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Selfsimilar profiles in large time asymptotics of solutions to damped wave equations

by

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Abstract. Large time behavior of solutions to the generalized damped wave equation $u_{tt} + Au_t + \nu Bu + F(x,t,u,u_t,\nabla u) = 0$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ is studied. First, we consider the linear nonhomogeneous equation, i.e. with $F = F(x,t)$ independent of u. We impose conditions on the operators A and B, on F, as well as on the initial data which lead to the selfsimilar large time asymptotics of solutions. Next, this abstract result is applied to the equation where $Au_t = u_t$, $Bu = -\Delta u$, and the nonlinear term is either $|u_t|^{q-1}u_t$ or $|u|^{\alpha-1}u$. In this case, the asymptotic profile of solutions is given by a multiple of the Gauss–Weierstrass kernel. Our method of proof does not require the smallness assumption on the initial conditions.

1. Introduction. The goal of this paper is to study the large time behavior of solutions to the initial value problem for the generalized semilinear wave equation with a dissipative term

$$u_{tt} + Au_t + \nu Bu + F(x,t,u,u_t,\nabla u) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

In the equation above, the pseudodifferential operators A and B are defined via the Fourier transform by the formulae

$$\widehat{Av}(\xi) = |\xi|^a\widehat{v}(\xi) \quad \text{and} \quad \widehat{Bv}(\xi) = |\xi|^b\widehat{v}(\xi) \tag{1.3}$$

for some real constants a and b satisfying $0 \le 2a < b$. Moreover, $\nu > 0$ is a fixed constant, and assumptions on the nonlinear term are specified in Section 2 below.

Our main purpose is to find conditions on the operators A and B, on the nonlinearity F, as well as on the initial data u_0 and u_1, which lead to the *selfsimilar large time behavior* of solutions. In the first step of our considerations, using the Fourier transform we solve the linear equation

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$u_{tt} + Au_t + \nu Bu = 0$ supplemented with the initial data (1.2). A careful analysis of the solution formula to this linear equation leads to the conclusion that they behave, as $t \to \infty$, like solutions of the diffusion equation $v_t + \nu A^{-1}Bv = 0$ supplemented with the initial condition $v(x,0) = u_0(x) + A^{-1}u_1(x)$ (assumptions, statement of results, and their proofs are given in Section 4, cf. Propositions 4.1 and 4.2).

Next, we assume that $a = 0$ (i.e. $Au_t = u_t$) and $F = -f(x,t)$ is independent of u, which reduces (1.1) to the linear nonhomogeneous equation. In this case, we prove that the large time asymptotics in $L^p(\mathbb{R}^n)$, $p \geq 2$, of solutions is described by a multiple MG_b of the fundamental solution G_b of the equation $u_t + \nu Bu = 0$. Note that such a fundamental solution has the selfsimilar form $t^{-n/b}G_b(x/t^{1/b})$. Moreover, the constant M depends on $f(x,t)$ and on the initial data (1.2) only (cf. Theorem 2.1 below).

Finally, we apply our results concerning the linear nonhomogeneous equation to the nonlinear problem (1.1)–(1.2) with $a = 0$ and $b = 2$ (i.e. $B = -\Delta$) obtaining the selfsimilar asymptotics of solutions in three different cases:

1. $x \in \mathbb{R}$, $F(x,t,u,u_t,\nabla u) = |u_t|^{q-1}u_t$ and *no smallness assumptions* imposed on the initial data (cf. Theorem 2.2);
2. n-dimensional case, $F(x,t,u,u_t,\nabla u)$ behaves like $|u|^{\alpha-1}u$, and again, we *do not assume* the initial data to be small (cf. Theorem 2.3);
3. general $F(x,t,u,u_t,\nabla u)$ and small initial data (1.2) (cf. Remark 2.3).

Fine estimates of oscillatory integrals which appear in the variation-of-constants formula for solutions to the linear nonhomogeneous equation $u_{tt} + Au_t + \nu Bu = f$ are the core of this paper. We systematically use tools from the harmonic analysis on $\mathbb{R}^n$ gathered in the monographs by Stein [28, 29]. In our considerations, we also need a certain variant of the Marcinkiewicz–Hörmander multiplier criteria for symbols having singularities away from the origin (cf. Remark 2.1). Let us also emphasize that more general operators A and B than those defined in (1.3) can be treated by methods from the paper. The only requirement is that their symbols are sufficiently smooth for $\xi \neq 0$ and behave asymptotically like $|\xi|^a$ and $|\xi|^b$, respectively, as $|\xi| \to 0$. Section 2 contains a detailed presentation and discussion of our results.

The proof of local and global existence of solutions to a large class of partial differential equations of the second order, similar to (1.1), can be found e.g. in [27]. We also refer the reader to [19] for a deeper discussion of the nonlinear wave equations. Let us remark that basic questions on the existence and uniqueness of solutions to (1.1)–(1.2) can also be studied via the semigroup method—an example of such a reasoning for a particular equation can be found in [10, Section 2.1].

There are several papers where asymptotic properties of solutions to some versions of equation (1.1) are considered.

The decay, in various norms, of solutions to the Cauchy problem for the semilinear wave equation with the first order dissipation $u_{tt} - \Delta u + u_t + f(u,u_t,\nabla u) = 0$ was studied by Matsumura [20] and by Kawashima *et al.* [18]. In [20] it is proved that, in some cases and for small initial data, decay rates of solutions agree with the analogous estimates to the linearized equation. This research was continued in [18] and similar results were obtained for $f(u,u_t,\nabla u) = |u|^\alpha u$, however, *without* smallness assumptions on the initial conditions.

Racke [26] discussed estimates of solutions to the damped system of the following type: $u_{tt} + cu_t + Bu = 0$ in an exterior domain and for B being the power of an elliptic operator in divergence form.

Partition of energy as time tends to infinity in a strongly damped generalized linear wave equation was proved by Biler [1]. Next, in the subsequent paper [2], Biler studied nonlinear wave-type equations with strong damping. An introduction of (pseudo)conformal invariants of the linear part of the equation allowed him to obtain optimal decay rates for solutions to the linearized problems. Then decay estimates for nonlinear problems were proved using scattering theory tools.

The idea that large time behavior of solutions to hyperbolic equations with damping is described by a selfsimilar diffusive profile is not new. Some results in this direction have already been obtained by Hsiao and Liu [11, 12], Nishihara [21–23], Nishihara and Yang [24], Yang and Milani [30]. Analyzing the long time behavior of a damped hyperbolic system, these authors showed that solutions of a conservative quasilinear wave equation converge, as $t \to \infty$, to solutions of the corresponding quasilinear heat equation.

Recently, Gallay and Raugel [10] studied large time behavior of small solutions to the nonlinear damped wave equation $\varepsilon u_{tt} + u_t = (a(x)u_x)_x + N(u,u_x,u_t)$, $x \in \mathbb{R}$, under the assumption that the diffusion coefficient $a(x)$ converges to positive limits $a_\pm$ as $x \to \pm\infty$. Introducing scaling variables and using various energy estimates, they found an asymptotic expansion of solutions as $t \to \infty$. They showed that this expansion is determined, up to the second order, by a linear parabolic equation whose form depends only on the limiting values $a_\pm$. On the other hand, in a successive paper [9], for $a \equiv 1$ and $N(u,u_x,u_t) = u - u^2$, they studied properties of travelling fronts, and they proved various stability results which are similar to those for the corresponding parabolic equation.

