trivial, we shall prove the sufficiency only. We need the following Lemma:

Let $\xi_n(a)$ be continuous functions in $-\infty<a<\infty$, satisfying the conditions:

1° *$\xi_n(a)$ are subadditive, i. e. $\xi_n(a'+a'')\leqslant \xi_n(a')+\xi_n(a'')$ for arbitrary a', a'',*

2° $\xi_n(-a)=\xi_n(a)$,

3° $\overline{\lim\limits_{n\to\infty}}\, \xi_n(a)<\infty$ *for every a.*

Then there exist a $\varrho>0$ and $K>0$ such that

$$\xi_n(a)\leqslant K \quad \text{for} \quad |a|\leqslant\varrho,\ n=1,2,\ldots$$

The proof may be carried out by the classical category method, similarly to the proof of the well-known theorem on sequences of linear operations.

Now we proceed to the proof of our theorem. If condition (o) is satisfied, then it is obvious that norm (10) satisfies conditions 1.1 (a), (b) and the space Z_a is complete. It remains to prove 1.1 (c'). For this purpose it is sufficient to prove that in condition (o) we may always assume that ϱ does not depend on a, $K(a)\leqslant K<\infty$, provided that a_0 is sufficiently small. In the contrary case there would exist sequences $a_n\to 0$, $y_n\to 0$ such that $\|a_n y_n\|/\|y_n\|\to\infty$. This, however, is impossible, for the functionals $\xi_n(a)=\|a y_n\|/\|y_n\|$ satisfy the conditions of the lemma (3° follows immediately by (o)), and the assertion of the lemma does not hold for these functionals.

Let Z_a^γ ($\gamma>1$) denote the set of all sequences $z=\{y_n\}$ for which the series

$$\sum_{n=1}^{\infty} \|y_n\|^\gamma < \infty.$$

In Z_a^γ we define the norm as

$$\|z\|_a^\gamma = \Big(\sum_{n=1}^{\infty} \|y_n\|^\gamma\Big)^{1/\gamma}. \tag{10'}$$

In the same way as for Z_a, one can prove

3.83. *The space Z_a^γ with norm* (10') *is a Fréchet space if and only if condition* (o) *is satisfied.*

Theorem 3.82 and 3.83 may be completed by the trivial remark that if Y is a Banach space, so are the spaces Z_a and Z_a^γ. Condition (o) is obviously satisfied if the norm is *monotonic, i. e.* satisfies the inequality $\|ya\|\leqslant\|y\|$ for $0\leqslant a\leqslant 1$, $y\in Y$ (this is the case in the spaces $\boldsymbol{S}, \boldsymbol{L}^a, \boldsymbol{l}^a$ $(0<a<1)$ which are not Banach spaces).

3.84. THEOREM 13. *Let X_s satisfy condition* (Σ_1); *then the set of those points x where the series*

$$\sum_{n=1}^{\infty} U_n(x)$$

converges perfectly (is perfectly bounded) is either of the first category, or identical with the space X_s.

A similar proposition holds also for the set of the points of absolute convergence and for the set of those points for which $\{U_n(x)\}\in Z_a^\gamma$, the norm $\|\ \|$ in Y being supposed to satisfy condition (o).

Proof. We shall prove the theorem for the set of points of perfect convergence. Suppose that the series converges perfectly in a set of the second category. Let us define an operation $V_k(x)$ from X_s to Z_p, setting

$$V_k(x)=\{U_1(x), U_2(x),\ldots, U_k(x), 0, 0, \ldots\}.$$

It is easily seen that if the series (*) converges perfectly, then the series $\sum_{n=k}^{\infty}\lambda_n y_n$ form a sequence convergent to 0 as $n\to\infty$ uniformly in Λ^*. Set $V(x)=\{U_n(x)\}$; then $\|V_k(x)-V(x)\|_p\to 0$ as $k\to\infty$ in a set of the second category, whence by Theorem 6' the sequence $\{V_k(x)\}$ converges in Z_p for every $x\in X_s$.

3.9. In this section we shall illustrate the applications of the foregoing theorems on sequences of operations by means of some examples.

