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The Heisenberg group and the group Fourier transform of regular homogeneous distributions

by

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Abstract. We calculate the group Fourier transform of regular homogeneous distributions defined on the Heisenberg group, $\mathbf{H}^n$. All such distributions can be written as an infinite sum of terms of the form $f(\theta)\overline{w}^{-k}P(z)$, where $(z,t)\in\mathbb{C}^n\times\mathbb{R}$, $w=|z|^2-it$, $\theta=\arg(\overline{w}/w)$ and $P(z)$ is an element of an orthonormal basis for the spherical harmonics. The formulas derived give the Fourier transform of the distribution in terms of a smooth kernel of the variable θ and the Weyl correspondent of P.

1. Introduction. In this paper we derive formulas for the group Fourier transform of regular homogeneous distributions on the Heisenberg group, $\mathbf{H}^n$. (We use coordinates $(z,t)\in\mathbb{C}^n\times\mathbb{R}$ on $\mathbf{H}^n$). It can be shown that all such distributions can be expressed as an infinite sum $\sum f_i(\theta)\overline{w}^{-k_i}P_i(z)$. Here, $w=|z|^2-it$, $\theta=\arg(\overline{w}/w)$ and the $P_i(z)$ are elements of an orthonormal basis for the spherical harmonics.

The group Fourier transform is a map from $L^1(\mathbf{H}^n)$ into the space of families of bounded operators defined on a Hilbert space. In many applications the Hilbert space is taken to be $L^2(\mathbb{R}^n)$. The domain of definition of the transform can be extended to include tempered distributions on $\mathbf{H}^n$. The group Fourier transform is of interest because it extends to a unitary map from $L^2(\mathbf{H}^n)$ to the space of families of Hilbert–Schmidt operators. Also, the group Fourier transform (which we will denote by $\widehat{\ }_H$) behaves nicely with respect to convolution defined by the group multiplication on $\mathbf{H}^n$. That is, $(f*g)\widehat{\ }_H=\widehat{f}_H\cdot\widehat{g}_H$, where the multiplication on the right is composition of operators.

The group Fourier transform is closely related to the Weyl correspondence. In fact, the formula we present gives the group Fourier transform of a regular homogeneous distribution in terms of the Weyl correspondent of P_i. This correspondent will be denoted by $\mathcal{W}(P_i)$. The set

$$\{\mathcal{W}(P)\mid P\text{ is a homogeneous harmonic polynomial}\}$$

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is of particular interest in its own right. It was shown by D. Geller [5] that the $\mathcal{W}(P)$ constitute operator analogues of spherical harmonics.

Since every regular homogeneous distribution $K(t,z)$ expands to a sum

$$\sum f_i(\theta)\overline{w}^{-k_i}P_i(z),$$

when calculating $\widehat{K}_H$ it is enough to consider terms of the form $f(\theta)\overline{w}^{-k}P$. We will see, in the case $P=1$, that the Fourier transform is a diagonal operator with respect to the usual Hermite basis.

There is a classical analogue [9] to the result proved herein. If K is a regular homogeneous distribution on $\mathbb{C}^n$, then K is an infinite sum of terms of the form $c|z|^{-2k}P(z)$. The result referred to states that

$$\mathcal{F}(\Gamma(k)|z|^{-2k}P(z)) = \Gamma(j)|\zeta|^{-2j}P(\zeta)$$

where $\mathcal{F}$ is the usual Fourier transform, and $j = n + \deg P - k$. The calculation in our case is complicated by the presence of the function $f(\theta)$ in the expression for K.

Sections 2, 3, and 4 contain introductory definitions and results; for more detail see [7], [8], and [9].

In Section 6, we calculate the group Fourier transform of K, homogeneous of degree $> -2n-2$ (hence, K defines a distribution). In Section 7 we consider K homogeneous of degree $\le -2n-2$. In this case K no longer defines a distribution; however, it is possible to define a distribution Λ_K which agrees with K away from the origin. We compute the Fourier transform of Λ_K.

I wish to thank D. Geller for many enlightening discussions.

2. The Heisenberg group. The *Heisenberg group*, $\mathbf{H}^n$, is a Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$. For (ζ,t) and (η,s) in $\mathbf{H}^n$, the multiplication is given by

$$(\zeta,t)\cdot(\eta,s) = (\zeta+\eta, t+s+2\,\mathrm{Im}(\zeta\cdot\overline{\eta})). \tag{1}$$

Let $\zeta = (z_1,\dots,z_n)$ and $z_j = x_j + iy_j$. Then the left invariant vector fields which agree with $\partial/\partial x_j$, $\partial/\partial y_j$, and $\partial/\partial t$ at the origin are, respectively, $X_j = \partial/\partial x_j + 2y_j\partial/\partial t$, $Y_j = \partial/\partial y_j - 2x_j\partial/\partial t$ and $T = \partial/\partial t$. These vector fields form a basis for the Lie algebra of $\mathbf{H}^n$ and they satisfy the following commutation relations:

$$[Y_j, X_k] = 4\delta_{jk}T.$$

All other commutators are zero.

We shall be interested in a particular class of unitary representations of the Heisenberg group and the corresponding representations of the Lie algebra. For all real λ different from zero, define a mapping R_λ from $\mathbf{H}^n$ to

the group of unitary operators on $L^2(\mathbb{R}^n)$ by

$$[R_\lambda(\zeta,t)f](x) = e^{2\pi i\lambda(u\cdot x + u\cdot v/2 + t/4)}f(x+v). \tag{2}$$

Here $\zeta = u + iv$ and $f \in L^2(\mathbb{R}^n)$. These representations are irreducible, and up to unitary equivalence these are all the irreducible, infinite-dimensional representations of the Heisenberg group.

We now turn to the connection between the above representation and the Weyl correspondence.

3. The Weyl correspondence. The Weyl correspondence was originally introduced in the development of quantum mechanics. Classical mechanics involves the study of functions dependent on $2n$ variables, $a(p_1,\dots,p_n,q_1,\dots,q_n)$. The quantum mechanic approach is to replace the p_j and q_j variables by operators P_j and Q_j acting on a Hilbert space H, satisfying the commutation relations

$$[P_j, Q_k] = (\lambda/(2\pi i))\delta_{jk}I.$$

The question then arises: how, in general, is the operator

$$a(P_1,\dots,P_n,Q_1,\dots,Q_n) = a(P,Q)$$

defined? Weyl answered the question in the following way.