Let us finally mention that there are a lot of papers containing similar results on large time behavior of solutions to conservation laws with dissipation and dispersion (cf. e.g. [3–5, 8, 14–17] and the references given there). In all those publications, the asymptotics of solutions is given by a selfsimilar diffusive profile. Our list of such papers is by no means exhaustive—we only cite the publications which had a direct influence on this paper; moreover, they used methods similar to ours.

Assumptions and statement of our main results can be found in Section 2. Section 3 contains technical lemmata concerning the asymptotic properties of the semigroup of linear operators e^{-tB}. Those facts will be useful in the proof of the first theorem in Section 2. We study the large time behavior of solutions to the linear nonhomogeneous equation $u_{tt}+Au_t+\nu Bu=f(x,t)$ in Section 4. The proofs of Theorems 2.2–2.3 can be found in Section 5.

Notation. Throughout this paper we use the notation $\|u\|_p$ for the Lebesgue $L^p(\mathbb{R}^n)$-norms of functions. As usual, for $p=\infty$, $1/p$ is understood as 0, and $2p/(2+p)$ as 2. The Fourier transform of v is given by $\mathcal{F}v(\xi)=\widehat{v}(\xi)\equiv\int_{\mathbb{R}^n}e^{-ix\xi}v(x)\,dx$. Given $l>0$ the operator D^l is defined in the Fourier variables as $\mathcal{F}[D^lv](\xi)=|\xi|^l\mathcal{F}[v](x)$. The constants independent of the solutions considered and of t (but perhaps depending on the initial values) will be denoted by the same letter C, even if they may vary from line to line. Occasionally, we write e.g. $C=C(\alpha,l)$ when we want to emphasize the dependence of C on parameters α and l.

2. Results and comments. Our considerations begin by the study of the asymptotic behavior of solutions to the problem

$$u_{tt}+Au_t+\nu Bu=f(x,t),\quad x\in\mathbb{R}^n,t>0, \tag{2.1}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), \tag{2.2}$$

where the pseudodifferential operators A and B are defined via the Fourier transform in (1.3), and the external force $f(x,t)$ satisfies assumptions specified below (cf. (2.11) and (2.12)). This linear nonhomogeneous equation can be solved following the standard procedure. Indeed, for sufficiently regular initial data and external forces, computing the Fourier transform of (2.1) with respect to x, solving the resulting ordinary differential equation with respect to t and inverting the Fourier transform, we obtain the variation-of-constants formula for solutions

$$u(x,t)=S(t)u_0(x)+T(t)u_1(x)+\int_0^t T(t-\tau)f(x,\tau)\,d\tau. \tag{2.3}$$

In the expression above, the linear operators $S(t)$ and $T(t)$ are defined as the Fourier multiplier operators

$$\mathcal{F}[S(t)u_0](\xi)=m_S(\xi,t)\widehat{u}_0(\xi)\quad\text{and}\quad\mathcal{F}[T(t)u_1](\xi)=m_T(\xi,t)\widehat{u}_1(\xi) \tag{2.4}$$

with the symbols

$$m_S(\xi,t)=e^{-(t/2)|\xi|^a(1-\sqrt{1-4\nu|\xi|^{b-2a}})}+\frac{-1+\sqrt{1-4\nu|\xi|^{b-2a}}}{\sqrt{1-4\nu|\xi|^{b-2a}}}e^{-(t/2)|\xi|^a}\sinh\Big(\frac{t}{2}\sqrt{|\xi|^{2a}-4\nu|\xi|^b}\Big), \tag{2.5}$$

$$m_T(\xi,t)=2e^{-(t/2)|\xi|^a}\frac{\sinh((t/2)\sqrt{|\xi|^{2a}-4\nu|\xi|^b})}{\sqrt{|\xi|^{2a}-4\nu|\xi|^b}}. \tag{2.6}$$

We refer the reader to Section 4 which contains the detailed analysis of the asymptotic properties of $u(x,t)$ defined by (2.3) under the assumption $0\le 2a<b$ (this is the so-called non-overdamped case). The results from that section lead to the conclusion that, for $a=0$, the solutions to (2.1)–(2.2) behave, for large t, like a multiple of a selfsimilar function. Before we formulate our first theorem let us recall some auxiliary facts.

Each solution $v=v(x,t)$, $x\in\mathbb{R}^n$, $t>0$, of the Cauchy problem

$$v_t+\nu Bv=0,\quad v(x,0)=v_0(x), \tag{2.7}$$

with sufficiently regular v_0, can be expressed via the Fourier transform as

$$v(x,t)=G_b(\nu t)*v_0(x)\equiv e^{-\nu tB}v_0(x), \tag{2.8}$$

where the kernel G_b is defined by

$$G_b(x,t)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-t|\xi|^b+ix\xi}\,d\xi. \tag{2.9}$$

It is easy to check (by a change of variables in (2.9)) that G_b has the selfsimilarity property

$$G_b(x,t)=t^{-n/b}G_b(xt^{-1/b},1). \tag{2.10}$$

Detailed analysis of asymptotic properties of solutions to (2.7) is contained in Section 3.

Now, we are in a position to present our first result stating that the large time asymptotics of solutions to (2.1)–(2.2) for $a=0$ is given by a multiple of the selfsimilar kernel $G_b(x,t)$.

THEOREM 2.1. *Let $a=0$ and $p\in[2,\infty]$. Assume that u is the solution to (2.1)–(2.2) corresponding to the initial data $u_0,u_1\in L^1(\mathbb{R}^n)$, satisfying moreover $D^{b/2}u_0,u_1\in L^p(\mathbb{R}^n)$. If $2\le p<2n/(n-1)$ suppose that there exists $\varepsilon>0$ such that*

$$\begin{aligned}\|f(\cdot,t)\|_1&\le C_1(1+t)^{-1-\varepsilon},\\ \|f(\cdot,t)\|_p&\le C_2(1+t)^{-(n/b)(1-1/p)-1-\varepsilon}\end{aligned} \tag{2.11}$$

for all $t>0$ and positive constants C_1,C_2 independent of t. If $2n/(n-1)\le p\le\infty$ we require, besides (2.11), a faster decay rate of the $L^{2p/(p+2)}$-norm of $f(x,t)$, namely,

$$\|f(\cdot,t)\|_{2p/(p+2)}\le C_3(1+t)^{-(n/b)(1-1/p)-1-\varepsilon} \tag{2.12}$$

for all $t > 0$ *and a positive constant* C_3. *Then*

$$\|u(\cdot,t) - MG_b(\cdot,\nu t)\|_p = o(t^{-(n/b)(1-1/p)}) \quad \text{as } t\to\infty, \tag{2.13}$$

where

$$M = \int_{\mathbb{R}^n} u_0(x)\,dx + \int_{\mathbb{R}^n} u_1(x)\,dx + \int_0^\infty \int_{\mathbb{R}^n} f(x,t)\,dx\,dt.$$

This theorem is an immediate consequence of Corollaries 4.1–4.2 and Proposition 4.3 from Section 4 which analyze, in detail, the behavior as $t\to\infty$ of each term on the right hand side of (2.3).