Let $A=(a_{in})$, $i, n=1,2,\ldots$ be a matrix of real numbers. Let us write

$$A_i=\sum_{n=1}^{\infty}|a_{in}|, \qquad A_i^k=\sum_{n=k}^{\infty}|a_{in}|.$$

In the sequel we shall deal with matrices satisfying some of the following conditions:

(α) $\sup_{(i)} A_i < \infty$,

(β) $\lim_{i\to\infty} \sum_{n=1}^{\infty} a_{in} = \sigma$,

(γ) $\lim_{i\to\infty} a_{in} = a_n$ for $n = 1, 2, \ldots$

As X_s let us take the space (I) of LO(I), *i. e.* the space of bounded sequences $x = \{a_n\}$, the norms being defined as

$$\|x\| = \sup_{(n)} |a_n|, \qquad \|x\|^* = \sum_{n=1}^{\infty} |a_n|/2^n.$$

This space is compact (for the l-convergence) and satisfies conditions (Σ_1) and (Σ_2). If we suppose that $A_i < \infty$ $(i = 1, 2, \ldots)$ then the functionals

(11) $$A_i(x) = \sum_{n=1}^{\infty} a_{in} a_n$$

are X_s-linear. It is easily seen that, given $\varrho > 0$, for sufficiently large k the following inequalities hold:

(12) $$\sup_{(i)} A_i^k \leqslant \sup_{(i), \|x\|^* \leqslant \varrho} |A_i(x)|,$$

(12′) $$\frac{1}{k} \sup_{(i)} A_i \leqslant \sup_{(i), \|x\|^* \leqslant \varrho} |A_i(x)|,$$

and given k, for sufficiently small ϱ,

(13) $$\sup_{(i), \|x\|^* \leqslant \varrho} |A_i(x)| \leqslant \frac{1}{k} \sup_{(i)} A_i + \sup_{(i)} A_i^k.$$

It follows that $\sup_{(i)} A_i < \infty$ implies

$$\mu_0 = \lim_{k\to\infty} \sup_{(i)} A_i^k.$$

Let us consider the conditions

(δ) $\mu_0 < \infty$, (ε) $\mu_0 = 0$.

By (12′) and (13) it follows that (α)⇄(δ); (ε) is equivalent to (α) and

(η) $$\lim_{k\to\infty} \sup_{(i)} A_i^k = 0.$$

Let us notice that, in virtue of (α), (δ) is equivalent to $B = X_s$. If (γ) is satisfied, then $\mu_0 = 0$ implies

(ϑ) $$\lim_{i\to\infty} \sum_{n=1}^{\infty} |a_{in} - a_n| = 0.$$

By theorem 1.6 and from the above remarks it follows that:

(a) *If the sequence $\{A_i(x)\}$ converges in a set of the second category, then it converges in the whole of X_s, $\mu_0 = 0$, and the conditions* (α), (β), (γ), (η), (ϑ) *are satisfied.*

(b) *If $0 < \mu_0 < \infty$, then the sequence $\{A_i(x)\}$ is bounded, but in a residual set its oscillation is greater than $\mu_0/2$.*

(c) *If $\mu_0 = \infty$, then the sequence $\{A_i(x)\}$ is unbounded in a residual set.*

(d) *If $\mu_0 = 0$ and condition* (γ) *is satisfied, then the sequence $\{A_i(x)\}$ converges in the entire space X_s*[5]).

3.91. Let us suppose now that the matrix A satisfies condition 3.9, (α)-(γ) with $\sigma = 1$, $a_n = 0$ for $n = 1, 2, \ldots$ Every matrix of this type defines a permanent (matrix)-method of summability, with the field of summability A^*[6]). By $A(x)$ we shall denote the A-limit of the sequence x; by A_b^*, A_{0b}^*, A_0^* we shall denote the subsets of A^* composed of bounded sequences, bounded sequences A-summable to 0, and sequences A-summable to 0 respectively. The term "summability method $A = (a_{in})$" will mean the method of summability corresponding to the matrix (a_{in}). Let us now consider the Saks space X_s (LO(I), p. 243-244) whose elements are sequences of A_{0b}^* with the following definition of norms:

$$\|x\| = \sup_{(n)} |a_n|, \qquad \|x\|^* = \sum_{n=1}^{\infty} |a_n|/2^n + \sup_{(n)} |A_n(x)|,$$

where $A_n(x)$ denotes the n-th transform of the sequence x, defined by formula (11). As shown in LO(I), p. 248-249, the space satisfies conditions (Σ_1) and (Σ_2).

THEOREM 14. *Let $A = (a_{in})$ and $B = (b_{in})$ be permanent methods of summability. If every bounded A-summable sequence is B-summable, then $A(x) = B(x)$ for every $x \in A_b^*$* (Mazur and Orlicz [5] and [3]).

Proof. The transforms $B_n(x)$ are linear functionals in the space X_s, and, by hypothesis, $B_n(x) \to B(x)$ in X_s. By theorem 7, $B(x)$ is a X_s-linear functional. The set of the sequences converging to 0 and with the terms absolutely less than 1 is dense in X_s (Mazur and Orlicz [5], p. 137), and since, for every sequence converging to 0, $B(x) = 0$, this relation must hold in the whole of X_s and thus in A_{0b}^*. To complete the proof, it suffices to write every sequence $x = \{a_n\} \in A_b^*$ in the form $a_n = a_n^0 + A(x)$, and notice that, by the permanency of method A, $\{a_n^0\} \in A_{0b}^b$.