First consider the function

$$a(p,q) = e^{2\pi i(u\cdot q + v\cdot p)}. \tag{3}$$

Since the operator $2\pi i(u\cdot Q + v\cdot P)$ is skew adjoint, the operator

$$\mathrm{W}(u,v) = e^{2\pi i(u\cdot Q + v\cdot P)}$$

is unitary and this is the operator assigned to the exponential function (3). Then by analogy with the Fourier transform and its inverse, to any function $a(p,q)$ assign

$$\mathcal{W}(a) = \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi i(u\cdot Q + v\cdot P)}\widehat{a}(u,v)\,du\,dv. \tag{4}$$

Here $\widehat{a}(u,v) = \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-2\pi i(x\cdot u + y\cdot v)}a(x,y)\,dx\,dy$. The operator $\mathcal{W}(a)$ is called the *Weyl correspondent* to $a(p,q)$. One realization of this scheme is to take $L^2(\mathbb{R}^n)$ as the Hilbert space, fix $\lambda = 1$, and set

$$(Q_j f)(x) = x_j f(x),$$

and

$$(P_j f)(x) = \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}(x).$$

Given this choice of H and P and Q, we can see how $\mathrm{W}(u,v)$ operates on a function f in $L^2(\mathbb{R}^n)$. First observe that

$$\mathrm{W}(us,vs)\cdot\mathrm{W}(ut,vt) = \mathrm{W}(u(s+t),v(s+t)) \tag{5}$$

and

$$\frac{\partial \mathrm{W}}{\partial s}(us, vs)\Big|_{s=0} = 2\pi i(u \cdot Q + v \cdot P). \tag{6}$$

Now, W defined by

$$[\mathrm{W}(u,v)f](x) = e^{2\pi i x \cdot u} e^{\pi i u \cdot v} f(x+v)$$

satisfies properties (5) and (6). On the other hand, these properties uniquely determine W.

More generally we work with operators which depend on the nonzero real parameter λ. That is, we have unbounded operators $Q_{j\lambda}$ and $P_{j\lambda}$ and these in turn give rise to operators $\mathrm{W}_\lambda(u,v)$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$ defined by

$$\mathrm{W}_\lambda(u,v) = e^{2\pi i(u\cdot Q_\lambda + v \cdot P_\lambda)}, \tag{7}$$

and corresponding Weyl correspondent $\mathcal{W}_\lambda(a)$. We can view the operators P_λ and Q_λ as operators on Hilbert spaces H_λ also indexed by the nonzero real parameter. However, when convenient we may identify these Hilbert spaces and drop the subscript. Again, if we identify the H_λ with $L^2(\mathbb{R}^n)$, then the $P_{j\lambda}$ and $Q_{j\lambda}$ can be defined by

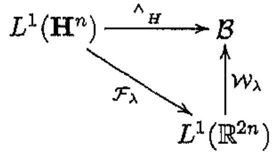
$$Q_{j\lambda} f = \lambda x_j f \quad \text{and} \quad P_{j\lambda} f = \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}.$$

Notice that in this case, $R_\lambda(\zeta, 0) = \mathrm{W}_\lambda(u,v)$.

4. The group Fourier transform. The operators defined in (7) allow us to define a mapping from $L^1(\mathbf{H}^n)$ into the space of families of bounded operators on H_λ. We denote this space of families by $\mathcal{B}$. The map is defined as follows:

$$\widehat{f}_H(\lambda) = \int_{\mathbf{H}^n} e^{\pi i \lambda t/2} \mathrm{W}_\lambda(u,v) f(\zeta, t)\, d\zeta\, dt. \tag{8}$$

The operator $\widehat{f}_H$ is called the *group Fourier transform of* f. Observe that the following diagram commutes:



The diagonal arrow is the Fourier transform

$$\mathcal{F}_\lambda(x,y) = \int_{\mathbf{H}^n} e^{2\pi i(u\cdot x + v \cdot y + t\lambda/4)} f(u,v,t)\, du\, dv\, dt.$$

In the case where H_λ is identified with $L^2(\mathbb{R}^n)$ and if $\phi, \psi \in L^2(\mathbb{R}^n)$, the operator $\widehat{f}_H$ is given by

$$\langle \widehat{f}_H(\lambda)\phi, \psi\rangle = \int_{\mathbf{H}^n} \langle R_\lambda(\zeta,t)\phi, \psi\rangle f(\zeta,t)\, d\zeta\, dt.$$

The group Fourier transform may also be defined in complex notation, and for what follows, complex notation will be more convenient.

Beginning with the Weyl correspondent, we wish to assign an operator to the function $f(\zeta, \overline{\zeta})$. To do this we start with unbounded operators $W_{1\lambda}, \ldots, W_{n\lambda}, W_{1\lambda}^+, \ldots, W_{n\lambda}^+$ on a Hilbert space H_λ satisfying

$$W_{j\lambda}^+ = W_{j\lambda}^*, \qquad [W_{j\lambda}^+, -W_{k\lambda}] = 2\lambda\delta_{jk} I.$$

The connection with the real case is given by the relation

$$W_\lambda = P_\lambda + iQ_\lambda.$$

Then with the function $e^{(-z\cdot\overline{\zeta} + \overline{z}\cdot\zeta)}$ associate the operator

$$\mathrm{W}_\lambda(z, \overline{z}) = e^{-z\cdot W_\lambda^+ + \overline{z}\cdot W_\lambda}.$$

The definition of the Weyl correspondent is analogous to (4). That is,

$$\mathcal{W}_\lambda(f) = \int_{\mathbb{C}^n} \mathrm{W}_\lambda(z,\overline{z})(F^{-1}f)(z)\, dV.$$

Here $F^{-1}f$ is the inverse Fourier transform $\int_{\mathbb{C}^n} \exp(z\cdot\overline{\zeta} - \overline{z}\cdot\zeta) f(\zeta)\, dV$. Now if $f \in L^1(\mathbf{H}^n)$, the group Fourier transform is given by

$$\widehat{f}_H(\lambda) = \int_{\mathbf{H}^n} e^{i\lambda t}\mathrm{W}_\lambda(z,\overline{z}) f(z,t)\, dV.$$