REMARK 2.1. Combining the Hölder inequality with (2.11) we obtain

$$\|f(\cdot,t)\|_{2p/(p+2)} \le C(1+t)^{-(n/b)(1/2-1/p)-1-\varepsilon}$$

for all $t > 0$ and a constant C. The stronger decay estimate (2.12) is required in the proof of Proposition 4.3, when the inequality

$$\|T_{\chi_2}(t)f(\cdot,\tau)\|_p \le C(t)\|f(\cdot,\tau)\|_p \tag{2.14}$$

fails to hold for $p \ge 2n/(n-1)$. Here, $T_{\chi_2}(t)$ is the Fourier multiplier operator defined by $\mathcal{F}[T(t)v](\xi) = m_T(\xi,t)\chi_2(\xi)\mathcal{F}[v](\xi)$ for $m_T(\xi,t)$ given in (2.6) with $a = 0$ and $\chi_2(\xi)$ being a smooth radial compactly supported function; cf. the beginning of Section 4. On the other hand, to get (2.14) for $2 \le p < 2n/(n-1)$, we apply a variant of the Marcinkiewicz–Hörmander multiplier criteria formulated and proved by Igari and Kuratsubo in [13], and the restrictions on p in Theorem 2.1 are required by their result.

Now, we present the way of using Theorem 2.1 in the study of the large time behavior of solutions to the nonlinear wave equation with the first order dissipation

$$u_{tt} + u_t - \nu\Delta u + F(u,u_t,\nabla u) = 0, \quad x\in\mathbb{R}^n, \tag{2.15}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{2.16}$$

We are going to apply known estimates of solutions to (2.15)–(2.16) in order to prove that the function $f(x,t) = -F(u(x,t),u_t(x,t),\nabla u(x,t))$ satisfies the estimates (2.11) and (2.12). Next, Theorem 2.1 will conclude the proof.

Let us first apply the decay estimates obtained by Matsumura [20] to get the selfsimilar large time behavior of solutions to (2.15)–(2.16) for a particular form of F.

THEOREM 2.2. *Let* $n = 1$ *and* $F(u,u_t,\nabla u) = |u_t|^{q-1}u_t$ *in* (2.15) *for some* $q > 2$. *Suppose that* $u_0, u_1 \in C_c^\infty(\mathbb{R})$. *Then the Cauchy problem* (2.15)–(2.16) *has a unique* C^2*-solution. This solution satisfies*

$$\|u(\cdot,t) - MG_2(\cdot,\nu t)\|_p = o(t^{-(1/2)(1-1/p)}) \quad \text{as } t\to\infty$$

for every $p\in[2,\infty]$ *and*

$$M = \int_{\mathbb{R}} u_0(x)\,dx + \int_{\mathbb{R}} u_1(x)\,dx - \int_0^\infty\int_{\mathbb{R}} (|u_t|^{q-1}u_t)(x,t)\,dx\,dt.$$

The next theorem improves results by Kawashima *et al.* [18].

THEOREM 2.3. *Let* $n\le 3$ *and* $F(u,u_t,\nabla u) = g(u)$ *in* (2.15), *where* $g(u)$ *is a continuous function on* $\mathbb{R}$ *satisfying*

$$g(u)u \ge k\widetilde{g}(u) \ge 0 \quad \text{for} \quad \widetilde{g}(u) = 2\int_0^u g(z)\,dz, \quad \text{and} \quad |g(u)| \le K_0|u|^{\alpha+1}$$

for some positive constants k, k_0 *and* α. *Assume that*

$$4/n < \alpha \le 2/(n-2) \quad \text{if } n = 3, \quad \text{and} \quad 4/n < \alpha < \infty \quad \text{if } n = 1,2.$$

Suppose that $u_0 \in H^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$. *Then the Cauchy problem* (2.15)–(2.16) *has a unique solution, and* u *satisfies*

$$\|u(\cdot,t) - MG_2(\cdot,\nu t)\|_p = o(t^{-(n/2)(1-1/p)}) \quad \text{as } t\to\infty,$$

for

$$M = \int_{\mathbb{R}^n} u_0(x)\,dx + \int_{\mathbb{R}^n} u_1(x)\,dx - \int_0^\infty\int_{\mathbb{R}^n} g(u(x,t))\,dx\,dt,$$

and for every p *satisfying* $2\le p < 2n/(n-2)$ *for* $n = 3$, $2\le p<\infty$ *for* $n = 2$, *and* $2\le p\le\infty$ *for* $n = 1$.

REMARK 2.2. First of all, we would like to emphasize that the selfsimilar asymptotics of solutions from Theorems 2.2 and 2.3 is obtained for initial data which are *of arbitrary size*. Previous papers (cf. [11, 12, 21–24, 30]) always required some smallness assumptions on the initial conditions.

REMARK 2.3. Using results from [20, Theorems 3 and 4] and following the proofs of Theorems 2.2 and 2.3, we can handle arbitrary nonlinearities $F(u,u_t,\nabla u)$ in (2.15) with $F(0,0,0) = 0$ obtaining selfsimilar asymptotics of solutions, however, for small initial conditions only.

REMARK 2.4. We do not need to limit ourselves to the one-dimensional case in Theorem 2.2. In order to extend that result to higher dimensions, it suffices to apply decay estimates of solutions obtained recently by Ono [25].

3. Preliminary estimates. Recall that for $l > 0$ the operator D^l is defined by the Fourier transform as $\mathcal{F}[D^l w](\xi) = |\xi|^l\widehat{w}(\xi)$. Note that, according to this definition, $D^2 = -\Delta$, $D^a = A$, and $D^b = B$. Using again the Fourier transform we immediately deduce that for sufficiently regular v_0 each solution of the Cauchy problem $v_t + D^l v = 0$, $v(x,0) = v_0(x)$ can be written as

$$(3.1)\qquad v(x,t)=G_l(t)*v_0(x)\equiv e^{-tD^l}v_0(x),$$

with the kernel

$$(3.2)\qquad G_l(x,t)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-t|\xi|^l+ix\xi}\,d\xi.$$

Moreover, G_l has the selfsimilar form

$$(3.3)\qquad G_l(x,t)=t^{-n/l}G_l(xt^{-1/l},1).$$

REMARK 3.1. Obviously, the kernel $G_l(x,1)$ is a C^∞-function as the Fourier transform of a function exponentially decaying at infinity. Moreover, for every multiindex $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\ge 0$, there exists a constant $C=C(\alpha)$, independent of x, such that the inequality

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}G_l(x,1)\right|\le C(1+|x|)^{-n-l-|\alpha|}$$

is valid for all $x\in\mathbb{R}^n$. (For a proof of a more general result we refer the reader to Dziubański [7, Theorem 2.6].) Hence $G_l(\cdot,1)\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Note, moreover, that for $l=2$, the heat kernel $G_2(x,t)=(4\pi t)^{-n/2}\exp(-|x|^2/(4t))$ decays exponentially in x. Furthermore, by (3.3) and by a change of variables we have

$$(3.4)\qquad \|G_l(\cdot,t)\|_p=t^{-n(1-1/p)/l}\|G_l(\cdot,1)\|_p$$

for every $p\in[1,\infty]$ and all $t>0$.

REMARK 3.2. If $0<l\le 2$, G_l is the density of the so-called symmetric stable semigroup of probabilistic measures, so is a nonnegative function. We refer the reader to [3, 4] for a fuller treatment of asymptotic properties of solutions to conservation laws with anomalous diffusion modelled by D^l, $0<l\le 2$, and its generalizations.

Our first lemma determines the decay rate of the L^p-norms of (3.1).

LEMMA 3.1. *For every $q\in[1,\infty]$ and $p\in[q,\infty]$ there exists a positive constant C independent of t such that*

$$(3.5)\qquad \|e^{-tD^l}v_0\|_p\le Ct^{-n(1/q-1/p)/l}\|v_0\|_q$$

for every $t>0$ provided $v_0\in L^q(\mathbb{R}^n)$.