[5]) Theorems (a), (c), and (d) are known.

[6]) Concerning the principal concepts of the theory of matrix-summability methods see, for instance, Mazur and Orlicz [5].

THEOREM 15. *Let $A=(a_{in})$ be a permanent method, non-equivalent to the identical method, consistent with every permanent method not weaker than A. Then there exists a bounded but divergent A-summable sequence* ([5],3.7.2).

Proof. The field A_0^* is an F-space if we define the norm as

$$\|x\|_0=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|a_n|}{1+|a_n|}+\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|A_n(x)|}{1+|A_n(x)|}+\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\sup\limits_{(k)}|A_{ik}(x)|}{1+\sup\limits_{(k)}|A_{ik}(x)|},$$

where

$$A_{ik}(x)=\sum_{n=1}^{k}a_{in}a_n\ ^{7}).$$

Now we define linear functionals in A_0^*:

$$U_n(x)=a_n \quad \text{for} \quad n=1,2,\ldots$$

The sequence $\{U_n(x)\}$ is not bounded in A_0^* (Mazur and Orlicz [5], p. 151). $U_n(x)\to 0$ if x is a sequence convergent to 0; and by hypothesis the sequences converging to 0 compose a dense set (in the sense of the topology induced by $\|\ \|_0$) in A_0^* (Mazur and Orlicz [5], p. 145). Thus the hypotheses of Theorem 10′ are satisfied, whence there exists an $x_0\in A_0^*$ such that the sequence $\{U_n(x_0)\}$ is bounded but divergent.

3.92. The very relevant method applied to the proof of Theorem 14 makes possible various generalizations. It can be applied to the problems of consistency for summability methods based of sequence-to--functions transformations, or to integral function-to-functions transformations, or to the transformations where a_{in} belong to a Banach space. To give an example more we shall consider continuous summability methods.

Let $\varphi_n(t)$ denote, for $n=1,2,\ldots$, continuous functions in $\langle t_0,\infty)$. Let us write

$$\Phi(t,x)=\sum_{n=1}^{\infty}\varphi_n(t)a_n. \tag{14}$$

The sequence x is called *summable by means of the functional continuous method* $\Phi\{\varphi_n\}$ (or briefly $\Phi\{\varphi_n\}$-*summable*) to $\Phi(x)$ if functions (14) are meaningful for $t\in\langle t_0,\infty)$ and

$$\lim_{t\to\infty}\Phi(t,x)=\Phi(x).$$

The method $\Phi\{\varphi_n\}$ is called *permanent* if $\lim a_n=a$ implies $\Phi\{\varphi_n\}$--summability of the sequence $x=\{a_n\}$, and moreover $\Phi(x)=a$. The field

[7]) Mazur and Orlicz [5], p. 134.

of the method $\Phi\{\varphi_n\}$, *i. e.* the set of all $\Phi\{\varphi_n\}$-summable sequences, will be denoted by Φ^*; its subset composed of those sequences for which $\Phi(x)=0$, or of those which are bounded, or of those which satisfy both of these conditions will be denoted by Φ_0^*,Φ_b^*, and Φ_{0b}^* respectively.

THEOREM 16. *Let $\Phi\{\varphi_n\}$ be a permanent continuous functional method of summability such that the series*

$$\sum_{n=1}^{\infty}|\varphi_n(t)| \tag{15}$$

converges uniformly in every interval $\langle t_0,t_1\rangle$ where $t_1<\infty$. Let the matrix--method $A=(a_{in})$ be permanent, and let every sequence $x\in\Phi_b^*$ be A-summable. Then $A(x)=\Phi(x)$, for every $x\in\Phi_b^*$.

The theorem remains true when the method A is replaced by a permanent continuous functional method $\Psi\{\psi_n\}$ [8]).

Proof. Let us denote by X_s the Saks space whose elements are the sequences of Φ_b^*, the norms being defined as

$$\|x\|=\sup_{(n)}|a_n|, \qquad \|x\|^*=\sum_{n=1}^{\infty}|a_n|/2^n+\sup_{t_0\leq t<\infty}|\Phi(t,x)|.$$

We shall prove that the sequences converging to 0 form a dense subset of X_s, and that conditions (Σ_1) and (Σ_2) are satisfied in X_s. Then the proof of our theorem may be carried out in the same way as that of Theorem 14.