If we again take as our Hilbert space $L^2(\mathbf{H}^n)$, then one realization of the operators $W_{j\lambda}$, $W_{j\lambda}^+$ is

$$W_{j\lambda} = i(2\lambda x_j + (1/2)\partial/\partial x_j), \tag{9}$$

$$W_{j\lambda}^+ = i(-2\lambda x_j + (1/2)\partial/\partial x_j). \tag{10}$$

Associated with each H_λ there is a preferred orthonormal basis $\{E_{\alpha\lambda}\}$ where $\alpha \in (\mathbb{Z}^+)^n$. For a fixed value of λ the basis of $L^2(\mathbf{H}^n)$, $\{E_{\alpha\lambda}\}$, associated with the operators W_λ, W_λ^+ is defined as follows. Set

$$E_{0\lambda}(x) = (|\lambda|/\pi)^{n/4} e^{-2|\lambda|x^2}.$$

Then for all $\alpha \in (\mathbb{Z}^+)^n$, define

$$E_{\alpha\lambda} = \begin{cases} (2|\lambda|)^{-|\alpha|/2}\alpha!^{-1/2}(W_\lambda^+)^\alpha E_{0\lambda} & \text{if } \lambda > 0, \\ (2|\lambda|)^{-|\alpha|/2}\alpha!^{-1/2} W_\lambda^\alpha E_{0\lambda} & \text{if } \lambda < 0. \end{cases}$$

In the case $\lambda = 1/4$, notice $\{(-i)^{|\alpha|}E_{\alpha,1/4}\}$ are the Hermite functions.

In general, for a fixed λ, the operators $W_{j\lambda}$ and $W_{j\lambda}^+$ act as weighted shift operators with respect to $\{E_{\alpha\lambda}\}$. That is,

$$W_{j\lambda}E_{\alpha\lambda} = (2\alpha_k|\lambda|)^{1/2}E_{\alpha-e_k}, \quad \text{zero} \quad \text{if } \alpha_k = 0,$$
$$W_{j\lambda}^+E_{\alpha\lambda} = [2(\alpha_k+1)|\lambda|]^{1/2}E_{\alpha+e_k}$$

for $\lambda > 0$. The right sides are reversed if $\lambda < 0$. Here, e_k denotes $(0,0,\ldots,1,\ldots,0) \in (\mathbb{Z}^+)^n$ with the 1 in the kth position.

Many of the nice properties of the usual Fourier transform have parallels for the group Fourier transform. For example,

$$(f*g)_H^\wedge(\lambda) = \widehat{f}_H(\lambda)\widehat{g}_H(\lambda), \qquad f,g \in L^1(\mathbf{H}^n).$$

Here, the convolution is with respect to the group multiplication, and the multiplication on the right is composition of operators. Also, if we set

$$\overline{Z}_j = (1/2)(X_j + iY_j), \tag{11}$$
$$Z_j = (1/2)(X_j - iY_j), \tag{12}$$

then

$$(Tf)_H^\wedge = -i\lambda\widehat{f}_H, \quad (Z_jf)_H^\wedge = \widehat{f}_H W_{j\lambda}^+, \quad (\overline{Z}_jf)_H^\wedge = -\widehat{f}_H W_{j\lambda}.$$

In the last three equations we assume that f is in the Schwartz space $S(\mathbf{H}^n)$.

There is also an analog of the Plancherel theorem. Let $\mathcal{B} = \{$bounded families R: for each λ, $R(\lambda)$ is a bounded operator on H_λ, $\|R\| = \sup_\lambda \|R(\lambda)\| < \infty$ and for all α, β the map $\lambda \mapsto (R(\lambda)E_{\alpha\lambda}, E_{\beta,\lambda})$ is measurable$\}$. Then the group Fourier transform (which henceforth we denote by $^\wedge$) is a map from $L^1(\mathbf{H}^n)$ into $\mathcal{B}$ and $\|\widehat{f}\| \le \|f\|_1$. Further, if we set

$$R_2 = \Big\{R : \text{for almost every } \lambda,\ \|R(\lambda)\|_2 < \infty \quad \text{and } \|R\|_2^2 = \int_{\mathbb{R}} \|R(\lambda)\|_2^2(2|\lambda|)^n\,d\lambda < \infty\Big\}$$

(here, $\|R(\lambda)\|_2$ is the Hilbert–Schmidt norm of $R(\lambda)$), then $^\wedge$ can be extended to a map from $L^2(\mathbf{H}^n)$ onto R_2 such that, if $f \in L^2(\mathbf{H}^n)$,

$$\|f\|_2^2 = (1/(2\pi^{n+1}))\|\widehat{f}\|_2^2.$$

There is a natural pairing of elements of $\mathcal{B}$. If $R, S \in \mathcal{B}$ and if

$$\int_{-\infty}^{\infty} \sum_\alpha |\langle R(\lambda)E_\alpha, S(\lambda)E_\alpha\rangle|(2|\lambda|)^n\,d\lambda < \infty, \tag{13}$$

then we set

$$(R\,|\,S) = \int_{-\infty}^{\infty} \sum_\alpha \langle R(\lambda)E_\alpha, S(\lambda)E_\alpha\rangle(2|\lambda|)^n\,d\lambda.$$

This pairing allows us to extend the definition of the group Fourier transform to the space of tempered distributions $S'(\mathbf{H}^n)$. Suppose that R is an operator family and that (13) holds for all $S = \widehat{f}$, $f \in S(\mathbf{H}^n)$. Then we say that for $F \in S'$, $\widehat{F} = R$ in the sense of tempered distributions if $F(f) = c_n(R|\widehat{f})$ for all $f \in S(\mathbf{H}^n)$.

5. Regular homogeneous distributions on $\mathbf{H}^n$. Consider the following dilations on the Heisenberg group: For $r > 0$, set $D_r(\zeta,t) = (r\zeta, r^2t)$. Notice that D_r is an automorphism of $\mathbf{H}^n$ whereas the usual dilation is not.

We wish to calculate the group Fourier transform of regular homogeneous distributions on $\mathbf{H}^n$. *Regular* means that the distribution agrees with a C^∞ function away from the origin. A distribution K on $\mathbf{H}^n$ is *homogeneous* of degree l if for all $\phi(\zeta,t) \in S$,

$$\langle K, r^{-2n-2}\phi(r^{-1}\zeta, r^{-2}t)\rangle = \langle r^lK, \phi\rangle.$$

If K is a function, this condition is equivalent to requiring $K(r\zeta, r^2t) = r^lK(\zeta,t)$.