Proof. The proof is based on the Young inequality

$$(3.6)\qquad \|h*g\|_p\le\|h\|_q\|g\|_r,$$

valid for every $p,q,r\in[1,\infty]$ satisfying $1+1/p=1/q+1/r$, and all $h\in L^q(\mathbb{R}^n)$, $g\in L^r(\mathbb{R}^n)$. We apply (3.6) to (3.1) and use (3.4). ■

The asymptotics of $e^{-tD^l}v_0$ with $v_0\in L^1(\mathbb{R}^n)$ in the inequality (3.5) is improved in the following lemma.

LEMMA 3.2. *Assume that $v_0\in L^1(\mathbb{R}^n)$. For every $p\in[1,\infty]$,*

$$(3.7)\qquad \Big\|e^{-tD^l}v_0-\Big(\int_{\mathbb{R}^n}v_0(x)\,dx\Big)G_l(\cdot,t)\Big\|_p=o(t^{-(n/l)(1-1/p)})$$

as $t\to\infty$.

Proof. This result is obtained from the inequality

$$(3.8)\qquad \Big\|h*g(\cdot)-\Big(\int_{\mathbb{R}^n}h(x)\,dx\Big)g(\cdot)\Big\|_p\le C\|\nabla g\|_p\|h\|_{L^1(\mathbb{R}^n,|x|dx)}$$

valid for each $p\in[1,\infty]$, all $h\in L^1(\mathbb{R}^n,|x|\,dx)$, $g\in C^1(\mathbb{R}^n)\cap W^{1,1}(\mathbb{R}^n)$, and a constant $C=C_p$ independent of g and h. This inequality is a particular case of a more general result proved in [6].

For the proof of the lemma, we apply (3.8) with $h(x)=v_0(x)$ and $g(x)=G_l(x,t)$, assuming first that $v_0\in L^1(\mathbb{R}^n,|x|\,dx)$. Here, one should use the equalities $\|\nabla G_l(\cdot,t)\|_p=t^{-n(1-1/p)/l-1/l}\|\nabla G_l(\cdot,1)\|_p$ for all $t>0$, which are a direct consequence of (3.3) and Remark 3.1. The general case of $v_0\in L^1(\mathbb{R}^n)$ can be handled by an approximation argument. Details of such a reasoning can be found in [3, Cor. 2.1 and 2.2]. ■

4. Asymptotics of solutions to the linear equation. First, note that $1-4\nu|\xi|^{b-2a}=0$ for $|\xi|=(4\nu)^{-1/(b-2a)}$. For fixed real constants R_1,R_2,R_3,R_4 satisfying

$$0<R_1<R_2<(4\nu)^{-1/(b-2a)}<R_3<R_4,$$

we choose nonnegative radial functions $\chi_1,\chi_2,\chi_3\in C^\infty(\mathbb{R}^n)$ such that

$$\chi_1(\xi)=\begin{cases}1 & \text{for } |\xi|\le R_1,\\ 0 & \text{for } |\xi|>R_2,\end{cases}\qquad \chi_3(\xi)=\begin{cases}0 & \text{for } |\xi|\le R_3,\\ 1 & \text{for } |\xi|>R_4,\end{cases}$$
$$\chi_2(\xi)=1-\chi_1(\xi)-\chi_3(\xi).$$

Using the cut-off functions χ_i we decompose $S(t)u_0$ and $T(t)u_1$ into

$$(4.1)\qquad S(t)u_0=\sum_{i=1}^3 S_{\chi_i}(t)u_0\quad\text{and}\quad T(t)u_1=\sum_{i=1}^3 T_{\chi_i}(t)u_1$$

where, analogously to (2.4), we define $\mathcal{F}[S_{\chi_i}(t)u_0](\xi)=\chi_i(\xi)m_S(\xi,t)\widehat{u}_0(\xi)$ and $\mathcal{F}[T_{\chi_i}(t)u_1](\xi)=\chi_i(\xi)m_T(\xi,t)\widehat{u}_1(\xi)$.

4.1. *Asymptotics of $S(t)u_0$.* Now, we are in a position to prove our preliminary results concerning the large time behavior of the first term on the right hand side of (2.3).

PROPOSITION 4.1. *Let $p \in [2,\infty]$ and $u_0 \in L^1(\mathbb{R}^n)$.*

(i) *Assume $0 < 2a < b$. There exist positive constants C, κ such that*

$$\begin{aligned}(4.2)\quad \|S(t)u_0 - e^{-\nu t D^{b-a}}u_0\|_p &\le C\|u_0\|_1((1+t)^{-(n/(b-a))(1-1/p)-(b-2a)/(b-a)}\\ &\quad + (1+t)^{-(n/a)(1-1/p)-(b-2a)/a}\\ &\quad + (t^{-(n/(b-a))(1-1/p)} + t^{-(n/a)(1-1/p)} + 1)e^{-\kappa t})\end{aligned}$$

for all $t > 0$.

(ii) *Assume $0 = a < b$. Let, moreover, $D^{b/2}u_0 \in L^p(\mathbb{R}^n)$. There exist positive constants C, κ such that*

$$\begin{aligned}(4.3)\quad \|S(t)u_0 - e^{-\nu t D^{b}}u_0\|_p &\le C\|u_0\|_1((1+t)^{-(n/b)(1-1/p)-1} + (t^{-(n/b)(1-1/p)} + 1)e^{-\kappa t})\\ &\quad + C\|D^{b/2}u_0\|_p e^{-\kappa t}\end{aligned}$$

for all $t > 0$.

COROLLARY 4.1. *Assume $u_0 \in L^1(\mathbb{R}^n)$ and let $M_0 = \int_{\mathbb{R}^n} u_0(x)\,dx$. Under the assumptions of Proposition 4.1, for $0 \le 2a < b$ and $p \in [2,\infty]$,*

$$(4.4)\quad \|S(t)u_0(\cdot) - M_0 G_{b-a}(\cdot,\nu t)\|_p = o(t^{-(n/(b-a))(1-1/p)}) \quad \text{as } t\to\infty.$$

Proof. This corollary, asserting the selfsimilar large time behavior of $S(t)u_0$, is an immediate consequence of Proposition 4.1 combined with Lemma 3.2. ∎

Proof of Proposition 4.1(i). The proof consists in finding appropriate estimates of $S_{\chi_i}(t)u_0$ in the decomposition (4.1).

Estimates of $S_{\chi_1}(t)u_0$. First recall the elementary inequality

$$(4.5)\quad 0 \le 1 - \frac{s}{2} - \sqrt{1-s} \le \frac{1}{2}s^2 \quad \text{for all } 0 \le s \le 1.$$

Moreover, it is easy to see that

$$(4.6)\quad |e^{-k} - e^{-l}| \le |k-l|e^{-\min\{k,l\}} \quad \text{for all } k,l \ge 0.$$

To estimate $\chi_1(\xi)m_S(\xi,t)$, note that for $|\xi| \le R_2$ (i.e. on the support of χ_1) straightforward calculations show

$$(4.7)\quad \nu|\xi|^{b-a} \le \frac{|\xi|^a}{2}(1-\sqrt{1-4\nu|\xi|^{b-2a}}) \le 2\nu|\xi|^{b-a}.$$

Next, using (4.5) for $s = 4\nu|\xi|^{b-2a}$ we get

$$(4.8)\quad \left|\frac{t|\xi|^a}{2}(1-\sqrt{1-4\nu|\xi|^{b-2a}}) - \nu t|\xi|^{b-a}\right| \le Ct|\xi|^{2b-3a}.$$

Hence, combining (4.8) with (4.6) and (4.7) we obtain

$$(4.9)\quad |e^{-(t/2)|\xi|^a(1-\sqrt{1-4\nu|\xi|^{b-2a}})} - e^{-\nu t|\xi|^{b-a}}| \le C|\xi|^{2b-3a}te^{-\nu t|\xi|^{b-a}}$$

for $|\xi| \le R_2$.