The permanency of the method $\Phi\{\varphi_n\}$ and the hypothesis on the series (15) imply

$$K=\sup_{t_0\leq t<\infty}\sum_{n=1}^{\infty}|\varphi_n(t)|<\infty. \tag{16}$$

Let $K(x_0,\varrho)$ be an arbitrary sphere in X_s, let $x_0=\{a_n^0\}$. To prove (Σ_1) (and (Σ_2)), it is sufficient, in virtue of 1.32, to prove that there exists an element $w\in K(x_0,\varrho)$ for which the condition (Σ_1) is satisfied. For $i=1,2,\ldots$ let us define the sequences

$$a_n^{(i)}=\begin{cases}a_n^0 & \text{for} \quad n=1,2,\ldots,i,\\ 0 & \text{elsewhere.}\end{cases}$$

Let $x_i=\{a_n^{(i)}\}$, $f_i(t)=\Phi(t,x_i)$, $f(t)=\Phi(t,x_0)$; by the hypotheses of our Theorem and by (16) it follows that the functions $f_i(t)$ and $f(t)$ satisfy the hypotheses of 1.51 (B). By this Lemma, for every positive integer s there exist non-negative numbers $a_s,a_{s+1},\ldots,a_{s+r}$ such that

$$\sum_{i=s}^{s+r}a_i=1 \quad \text{and} \quad \sup_{t_0\leq t<\infty}\Big|\sum_{i=s}^{s+r}a_i\Phi(t,x_i)-\Phi(t,x_0)\Big|<\varrho/4.$$

[8]) Under analogous hypotheses, this theorem was first proved by M. Altman [2].

Choose s so large that

$$\sum_{n=s+1}^{\infty} \frac{1}{2^n} \leqslant \frac{\varrho}{8},$$

and set

$$\delta = \frac{1}{2^{r+s}} \frac{\varrho}{4K}.$$

Let us consider the sequence $w=\{b_n\}$ with terms

$$b_n = \left(1 - \frac{\varrho}{4k}\right) \sum_{i=s}^{s+r} a_i a_n^{(i)} \qquad (n=1,2,\ldots).$$

Analogous estimations to those in LO(I), p. 249 (we suppose, moreover, that $K>1$), show that for every $y=\{a_n\} \epsilon X_s$ such that $\|y\|^* < \delta$, the element $y+w$ belongs to X_s and

(17) $$\|w - x_0\|^* < \varrho.$$

Since almost all the terms of the sequence w vanish, (17) easily implies also that the space X_s is separable.

3.93. Now let X_s be space (III) of LO(I), *i. e.* the space the elements of which are measurable functions in $\langle a,b\rangle$, bounded almost everywhere, and in which the norms are defined by

$$\|x\| = \operatorname*{sup\,ess}_{a \leqslant t \leqslant b} |x(t)|, \qquad \|x\|^* = \int_a^b |x(t)|\,dt.$$

X_s is a separable space satisfying conditions (Σ_1) and (Σ_2). Given a sequence $y_n(t)$ of integrable functions in $\langle a,b\rangle$, we shall need the following conditions:

(α) $\sup_{(n)} \int_a^b |y_n(t)|\,dt < \infty$,

(β) $\lim_{n\to\infty} \int_a^\tau y_n(t)\,dt$ exists for $a \leqslant \tau \leqslant b$,

(γ) the functions $\Phi_n(\tau) = \int_a^\tau |y_n(t)|\,dt$ are equicontinuous in $\langle a,b\rangle$.

Let us write

$$\Phi_n(E) = \int_E |y_n(t)|\,dt \quad \text{for} \quad E \subset \langle a,b\rangle,$$

$$\Phi = \lim_{\varrho\to 0} \sup_{(n),|E|\leqslant\varrho} \Phi_n(E).$$

Condition (γ) is obviously equivalent to

(γ') $$\Phi = 0.$$

Let us consider the sequence of linear functionals in X_s

(18) $$T_n(x) = \int_a^b x(t) y_n(t)\,dt.$$

The following propositions are valid for the sequence $\{T_n(x)\}$:

(δ) *condition* (α) *implies* $\Phi = \mu_0$,

(ε) *condition* (α) *is equivalent to* $\mu_0 < \infty$,

(η) *condition* (γ) *is equivalent to* $\mu_0 = 0$.

Hence theorems 1, 6, 9 imply that

(a) *The sequence* $\{T_n(x)\}$ *converges in the whole of* X_s *if it converges in a set of the second category. Moreover, in this case,* $\mu_0 = \Phi = 0$ *and conditions* (α)-(γ) *are satisfied.*

(b) *If* $0 < \mu_0 < \infty$, *then the sequence* $\{T_n(x)\}$ *is bounded; in a residual set, however, its oscillation is not less than* $\Phi/2$.

(c) $\mu_0 = \infty$ *implies the unboundedness of the sequence* $\{T_n(x)\}$ *in a residual set.*

(d) *If condition* (β) *is satisfied and* $\mu_0 = \Phi = 0$, *then the sequence* $\{T_n(x)\}$ *converges in the entire space*[9]).

[9]) Theorems (a), (c), and (d) are known.

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