It can be shown [3] that every regular homogeneous distribution on $\mathbf{H}^n$ can be expressed in the form

$$K(t,z) = \sum_i K_i(t,|z|^2)P_i(z) \tag{14}$$

where $\{P_i\}$ form an orthonormal basis for the bigraded spherical harmonics and the K_i are homogeneous of the appropriate degree. We can write

$$K_i(t,|z|^2) = f_i(\theta)\overline{w}^{-k_i} \tag{15}$$

with $w = w(t,z) = |z|^2 - it$, $\theta = \arg(\overline{w}/w)$ and $-2k_i + \deg P_i = \deg K$ ($f_i(\theta)$ is homogeneous of degree zero).

In view of (14) and (15), when calculating the group Fourier transform of a regular homogeneous distribution, it is enough to consider distributions of the form

$$f(\theta)\overline{w}^{-k}P(z). \tag{16}$$

This was done in [5] for the case where $K(t,z) = \overline{w}^{\gamma-k}w^{-\gamma}P(z)$, that is, for $f(\theta) = e^{i\gamma\theta}$. The precise result is stated below in Theorem 5.2, but first we need to consider the Weyl correspondent of a polynomial $Q(z)$.

The proof of the following theorem is given in [5].

THEOREM 5.1. *Suppose P is a harmonic polynomial in ζ, $\overline{\zeta}$ where $\zeta \in \mathbb{C}^n$. If $P = \sum a_{\varrho\gamma}\zeta^\varrho\overline{\zeta}^\gamma$ then $\mathcal{W}(P) = \sum a_{\varrho\gamma}W^\varrho(W^+)^\gamma = \sum a_{\varrho\gamma}(W^+)^\gamma W^\varrho$.*

In equation (14) we stated that the P_i were elements of an orthonormal basis for the bigraded spherical harmonics. We wish now to clarify that terminology. Every polynomial P in the variables ζ and $\overline{\zeta}$ can be written in the form

$$P(\zeta,\overline{\zeta}) = \sum a_{\varrho\gamma}\zeta^{\varrho}\overline{\zeta}^{\gamma},$$

where the sum is taken over all multi-indices $\varrho,\gamma \in (\mathbb{Z}^+)^n$, and $a_{\varrho\gamma}=0$ for all but finitely many (ϱ,γ). Let

$$P_{pq} = \{\text{polynomials } P(\zeta,\overline{\zeta}) : a_{\varrho\gamma}=0 \text{ unless } |\varrho|=p,\ |\gamma|=q\}.$$

Further, let Δ denote the Laplacian. Then

$$\Delta : P_{pq} \to P_{p-1,q-1}.$$

Set

$$H_{pq} = \{P \in P_{pq} : \Delta P = 0\}.$$

The elements of H_{pq} are called the *(solid) bigraded spherical harmonics.*

We are now in a position to state the main result of the paper. The proof will be given in Section 6.

THEOREM 6.6. *Given the regular homogeneous distribution* $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$, *with* $P\in H_{pq}$, $p+q=\kappa$ *and* $-2n-2<\kappa-2\operatorname{Re}k<0$, *the group Fourier transform of* K *is* $\widehat{K}(\lambda)=J(\lambda)$, *where* J *is defined by*

$$J(\lambda)E_{\alpha\lambda} = C\Big(\int_{-\pi}^{\pi} f(\theta)\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_{\alpha\lambda}.$$

Here $M=|\alpha|-p$ *if* $\lambda>0$, $M=|\alpha|-q$ *if* $\lambda<0$, *and*

$$C = (-1)^q\pi^{n+1}2^{1-n-\kappa}|\lambda|^{-j},$$

where $j=n+\kappa+1-k$. *The function* K_M *is a smooth function of* θ *defined in equation* (17) *below.*

The following theorem was proved in [5] as Proposition 7.1.

THEOREM 5.2. *Suppose* $k,\gamma\in\mathbb{C}$, $\kappa=p+q$. *Suppose* γ *and* $k-\gamma$ *are not equal to* $0,-1,-2,\ldots$, *and that*

$$-2n-2<\kappa-2\operatorname{Re}k<0,$$

or that

$$\kappa\ge 1,\qquad \kappa-2k=-2n-2,$$

Define

$$G_{k\gamma}(w) = \Gamma(\gamma)\Gamma(k-\gamma)\overline{w}^{\gamma-k}w^{-\gamma}$$

and

$$K_{k\gamma P}(u) = G_{k\gamma}(w(u))P(z).$$

Let $j=n+\kappa+1-k$. *Then* $\widehat{K}_{k\gamma P}=J_{j\gamma P}$, *defined by*

$$J_{j\gamma P}(\lambda)E_{\alpha\lambda} = (-1)^q\pi^{n+1}2^{1-n-\kappa}c_{j\gamma}(|\alpha|,\lambda)\mathcal{W}_\lambda(P)E_{\alpha\lambda},$$

where the $c_{j\gamma}$ *are given as follows: Let* $p'=p$ *if* $\lambda>0$, $p'=q$ *if* $\lambda<0$, $\gamma'=\gamma$ *if* $\lambda>0$, $\gamma'=k-\gamma$ *if* $\lambda<0$, *and if* $M=|\alpha|-p'\ge 0$ *then*

$$c_{j\gamma}(|\alpha|,\lambda) = |\lambda|^{-j}\Gamma(M+\gamma')\Gamma(j)\Gamma(M+\gamma'+j)^{-1}.$$

REMARKS. The hypothesis requiring that $-2n-2<\kappa-2\operatorname{Re}k$ ensures that $K(u)$ is locally integrable and hence defines a distribution. If $\kappa\ge1$, $\kappa-2k=-2n-2$, then the distribution associated with $K(u)$ is a principal value distribution. The requirement that γ and $k-\gamma$ are not in $\mathbb{Z}^-$ is assumed in order that $\Gamma(\gamma)$ and $\Gamma(k-\gamma)$ be defined, but this last assumption can be dropped if in either of these cases we set

$$G_{k\gamma}(w) = (-1)^l\Gamma(k+l)(l!)^{-1}\overline{w}^{\gamma-k}w^{-\gamma}$$

and

$$c_{j\gamma}(|\alpha|,\lambda) = |\lambda|^{-j}(-1)^{l-M}\Gamma(j)[\Gamma(M+j-l)(l-M)!]^{-1}.$$

Here $l=-\gamma$ if $\gamma\in\mathbb{Z}^-$ and $\lambda>0$ or $l=\gamma-k$ if $k-\gamma\in\mathbb{Z}^-$ and $\lambda<0$. Also, M must be less than or equal to l. For all other values of M and λ we have $c_{j\gamma}=0$. (γ and $k-\gamma$ cannot simultaneously be contained in $\mathbb{Z}^-$ since $\operatorname{Re}k>0$.) For details see [5], Proposition 7.2.