Now, let us estimate the second term in the definition of $m_S(\xi,t)$ (cf. (2.5)). According to the inequalities

$$\left|\sinh\left(\frac{t}{2}\sqrt{|\xi|^{2a}-4\nu|\xi|^b}\right)\right| \le C(R_2)t|\xi|^a$$

and

$$|-1+\sqrt{1-4\nu|\xi|^{b-2a}}| \le 4\nu|\xi|^{b-2a}$$

(the latter follows from (4.7)), which are valid for $|\xi| \le R_2$, we have

$$\begin{aligned}(4.10)\quad \left|\frac{-1+\sqrt{1-4\nu|\xi|^{b-2a}}}{\sqrt{1-4\nu|\xi|^{b-2a}}}e^{-(t/2)|\xi|^a}\sinh\left(\frac{t}{2}\sqrt{|\xi|^{2a}-4\nu|\xi|^b}\right)\right|&\\ \le 4\nu Ct|\xi|^{b-a}e^{-(t/2)|\xi|^a}&.\end{aligned}$$

Next, we will use the Hausdorff–Young inequality

$$(4.11)\quad \|\widehat{v}\|_p \le C\|v\|_q,$$

valid for every $1 \le q \le 2 \le p \le \infty$ such that $1/p + 1/q = 1$. In view of the definitions of $S_{\chi_1}(t)$ and $e^{-\nu t D^{b-a}}$, for every $p \in [2,\infty]$, it follows from (4.11), (4.9) and (4.10) that

$$\begin{aligned}(4.12)\quad \|S_{\chi_1}(t)u_0 - e^{-\nu tD^{b-a}}u_0\|_p^q &\le C\int_{\mathbb{R}^n} |\chi_1(\xi)(m_S(\xi,t) - e^{-\nu t|\xi|^{b-a}})|^q|\widehat{u}_0(\xi)|^q\,d\xi\\ &\quad + C\int_{\mathbb{R}^n} |(1-\chi_1(\xi))e^{-\nu t|\xi|^{b-a}}|^q|\widehat{u}_0(\xi)|^q\,d\xi\\ &\le C\|\widehat{u}_0\|_\infty^q\Big(\int_{|\xi|\le R_2} (|\xi|^{2b-3a}te^{-\nu t|\xi|^{b-a}})^q + (t|\xi|^{b-a}e^{-(t/2)|\xi|^a})^q\,d\xi\\ &\quad + \int_{|\xi|>R_2} e^{-\nu qt|\xi|^{b-a}}\,d\xi\Big)\\ &\le C\|u_0\|_1^q((1+t)^{-n/(b-a)-(b-2a)q/(b-a)}\\ &\quad + (1+t)^{-n/a-(b-2a)q/a} + t^{-n/(b-a)}e^{-\kappa t})\end{aligned}$$

for all $t > 0$, and some positive constants C and κ independent of t. Here, to get the last term on the right hand side of (4.12), we use the inequalities

$$\int_{|\xi|>R_2} e^{-\nu qt|\xi|^{b-a}}\,d\xi \le e^{-\nu qtR_2^{b-a}/2}\int_{\mathbb{R}^n} e^{-\nu qt|\xi|^{b-a}/2}\,d\xi = Ct^{-n/(b-a)}e^{-\kappa t}$$

for $C=\int_{\mathbb{R}^n} e^{-\nu q|w|^{b-a}/2}\,dw$ and $\kappa=\nu qR_2^{b-a}/2$.

Estimates of $S_{\chi_2}(t)u_0$. Since, in this case $R_1\le|\xi|\le R_4$, there exists a positive constant κ such that

$$|e^{-(t/2)|\xi|^a(1-\sqrt{1-4\nu|\xi|^{b-2a}})}|\le e^{-\kappa t}.$$

Moreover, the function $(\sinh z)/z$ is locally bounded for complex z, hence we immediately obtain

$$\left|\frac{-1+\sqrt{1-4\nu|\xi|^{b-2a}}}{\sqrt{1-4\nu|\xi|^{b-2a}}}e^{-(t/2)|\xi|^a}\sinh\left(\frac{t}{2}\sqrt{|\xi|^{2a}-4\nu|\xi|^b}\right)\right| \le C(R_1,R_4)t|\xi|^a e^{-(t/2)|\xi|^a}\le Ce^{-\kappa t}$$

for $R_1\le|\xi|\le R_4$ and some constants $C>0$ and $\kappa\in(0,R_1^a/2)$. Hence, by (4.11), we have

$$(4.13)\qquad \|S_{\chi_2}(t)u_0\|_p^q \le C\int_{\mathbb{R}^N}|\chi_2(\xi)m_S(\xi,t)|^q|\widehat{u}_0(\xi)|^q\,d\xi \le C\|u_0\|_1^q e^{-\kappa qt}\int_{R_1\le|\xi|\le R_4} d\xi \le C\|u_0\|_1^q e^{-\kappa qt}.$$

Estimates of $S_{\chi_3}(t)u_0$. Here the assumption $|\xi|\ge R_3$ gives the inequality $|\xi|^{2a}-4\nu|\xi|^b<0$, which implies immediately that

$$\left|\sinh\left(\frac{t}{2}\sqrt{|\xi|^{2a}-4\nu|\xi|^b}\right)\right|\le 1.$$

Hence, we bound the symbol of $S_{\chi_3}(t)$ directly as follows:

$$|\chi_3(\xi)m_S(\xi,t)|\le C(R_3)\chi_3(\xi)e^{-(t/2)|\xi|^a}$$

with a positive constant $C(R_3)$ independent of ξ and t. This leads, by (4.11), to

$$(4.14)\qquad \|S_{\chi_3}(t)u_0\|_p^q\le C\|\widehat{u}_0\|_\infty^q\int_{|\xi|\ge R_1} e^{-q(t/2)|\xi|^a}\,d\xi\le C\|u_0\|_1^q t^{-n/a}e^{-\kappa t}$$

for all $t>0$, and positive constants C and κ independent of t.

Combining (4.12)–(4.14) we obtain (4.2), which completes the proof of the first part of Proposition 4.1.

Proof of Proposition 4.1(ii). Here, the reasoning is analogous to that in the proof of part (i). It follows from (4.9) and (4.10) that for $a=0$ we have

$$(4.15)\qquad |\chi_1(\xi)(m_S(\xi,t)-e^{-\nu t|\xi|^b})| \le C(R_2)\chi_1(\xi)(|\xi|^{2b}te^{-\nu t|\xi|^b}+t|\xi|^b e^{-t/2}) \le C\chi_1(\xi)(|\xi|^{2b}te^{-\nu t|\xi|^b}+e^{-\kappa t})$$

for all $|\xi|\le R_2$, $t>0$, and positive constants C and κ. In the same manner, straightforward calculations give

$$(4.16)\qquad |\chi_2(\xi)m_S(\xi,t)|\le C(R_1,R_4)e^{-\kappa t}\chi_2(\xi)$$

for a positive constant κ.