Consider again the functions

$$G_{k\gamma}(w) = \Gamma(\gamma)\Gamma(k-\gamma)\overline{w}^{\gamma-k}w^{-\gamma} = \Gamma(\gamma)\Gamma(k-\gamma)(\overline{w}/w)^{\gamma}\overline{w}^{-k}.$$

So $G_{k\gamma}=\Gamma(\gamma)\Gamma(k-\gamma)e^{i\theta\gamma}\overline{w}^{-k}$, where θ is the argument of $\overline{w}/w$. Now $K_{k\gamma P}=G_{k\gamma}P$ has the form of the distribution given in (16).

We will show that there exists a function $\mathrm{K}_{Mjk}(\theta)$ such that

$$\int_{-\pi}^{\pi}\mathrm{K}_{Mjk}(\theta)\Gamma(k-\gamma)\Gamma(\gamma)e^{i\gamma\theta}\,d\theta = \Gamma(\gamma+M)\Gamma(j)[\Gamma(\gamma+j+M)]^{-1}.$$

The group Fourier transform of $K_{k\gamma P}$ may then be expressed in terms of this kernel $\mathrm{K}_M(\theta)$. First, we need an identity proved in [5] and we recall some properties of hypergeometric functions, the details of which are given in [1].

LEMMA 5.3. *Suppose* $\gamma\in\mathbb{C}$ *and* $\operatorname{Re}k>1$. *Then*

$$\int_{-\pi}^{\pi} e^{i(\gamma-k+1)\theta}(e^{i\theta}+1)^{k-2}\,d\theta = 2\pi\Gamma(k-1)[\Gamma(\gamma)\Gamma(k-\gamma)]^{-1}.$$

DEFINITION 5.4. Set

$$(a)_n = \Gamma(a+n)/\Gamma(a),$$

i.e.,

$$(a)_0=1,\qquad (a)_n=a(a+1)\ldots(a+n-1),\qquad n=1,2,\ldots$$

If $c \neq 0, -1, -2, \ldots$, then define the hypergeometric function F by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n (b)_n z^n / [(c)_n n!]$$

If a or $b = -l$, $l \in \mathbb{Z}^+$, then $(-l)_n = (-1)^n l!/(l-n)!$, $n \le l$, so in these cases the sum is finite.

PROPOSITION 5.5. *If $c \notin \mathbb{Z}^-$ and $\operatorname{Re}(c-a-b) > 0$, then for $z = 1$,*

$$F(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)[\Gamma(c-a)\Gamma(c-b)]^{-1}.$$

THEOREM 5.6. *Suppose $\operatorname{Re}\gamma, \operatorname{Re}(k-\gamma) > 0$ and M is a positive integer. Then*

$$\int_{-\pi}^{\pi} \Gamma(k-\gamma)\Gamma(\gamma) e^{i\gamma\theta} \mathrm{K}_M(\theta)\, d\theta = \Gamma(\gamma+M)\Gamma(j)[\Gamma(\gamma+j+M)]^{-1}$$

where K_M is defined by

$$\mathrm{K}_M(\theta) = (2\pi)^{-1} e^{i(1-k)\theta}(e^{i\theta}+1)^{k+j-2} \times \Gamma(j)[\Gamma(k+j-1)]^{-1} F(-M,j;k+j-1;e^{i\theta}+1). \tag{17}$$

Proof. By Definition 5.4,

$$\int_{-\pi}^{\pi} \Gamma(k-\gamma)\Gamma(\gamma)e^{i\gamma\theta}(2\pi)^{-1}e^{i(1-k)\theta}(e^{i\theta}+1)^{k+j-2}\Gamma(j)[\Gamma(k+j-1)]^{-1} \times F(-M,j;k+j-1;e^{i\theta}+1)\, d\theta$$

$$= \int_{-\pi}^{\pi} (2\pi)^{-1}\Gamma(k-\gamma)\Gamma(\gamma)e^{i\gamma\theta}e^{i(1-k)\theta}(e^{i\theta}+1)^{k+j-2}\Gamma(j) \times \sum_{m=0}^{M} \frac{1}{m!}(-M)_m(j)_m \frac{1}{\Gamma(k+m+j-1)}(e^{i\theta}+1)^m\, d\theta$$

$$= (2\pi)^{-1}\Gamma(k-\gamma)\Gamma(\gamma)\Gamma(j)\sum_{m=0}^{M}\frac{1}{m!}(-M)_m(j)_m\frac{1}{\Gamma(k+n+j-1)} \times \int_{-\pi}^{\pi} e^{i(\gamma+j+m)\theta}e^{i(1-(k+m+j))\theta}(e^{i\theta}+1)^{k+m+j-2}\, d\theta$$

$$= (2\pi)^{-1}\Gamma(k-\gamma)\Gamma(\gamma)\Gamma(j)\sum_{m=0}^{M}\frac{1}{m!}(-M)_m(j)_m\frac{1}{\Gamma(k+m+j-1)} \times \frac{2\pi\Gamma(k+m+j-1)}{\Gamma(\gamma+m+j)\Gamma(k+m+j-(\gamma+m+j))}$$

$$= \frac{\Gamma(\gamma)\Gamma(j)}{\Gamma(\gamma+j)}\sum_{m=0}^{M}\frac{1}{n!}\frac{(-M)_m(j)_m}{(\gamma+j)_m}.$$

The second to last equality follows from Lemma 5.3.

Now, by Definition 5.4 and using Proposition 5.5 where we take $a = -M$, $b = j$ and $c = \gamma + j$ we obtain

$$\frac{\Gamma(\gamma)\Gamma(j)}{\Gamma(\gamma+j)}\sum_{m=0}^{M}\frac{1}{n!}\frac{(-M)_m(j)_m}{(\gamma+j)_m} = \frac{\Gamma(\gamma)\Gamma(j)}{\Gamma(\gamma+j)}\frac{\Gamma(\gamma+j)\Gamma(\gamma+j-(-M)-j)}{\Gamma(\gamma+j-j)\Gamma(\gamma+j-(-M))} = \frac{\Gamma(\gamma+M)\Gamma(j)}{\Gamma(\gamma+j+M)},$$

and the proof is complete.

COROLLARY 5.7. *Suppose $K_{k\gamma P}$ is defined as in Theorem 5.2. Then for all $\gamma, k-\gamma \in \mathbb{C} - \mathbb{Z}^-$,*

$$\widehat{K}_{k\gamma P}E_\alpha = \begin{cases} C\Big(\int_{-\pi}^{\pi} \Gamma(k-\gamma)\Gamma(\gamma)e^{i\gamma\theta}\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_\alpha, & \lambda > 0,\\ C\Big(\int_{-\pi}^{\pi} \Gamma(k-\gamma)\Gamma(\gamma)e^{i(k-\gamma)\theta}\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_\alpha, & \lambda < 0,\end{cases}$$

where $C = (-1)^q \pi^{n+1} 2^{1-n-\kappa}|\lambda|^{-j}$. For either γ or $k-\gamma \in \mathbb{Z}^-$, $\Gamma(k-\gamma)\Gamma(\gamma)$ is replaced by $(-1)^l\Gamma(k+l)(l!)^{-1}$, where l is defined in the remarks after 5.2.

Proof. The corollary follows directly from the theorem for $\operatorname{Re}\gamma > 0$ and $\operatorname{Re} k - \gamma > 0$ but the result holds for all $\gamma, k-\gamma \in \mathbb{C}$ by analytic continuation. That is, both sides of the equation

$$\int_{-\pi}^{\pi} \Gamma(k-\gamma)\Gamma(\gamma)e^{i\gamma'\theta}\mathrm{K}_M(\theta)\,d\theta = \Gamma(\gamma'+M)\Gamma(j)[\Gamma(\gamma'+j+M)]^{-1}$$

have analytic extensions as functions of γ' to the entire complex plane and since they agree for $\operatorname{Re}\gamma' > 0$ they are equal for all γ'. (Here γ' is as in Theorem 5.2.)

6. The group Fourier transform of regular homogeneous distributions. The goal of this section is to show that functions $f(\theta) \in C^l([-\pi,\pi])$ can be approximated in C^l by linear combinations over C of functions of the form $e^{i\gamma\theta}$ for $\gamma \in \mathbb{C}$. We will denote the algebra of all such functions by A. Then the group Fourier transform of distributions of the form given in (16) will be calculated in terms of the kernel $\mathrm{K}_M(\theta)$.

First we need a lemma:

LEMMA 6.1. *Every polynomial $p(\theta)$ can be approximated in C^l by elements of the algebra A.*

Set $S^N = \{f \in C^N[-\pi,\pi] : f^{(k)}(-\pi) = f^{(k)}(\pi) \text{ for } 0 \le k \le N\}$. We know that if $f \in S^N$, the Fourier series for f converges to f in C^{N-2}. So, given $p(\theta)$, it is enough to show that there exists an $F \in A$ such that $p(\theta) - F(\theta) \in S^{l+2}$. That is, we need an $F \in A$ such that $F^{(n)}(-\pi) - F^{(n)}(\pi) = p^{(n)}(-\pi) - p^{(n)}(\pi)$ for $0 \le n \le l+2$. Denote the constants $p^{(n)}(-\pi) - p^{(n)}(\pi)$ by C_n. Next, choose constants $\gamma_0, \gamma_1, \ldots, \gamma_{l+2} \in \mathbb{C}$ distinct and not integers. Set

$$F(\theta) = \sum_{k=0}^{l+2} c_k e^{i\gamma_k\theta},$$

the c_k to be determined. Then

$$F^n(\theta) = \sum_{k=0}^{l+2} c_k (i\gamma_k)^n e^{i\gamma_k\theta}.$$

Hence,

$$F^n(-\pi) - F^n(\pi) = \sum_{k=0}^{l+2} c_k (i\gamma_k)^n [e^{-i\pi\gamma_k} - e^{i\pi\gamma_k}].$$

If we set $d_k = c_k[e^{-i\pi\gamma_k} - e^{i\pi\gamma_k}]$, we have the system of equations

$$\sum_{k=0}^{l+2} d_k (i\gamma_k)^n = C_n.$$

The matrix associated with this system is the Vandermonde matrix so the system has a solution.

PROPOSITION 6.2. *For any continuous function $f(\theta) \in C^l([-\pi,\pi])$, there exist functions $f_k(\theta) = \sum_{j=0}^{N_k} c_{jk} e^{i\gamma_{jk}\theta}$ such that f_k converges to f in C^l.*

Proof. Notice that the algebra, A, of functions consisting of the linear combinations of the $e^{i\gamma\theta}$, $\theta \in [-\pi,\pi]$, separates points and is closed under complex conjugation. Hence, by the complex Stone–Weierstrass theorem every continuous function $f(\theta)$ can be uniformly approximated by functions in this algebra.

Next, consider a function $f(\theta) \in C^l$. Its lth derivative $f^{(l)}$ is continuous and so there exists a sequence, $f_{lk} \in A$, which converges uniformly to $f^{(l)}$. Set $F_k(\theta) = \int_0^\theta f_{lk}(\phi)\,d\phi$. Then the $F_k(\theta)$ converge uniformly to $\int_0^\theta f^l(\phi)\,d\phi$. Thus, $f^{(l-1)} = c + \lim_{k\to\infty} F_k$, and the convergence is uniform. By repeated integrations, we obtain a sequence of functions in A which converges to $f(\theta) - p(\theta) \in C^l$ so by the lemma there exists $f_k \in A$ which converges to $f \in C^l$.

PROPOSITION 6.3. *The distribution $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$ is contained in $C^l(\mathbf{H}^n - \{0\})$ if and only if f is contained in C^l.*

Proof. First, assume $f \in C^l$. Recall that $\theta = \arg(\overline{w}/w)$. Set $\theta = -i(\ln\overline{w} - \ln w)$, where the natural logarithm is defined on the principal branch. (That is, $\ln z$ is analytic away from the negative real axis.)

Now $w = |z|^2 - it$, hence $\operatorname{Re} w = \operatorname{Re}\overline{w} \ge 0$ for all z, t. Thus, $\ln w$, $\ln\overline{w}$ are smooth for all $w \ne 0$. So, K is a product and composition of functions at least C^l.