Let us study $\chi_3(\xi)m_S(\xi,t)$ more carefully. Recall that the symbol $m_S(\xi,t)$ for $a=0$ can be written as

$$m_S(\xi,t)=e^{-t/2}e^{it/2\sqrt{4\nu|\xi|^b-1}} + \left(1+\frac{i}{\sqrt{4\nu|\xi|^b-1}}\right)e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4\nu|\xi|^b-1}\right).$$

Define

$$g(\xi,t)=e^{\kappa t}|\xi|^{-b/2}m_S(\xi,t)\chi_3(\xi)$$

for some positive $\kappa<1/2$. Obviously, $g(\xi,t)$ is a C^∞-function with respect to ξ, because $\chi_3(\xi)=0$ for $|\xi|\le R_3$. An easy computation shows that for every multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\ge0$, there exists a constant $C=C(\alpha)$, independent of ξ and t, such that

$$(4.17)\qquad \left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}g(\xi,t)\right|\le C(1+|\xi|)^{-b/2-|\alpha|}$$

for all $\xi\in\mathbb{R}^n$ and $t>0$. Hence, by [29, Prop. 1, Ch. VI, Sec. 4], the Fourier transform of $g(\xi,t)$ with respect to ξ agrees with a function which is C^∞ away from the origin. Moreover,

$$(4.18)\qquad \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathcal{F}^{-1}[g(\cdot,t)](x)\right|\le C(\alpha,N)|x|^{-n+b/2-|\alpha|-N}$$

for all multiindices α and all $N\ge0$ so that $n-b/2+|\alpha|+N>0$, and constants $C(\alpha,N)$ independent of x and t.

In particular, it follows from (4.18) that $\mathcal{F}^{-1}[g(\cdot,t)]\in L^1(\mathbb{R}^n)$ with the L^1-norm uniformly bounded with respect to t. Hence, by the Young inequality (3.6), we obtain

$$(4.19)\qquad \|\mathcal{F}^{-1}[m_S(\cdot,t)\chi_3(\cdot)\widehat{u}_0]\|_p = e^{-\kappa t}\|\mathcal{F}^{-1}[g(\cdot,t)|\xi|^{b/2}\widehat{u}_0]\|_p = e^{-\kappa t}\|\mathcal{F}^{-1}[g(\cdot,t)]*D^{b/2}u_0\|_p \le Ce^{-\kappa t}\|D^{b/2}u_0\|_p$$

for all $t>0$ and positive constants C and $\kappa<1/2$, independent of t.

The proof is now completed by considerations similar to those in the proof of the first part, hence we skip the details. ■

4.2. *Asymptotics of* $T(t)u_1$. In the case of $T(t)u_1$, for $a > 0$, we need the notion of the Riesz potential I_a defined in the Fourier variables as

$$\mathcal{F}[I_a v](\xi) = |\xi|^{-a}\widehat{v}(\xi) \quad \text{for every } 0 < a < n.$$

The properties of the operator I_a can be found in Stein's monograph [28].

PROPOSITION 4.2. *Let* $p \in [2,\infty]$.

(i) *Assume that* $0 < 2a < b$, $0 < a < n$, *and* $I_a u_1 \in L^1(\mathbb{R}^n)$. *There exist positive constants* C *and* κ *such that*

$$\begin{aligned}(4.20)\quad &\|T(t)u_1 - e^{-\nu t D^{b-a}} I_a u_1\|_p \\ &\le C\|I_a u_1\|_1((1+t)^{-(n/(b-a))(1-1/p)-(b-2a)/(b-a)} \\ &\quad + (1+t)^{-(n/a)(1-1/p)} \\ &\quad + (t^{-(n/(b-a))(1-1/p)} + t^{-(n/a)(1-1/p)} + 1)e^{-\kappa t})\end{aligned}$$

for all $t > 0$.

(ii) *Assume that* $0 = a < b$ *and* $u_1 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. *There exist positive constants* C *and* κ *such that*

$$\begin{aligned}(4.21)\quad &\|T(t)u_1 - e^{-\nu t D^b} u_1\|_p \\ &\le C\|u_1\|_1((1+t)^{-(n/b)(1-1/p)-1} + (t^{-(n/b)(1-1/p)} + 1)e^{-\kappa t}) \\ &\quad + C\|u_1\|_p e^{-\kappa t}\end{aligned}$$

for all $t > 0$.

COROLLARY 4.2. *Let* $a = 0$, $u_1 \in L^1(\mathbb{R}^n)$, *and* $M_1 = \int_{\mathbb{R}^n} u_1(x)\,dx$. *Under the assumptions of Proposition* 4.2(ii), *for every* $p \in [2,\infty]$,

$$(4.22)\quad \|T(t)u_1(\cdot) - M_1 G_b(\cdot,\nu t)\|_p = o(t^{-(n/b)(1-1/p)}) \quad \text{as } t \to \infty.$$

Proof. As in the case of Corollary 4.1, we apply Lemma 3.2 to (4.21). ■

Proof of Proposition 4.2(i). First, rewrite the definition of $T(t)$ (cf. (2.4)) as

$$\mathcal{F}[T(t)u_1](\xi) = |\xi|^a m_T(\xi,t)\mathcal{F}[I_a u_1](\xi)$$

with

$$|\xi|^a m_T(\xi,t) = \frac{e^{-(t/2)|\xi|^a(1-\sqrt{1-4\nu|\xi|^{b-2a}})} - e^{-(t/2)|\xi|^a(1+\sqrt{1-4\nu|\xi|^{b-2a}})}}{\sqrt{1-4\nu|\xi|^{b-2a}}}.$$

From now on, the reasoning is similar to that in the proof of Proposition 4.1, hence we shall be brief in details.

As in (4.7) and (4.9), for $|\xi| \le R_2$ we have

$$\begin{aligned}(4.23)\quad &\left|\frac{e^{-(t/2)|\xi|^a(1-\sqrt{1-4\nu|\xi|^{b-2a}})}}{\sqrt{1-4\nu|\xi|^{b-2a}}} - e^{-\nu t|\xi|^{b-a}}\right| \\ &\qquad \le C(R_2)(|\xi|^{2b-3a}t + |\xi|^{b-2a})e^{-\nu t|\xi|^{b-a}};\end{aligned}$$

moreover,

$$(4.24)\quad \left|\frac{e^{-(t/2)|\xi|^a(1+\sqrt{1-4\nu|\xi|^{b-2a}})}}{\sqrt{1-4\nu|\xi|^{b-2a}}}\right| \le C(R_2)e^{-(t/2)|\xi|^a}.$$

Next, the argument used in the proof of Proposition 4.1 (cf. the cases $R_1 \le |\xi| \le R_4$ and $|\xi| \ge R_3$) gives again

$$|(\chi_2(\xi) + \chi_3(\xi))|\xi|^a m_T(\xi,t)| \le C(e^{-\kappa t}\chi_2(\xi) + e^{-\kappa t|\xi|^a}\chi_3(\xi))$$

for all $t > 0$, and nonnegative constants C and κ independent of ξ and t.

The remainder of the proof follows the analogous part of the proof of Proposition 4.1(i), hence we skip the details. ■

Proof of Proposition 4.2(ii). Although the reasoning here is similar to our previous considerations, let us write down the estimates which lead to (4.21), because they will be useful in the proof of Proposition 4.3 below. Recall that for $a = 0$,

$$m_T(\xi,t) = 2e^{-t/2}\frac{\sinh((t/2)\sqrt{1-4\nu|\xi|^b})}{\sqrt{1-4\nu|\xi|^b}}.$$

We use the decomposition of $T(t)u_1$ given in (4.1).