Going the other way, suppose $K(t,z) \in C^l(\mathbf{H}^n - \{0\})$. Now,

$$f(\theta) = \frac{K(t,z)\overline{w}^{-k}}{P(z)}.$$

Fix z_0 and consider

$$\theta_0 = -i(\ln(|z_0|^2 + it) - \ln(|z_0|^2 - it)).$$

We wish to show that f is continuous at θ_0.

Suppose $|z_0| = r$. Then

$$\theta_0 = -i(\ln(|z|^2 + it) - \ln(|z|^2 - it)),$$

for all $|z| = r$. Recall that $P(z)$ is a basis element for the space of spherical harmonics, so $P(z)$ is not identically zero on the sphere $|z| = r$. Choose z_1 such that $|z_1| = r$ and $P(z_1) \ne 0$. Then

$$f(\theta_0) = \frac{K(t,z_1)\overline{w}^{-k}}{P(z_1)}$$

is defined and f is continuous at θ_0 since K, $\overline{w}$, and P are continuous at z_1 and $P(z_1) \ne 0$. Similarly it can be shown that the derivatives of f through l are continuous.

COROLLARY 6.4. *For a given distribution $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$, there exist distributions $K_n(t,z) = f_n(\theta)\overline{w}^{-k}P(z)$ such that $f_n(\theta) \in A$ and $\|K - K_n\|_{C^l} \to 0$.*

Proof. This follows directly from Propositions 6.3 and 6.2.

Using Corollaries 6.4 and 5.7, we will show, for $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$,

$$\widehat{K}E_\alpha = C\Big(\int_{-\pi}^{\pi} f(\theta)\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}(P)E_\alpha.$$

First we need an estimate proven in [5].

Define Hilbert spaces H_λ^k, $k \in \mathbb{R}$, as follows. Consider vectors $v = \sum_\alpha v_\alpha E_{\alpha\lambda}$ such that $\sum_\alpha(|\alpha| + 1)^k|v_\alpha|^2 = \|v\|_k^2 < \infty$. These vectors with this norm form a Hilbert space for $k \in \mathbb{R}$. If $k \in \mathbb{C}$, set $H_\lambda^k = H_\lambda^{\operatorname{Re} k}$.

PROPOSITION 6.5. *Suppose K is a regular homogeneous distribution of order k and $\widehat{K} = J$. Then each $J(\lambda)$ has an extension as a bounded operator from H_λ^{-k} to H_λ^0. Further, there exist constants c and l such that $\|J\|_{-k,0} < c\|K\|_{C^l}$. (Here, $\| \|_{-k,0}$ denotes the operator norm from H^{-k} to H^0 and $\| \|_{C^l}$ denotes the C^l norm over $\{1 \le |u| \le 2\}$).*

THEOREM 6.6. *For the regular homogeneous distribution*

$$K(t,z) = f(\theta)\overline{w}^{-k}P(z),$$

$P \in H_{pq}$, $p+q=\kappa$ and $-2n-2 < \kappa - 2\,\mathrm{Re}\,k < 0$, the group Fourier transform of K is $\widehat{K}(\lambda) = J(\lambda)$, where J is defined by

$$J(\lambda)E_{\alpha\lambda} = C\Big(\int_{-\pi}^{\pi} f(\theta)\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_{\alpha\lambda}.$$

Here, $M=|\alpha|-p$ if $\lambda>0$, $M=|\alpha|-q$ if $\lambda<0$, $C=(-1)^q\pi^{n+1}2^{1-n-\kappa}|\lambda|^{-j}$, where $j=n+\kappa+1-k$. The function K_M is a smooth function of θ defined in equation (17) above.

Proof. By Proposition 6.2 and Corollary 6.4 there exist distributions $K_n = f_n(\theta)\overline{w}^{-k}P(z)$ such that $f_n(\theta) \to f(\theta)$ in $C^l[-\pi,\pi]$ and $K_n \to K$ in $C^l(\mathbf{H}^n - \{0\})$ for l arbitrarily large (since K is a regular distribution).

By Corollary 5.7, we have

$$\widehat{K}_n E_{\alpha\lambda} = C\Big(\int_{-\pi}^{\pi} f_n(\theta)\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_{\alpha\lambda}.$$

Hence, for all α, β,

$$\langle \widehat{K}_n E_{\alpha\lambda}, E_{\beta\lambda}\rangle \to \Big\langle C\Big(\int_{-\pi}^{\pi} f(\theta)\mathrm{K}_M(\theta)\,d\theta\Big)\mathcal{W}_\lambda(P)E_{\alpha\lambda}, E_{\beta\lambda}\Big\rangle.$$

However, by Proposition 6.5, $\widehat{K}_n \to \widehat{K}$ in $\| \|_{-(\deg K),0}$, but this implies

$$\langle \widehat{K}_n E_{\alpha\lambda}, E_{\beta\lambda}\rangle \to \langle \widehat{K} E_{\alpha\lambda}, E_{\beta\lambda}\rangle$$

for all α, β. So $\widehat{K} = J$.

7. Functions which are not locally integrable about the origin. In the previous section, we calculated the group Fourier transform of homogeneous distributions $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$ where $\kappa - 2\,\mathrm{Re}\,k > -2n-2$. This hypothesis ensured that K was locally integrable and hence defined a distribution. We now consider K such that $\kappa - 2\,\mathrm{Re}\,k \le -2n-2$. In this case K no longer defines a distribution, but it is possible to define a distribution which agrees with K away from 0. We will investigate the group Fourier transform of this new class of distributions. First we need some results and definitions given in [6].

Recall that $\mathcal{B}$ is the set of families of operators $R(\lambda)$ where each $R(\lambda)$, $\lambda \in \mathbb{R}^*$, is a bounded operator on H_λ. Let $\{R_{\alpha\beta}(\lambda)\}$ be the matrix of $R(\lambda)$ with respect to the orthonormal basis $E_{\alpha\lambda}$; that is, $R(\lambda)E_{\beta\lambda} = \sum R_{\alpha\beta}(\lambda)E_{\alpha\lambda}$. We denote by $\mathcal{Q}$ the subset of $\mathcal{B}$ defined by

$$\mathcal{Q} = \{R(\lambda) : R_{\alpha\beta}(\lambda) \in C_0^\infty(\mathbb{R}^*) \text{ for all } \alpha,\beta, \text{ and for some } N \in \mathbb{N},\ R_{\alpha\beta}(\lambda) = 0 \text{ if } |\alpha|+|\beta| > N\}.$$

PROPOSITION 7.1. *For any $R \in \mathcal{Q}$, there exists $f \in S(\mathbf{H}^n)$ such that $\widehat{f} = R$.*

PROPOSITION 7.2. *Suppose G is a homogeneous function of degree j on $\mathbf{H}^n$ which is locally integrable away from 0. Then there exists a number $M(G)$ such that for any $0 < A < B$ we have*

$$\int_{A<|x|<B} G(x)\,dx = \begin{cases} M(G)(2n+2+j)^{-1}(B^{2n+2+j} - A^{2n+2+j}) & \text{if } j \ne -2n-2,\\ M(G)\log(B/A) & \text{if } j = -2n-2.\end{cases}$$

Suppose G is a homogeneous function of degree $j \in \mathbb{C}$ which is smooth away from 0. The preceding proposition allows us to define a distribution $\Lambda_G \in S'$ by

$$\Lambda_G(\phi) = \int_{|\zeta|\le 1}\Big(\phi(\zeta) - \sum_{|\alpha|\le N}\phi^{(\alpha)}(0)\zeta^\alpha/\alpha!\Big)G(\zeta)\,d\zeta + \sum_{|\alpha|\le N,\,|\alpha|\ne -2n-2-j}(2n+2+|\alpha|+j)^{-1}M(\zeta^\alpha G)\phi^{(\alpha)}(0)/\alpha! + \int_{|\zeta|>1}\phi(\zeta)G(\zeta)\,d\zeta$$

where N is chosen arbitrarily with $N \ge -2n-2-\mathrm{Re}\,j-1$. Notice that, if $\mathrm{Re}\,j > -2n-2$, we can choose $N=-1$ so $\Lambda_G = G$. If $\mathrm{Re}\,j \le -2n-2$ and $-j-2n-2 \notin \mathbb{Z}^+$, then Λ_G is a distribution which agrees with G away from 0.

THEOREM 7.3. *Suppose $K(t,z) = f(\theta)\overline{w}^{-k}P(z)$ where $\kappa - 2\,\mathrm{Re}\,k \le -2n-2$. Further, suppose that $\mathrm{Im}\,k \ne 0$ then*

$$(\widehat{\Lambda}_K \mid \widehat{\phi}) = \int_{-\infty}^{\infty}\sum_\alpha \langle J(\lambda)E_{\alpha\lambda}, \widehat{\phi}E_{\alpha\lambda}\rangle(2|\lambda|)^n\,d\lambda,$$

where J and κ are defined in Theorem 6.6, and $\widehat{\phi} \in \mathcal{Q}$.

Proof. The result holds for $\kappa - 2\,\mathrm{Re}\,k > -2n-2$ by Theorem 6.6, since in this case $\Lambda_K = K$.

Now, fix k_0 such that $\operatorname{Im} k_0 \neq 0$ and $\kappa - 2\operatorname{Re} k_0 \leq -2n-2$. Next, fix N_0 such that $N_0 > -2n-2-(\kappa - 2\operatorname{Re} k_0) - 1$. Let $\Lambda_K^{N_0}$ denote Λ_K with the particular choice of $N = N_0$. Then $\Lambda_K^{N_0}(\phi)$ depends analytically on k where $\operatorname{Im} k \neq 0$ and $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$.

For all $\widehat{\phi} \in \mathcal{Q}$, we have $\Lambda_K(\phi) = c_n(\widehat{\Lambda}_K \mid \widehat{\phi})$. But for k satisfying $\kappa - 2\operatorname{Re} k > -2n-2$,

$$(\widehat{\Lambda}_K^{N_0} \mid \widehat{\phi}) = \int_{-\infty}^{\infty} \sum_{\alpha} \langle J(\lambda) E_{\alpha\lambda}, \widehat{\phi} E_{\alpha\lambda}\rangle (2|\lambda|)^n \, d\lambda,$$

where the right hand side depends analytically on k for $\operatorname{Im} k \neq 0$ and $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$. Hence, by analytic continuation, the statement of the theorem holds for all k satisfying $\operatorname{Im} k \neq 0$, $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$. So in particular the result holds for k_0, but k_0 was an arbitrary complex number satisfying $\operatorname{Im} k_0 \neq 0$, $\kappa - \operatorname{Re} k_0 \leq -2n-2$. So the theorem is proved.

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Sobolev embeddings with variable exponent

by

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Abstract. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let $p : \overline{\Omega} \to [1,\infty)$ be Lipschitz-continuous. We consider the generalised Lebesgue space $L^{p(x)}(\Omega)$ and the corresponding Sobolev space $W^{1,p(x)}(\Omega)$, consisting of all $f \in L^{p(x)}(\Omega)$ with first-order distributional derivatives in $L^{p(x)}(\Omega)$. It is shown that if $1 \leq p(x) < n$ for all $x \in \Omega$, then there is a constant $c > 0$ such that for all $f \in W^{1,p(x)}(\Omega)$,

$$\|f\|_{M,\Omega} \leq c\|f\|_{1,p,\Omega}.$$

Here $\|\cdot\|_{M,\Omega}$ is the norm on an appropriate space of Orlicz–Musielak type and $\|\cdot\|_{1,p,\Omega}$ is the norm on $W^{1,p(x)}(\Omega)$. The inequality reduces to the usual Sobolev inequality if $\sup_\Omega p < n$. Corresponding results are proved for the case in which $p(x) > n$ for all $x \in \Omega$.

1. Introduction. The most common assumptions in existence theorems for the Dirichlet boundary-value problem for the quasi-linear equation

$$-\sum_{i=1}^{n} D_i a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x), \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$, involve the polynomial growth of coefficients:

$$|a_i(x,\xi)| \leq g(x) + c|\xi|^{q-1}, \quad g \in L^{q'}(\Omega),$$
$$\sum_{i=0}^{n} a_i(x,\xi)\xi_i \geq c_1|\xi|^p - c_2,$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$.

Similarly, regularity problems for variational integrals $\int_\Omega F(\nabla u(x))\,dx$ are solved under the assumption

$$c_1|\xi|^p \leq F(\xi) \leq c_2(1+|\xi|)^q, \quad \xi \in \mathbb{R}^n.$$

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