By (4.23) and (4.24) with $a = 0$, and by the Hausdorff–Young inequality (4.11), following the calculations in (4.12), we have

$$\begin{aligned}(4.25)\quad &\|T_{\chi_1}(t)u_1 - e^{-\nu t D^b}u_1\|_p \\ &\qquad \le C\|u_1\|_1((1+t)^{-(n/b)(1-1/p)-1} + (t^{-(n/b)(1-1/p)} + 1)e^{-\kappa t})\end{aligned}$$

for every $p \in [2,\infty]$, all $t > 0$, and some positive constants C and κ.

Since the function $(\sinh z)/z$ is locally bounded for complex z, we immediately obtain $|\chi_2(\xi)m_T(\xi,t)| \le Cte^{-t/2}\chi_2(\xi)$. This leads by (4.11) (cf. also (4.13)) to

$$(4.26)\quad \|T_{\chi_2}(t)u_1\|_p \le Ce^{-\kappa t}\|u_1\|_1$$

for some positive $\kappa < 1/2$.

A reasoning completely analogous to that in (4.19) with $g(\xi,t) = e^{\kappa t}\chi_3(\xi)m_T(\xi,t)$ applies here to $T_{\chi_3}(t)u_1$ and gives

$$(4.27)\quad \|T_{\chi_3}(t)u_1\|_p \le Ce^{-\kappa t}\|u_1\|_p$$

for every $p \in [1,\infty]$ and all $t > 0$, and positive constants C and κ.

By the decomposition (4.1), combining (4.25)–(4.27), we complete the proof of the inequality (4.21). ■

4.3. *Asymptotics of* $\int_0^t T(t-\tau)f(\cdot,\tau)\,d\tau$. Next, we discuss the asymptotic properties of the third term on the right hand side of (2.3). In the proposition below, we show that appropriate decay estimates on the forcing term $f(x,t)$ guarantee the selfsimilar large time behavior of $\int_0^t T(t-\tau)f(\cdot,\tau)\,d\tau$.

PROPOSITION 4.3. *Let $a=0$ and $b>n/2$. Fix $p\in[2,\infty]$. Under the assumptions of Theorem 2.1 imposed on $f(x,t)$ we have*

$$\Big\|\int_0^t T(t-\tau)f(\cdot,\tau)\,d\tau-\Big(\int_0^\infty\int_{\mathbb{R}^n} f(y,\tau)\,dy\,d\tau\Big)G_b(\cdot,\nu t)\Big\|_p = o(t^{-(n/b)(1-1/p)}) \quad \text{as } t\to\infty. \tag{4.28}$$

Proof. As in the proofs of Propositions 4.1 and 4.2, we use the decomposition $T(t-\tau)f(\tau)=\sum_{i=1}^3 T_{\chi_i}(t-\tau)f(\tau)$ where $\mathcal{F}[T_{\chi_i}(t-\tau)f(\tau)](\xi)=\chi_i(\xi)m_T(\xi,t-\tau)\widehat{f}(\xi,\tau)$.

First, we prove that

$$\Big\|\int_0^t T_{\chi_1}(t-\tau)f(\cdot,\tau)\,d\tau-\Big(\int_0^\infty\int_{\mathbb{R}^n} f(y,\tau)\,dy\,d\tau\Big)G_b(\cdot,t)\Big\|_p = o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty. \tag{4.29}$$

Note that the integral $\int_0^\infty\int_{\mathbb{R}^n} f(y,\tau)\,dy\,d\tau$ is finite, which is guaranteed by the first inequality in (2.11).

Straightforward calculations show that for every multiindex α there is a constant $C(\alpha)$ independent of t such that

$$\Big|\frac{\partial^\alpha}{\partial\xi^\alpha}(\chi_1(\xi)m_T(\xi,t))\Big|\le C(\alpha)|\xi|^{-|\alpha|}.$$

Hence, it follows from the Marcinkiewicz multiplier theorem [28] that for every $t>0$, $T_{\chi_1}(t)$ is the bounded Fourier multiplier operator on $L^p(\mathbb{R}^n)$ for every $p\in(1,\infty)$, i.e. $\|T_{\chi_1}(t)v\|_p\le C(p)\|v\|_p$. Moreover, the constant $C(p)$ can be assumed to be independent of t. Therefore, by the assumption (2.11) we obtain

$$\begin{aligned}\Big\|\int_{t/2}^t T_{\chi_1}(t-\tau)f(\cdot,\tau)\,d\tau\Big\|_p &\le C\int_{t/2}^t \|f(\cdot,\tau)\|_p\,d\tau\\ &\le C\int_{t/2}^t \tau^{-n(1-1/p)/b-1-\varepsilon}\,d\tau\\ &= o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty\end{aligned}$$

for every $p\in(1,\infty)$.

We use (2.12) to find an analogous estimate of the L^∞-norm. Indeed, since $\chi_1(\xi)m_T(\xi,t)$ is a bounded function and compactly supported with respect to ξ, we have $\|\chi_1(\cdot)m_T(\cdot,t)\|_2\le C$ with C independent of t. Hence, combining the Hölder inequality with the Plancherel equality and (2.12), we obtain

$$\begin{aligned}\Big\|\int_{t/2}^t T_{\chi_1}(t-\tau)f(\cdot,\tau)\,d\tau\Big\|_\infty &\le C\int_{t/2}^t \|\chi_1(\cdot)m_T(\cdot,t-\tau)\|_2\|f(\cdot,\tau)\|_2\,d\tau\\ &\le C\int_{t/2}^t \tau^{-n/b-1-\varepsilon}\,d\tau\\ &= o(t^{-n/b}) \quad \text{as } t\to\infty.\end{aligned}$$

A similar reasoning, based on (3.5), gives

$$\Big\|\int_{t/2}^t e^{-(t-\tau)\nu D^b}f(\cdot,\tau)\,d\tau\Big\|_p = o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty$$

for every $p\in[1,\infty]$.

Next, it follows from (4.25) that for $p\in[2,\infty]$ we have

$$\|T_{\chi_1}(t-\tau)f(\cdot,\tau)-e^{-(t-\tau)\nu D^b}f(\cdot,\tau)\|_p \le C(t-\tau)^{-n(1-1/p)/b-1}\|f(\cdot,\tau)\|_1 \tag{4.30}$$

for all $t>0$, $0<\tau<t$, and a constant C independent of t and τ. The inequality (4.30) implies that $\int_0^{t/2} T_{\chi_1}(t-\tau)f(\cdot,\tau)\,d\tau$ is well approximated by $\int_0^{t/2} e^{-(t-\tau)\nu D^b}f(\cdot,\tau)\,d\tau$. Indeed, by (4.30) and (2.11) we have

$$\begin{aligned}\Big\|\int_0^{t/2} T_{\chi_1}(t-\tau)f(\cdot,\tau)\,d\tau-\int_0^{t/2} e^{-(t-\tau)\nu D^b}f(\cdot,\tau)\,d\tau\Big\|_p &\le C\int_0^{t/2}(t-\tau)^{-n(1-1/p)/b-1}(1+\tau)^{-1-\varepsilon}\,d\tau\\ &\le C(t/2)^{-n(1-1/p)/b-1}\int_0^\infty (1+\tau)^{-1-\varepsilon}\,d\tau\\ &= o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty.\end{aligned}$$

But, by (2.11) and (3.4), we obtain

$$\Big\|\Big(\int_{t/2}^{\infty}\int_{\mathbb{R}^n} f(y,\tau)\,dy\,d\tau\Big)G_b(\cdot,\nu t)\Big\|_p \le Ct^{-n(1-1/p)/b}\int_{t/2}^{\infty}\|f(\cdot,\tau)\|_1\,d\tau$$
$$= o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty,$$

therefore the proof of (4.29) will be completed by showing that the quantity

$$(4.31)\qquad t^{n(1-1/p)/b}\Big\|\int_0^{t/2} e^{-(t-\tau)\nu D^b} f(\cdot,\tau)\,d\tau - \Big(\int_0^{t/2}\int_{\mathbb{R}^n} f(y,\tau)\,dy\,d\tau\Big)G_b(\cdot,\nu t)\Big\|_p$$
$$= t^{n(1-1/p)/b}\Big\|\int_0^{t/2}\int_{\mathbb{R}^n}(G_b(\cdot-y,\nu(t-\tau))-G_b(\cdot,\nu t))f(y,\tau)\,dy\,d\tau\Big\|_p$$

tends to 0 as $t\to\infty$. To prove this assertion, we fix $\delta\in(0,1/2)$ and decompose the integration range $[0,t/2]\times\mathbb{R}^n$ in (4.31) into two parts Ω_1 and Ω_2, where

$$\Omega_1=[0,\delta t]\times\{y\in\mathbb{R}^n : |y|\le\delta t^{1/b}\},\qquad \Omega_2=([0,t/2]\times\mathbb{R}^n)\setminus\Omega_1.$$

We estimate the L^p-norm of the integral in (4.31) over Ω_2 in a straightforward manner by the following quantity:

$$(4.32)\qquad \iint_{\Omega_2}(\|G_b(\cdot-y,\nu(t-\tau))\|_p+\|G_b(\cdot,\nu t)\|_p)|f(y,\tau)|\,dy\,d\tau$$
$$\le C\Big(\iint_{\Omega_2}(t-\tau)^{-n(1-1/p)/b}|f(y,\tau)|\,dy\,d\tau + t^{-n(1-1/p)/b}\iint_{\Omega_2}|f(y,\tau)|\,dy\,d\tau\Big)$$
$$\le Ct^{-n(1-1/p)/b}\iint_{\Omega_2}|f(y,\tau))|\,dy\,d\tau.$$

Now, it follows easily from the first assumption on f in (2.11) that

$$\iint_{\Omega_2}|f(y,\tau)|\,dy\,d\tau\to 0\quad \text{as } t\to\infty.$$

Next, we estimate the integral in (4.31) over Ω_1. The selfsimilar form of $G_b(y,\tau)$, the continuity of translations in $L^p(\mathbb{R}^n)$, and the continuity of $G_b(y,\tau)$ with respect to τ imply that for $\eta>0$ there is $\delta\in(0,1/2)$ such

that

$$t^{n(1-1/p)/b}\sup_{\substack{|y|\le\delta t^{1/b}\\ 0<\tau\le\delta t}}\|G_b(\cdot-y,\nu(t-\tau))-G_b(\cdot,\nu t)\|_p = \sup_{\substack{|z|\le\delta\\ 0<s\le\delta}}\|G_b(\cdot-z,\nu(1-s))-G_b(\cdot,\nu)\|_p\le\eta$$

for each $p\in[2,\infty]$, $\delta\in(0,1/2)$, and $t>0$. Applying this estimate to (4.31) with Ω_1 as the integration domain we see that the integral is bounded by

$$(4.33)\qquad \eta C\iint_{\Omega_1}|f(y,\tau)|\,dy\,d\tau.$$

Since η was an arbitrary number, in view of (4.33) and (4.32), we conclude that, for every $p\in[2,\infty]$, the quantity in (4.31) tends to 0 as $t\to\infty$. This completes the proof of (4.29).

To deal with $\int_0^t T_{\chi_2}(t-\tau)f(\cdot,\tau)\,d\tau$, note that the function $\varphi(r,t)=m_T(r,t)\chi_2(r)$ (here $r=|\xi|$) satisfies the inequality

$$\sup_{r>0}|\varphi(r,t)|+\Big(\sup_{R>0}\int_R^{2R}\Big|\frac{\partial}{\partial r}\varphi(r,t)\Big|^2\,dr\Big)^{1/2}\le Ce^{-\kappa t}$$

for some positive constants C and $\kappa<1/2$. By [13], this condition is sufficient for $T_{\chi_2}(t)$ to be a Fourier multiplier operator bounded on $L^p(\mathbb{R}^n)$ for $2\le p<2n/(n-1)$. Hence, in this range of p, by (2.11), we have

$$\int_0^t\|T_{\chi_2}(t-\tau)f(\cdot,\tau)\|_p\,d\tau\le C\int_0^t e^{-\kappa(t-\tau)}\|f(\cdot,\tau)\|_p\,d\tau$$
$$\le C\int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-n(1-1/p)/b-1-\varepsilon}\,d\tau.$$

If $2n/(n-1)\le p$, we define $K_2(\xi,t-\tau)=\mathcal{F}^{-1}[\chi_2(\cdot)m_T(\xi,t-\tau)](x)$. By the Plancherel formula and estimates of $m_T(\xi,t)$ on the support of χ_2 we have $\|K_2(\cdot,t-\tau)\|_2\le Ce^{-\kappa(t-\tau)}$ for all $t>0$, $\tau\in(0,t)$, and positive constants C and κ, independent of t and τ. Hence, it follows from the Young inequality (3.6) that

$$\int_0^t\|T_{\chi_2}(t-\tau)f(\cdot,\tau)\|_p\,d\tau\le C\int_0^t\|K_2(\cdot,t-\tau)\|_2\|f(\cdot,\tau)\|_{2p/(p+2)}\,d\tau$$
$$\le C\int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-n(1-1/p)/b-1-\varepsilon}\,d\tau$$

by the assumption (2.12). Finally, straightforward calculations based on the splitting $\int_0^t\ldots\,d\tau=\int_0^{t/2}\ldots\,d\tau+\int_{t/2}^t\ldots\,d\tau$ show that the last displayed integral tends to 0 like $o(t^{-n(1-1/p)/b})$ as $t\to\infty$.

The reasoning in the case of $T_{\chi_3}(t)$ is much shorter. By (4.27), $T_{\chi_3}(t)$ is a bounded operator on $L^p(\mathbb{R}^n)$ for every $p \in [1,\infty]$ with the L^p-norm decaying exponentially in t. Consequently, we have

$$\Big\| \int_0^t T_{\chi_i}(t-\tau) f(\cdot,\tau)\,d\tau \Big\|_p \le C \int_0^t e^{-\kappa(t-\tau)}(1+\tau)^{-n(1-1/p)/b-1-\varepsilon}\,d\tau = o(t^{-n(1-1/p)/b}) \quad \text{as } t\to\infty.$$

Now, the proof of Proposition 4.3 is complete. ∎

5. Nonlinear wave equation

Proof of Theorem 2.2. We use [20, Theorem 5] which, in particular, ensures the existence of the unique classical solution satisfying

$$\text{(5.1)} \qquad \begin{aligned} &\|u(\cdot,t)\|_2 \le C(1+t)^{-1/4}, \quad &&\|u(\cdot,t)\|_\infty \le C(1+t)^{-1/2},\\ &\|u_t(\cdot,t)\|_2 \le C(1+t)^{-3/4}, \quad &&\|u_t(\cdot,t)\|_\infty \le C(1+t)^{-3/2}\end{aligned}$$

for all $t \ge 0$ and constants C independent of t. Therefore, the interpolation inequality gives

$$\text{(5.2)} \qquad \|u_t(\cdot,t)\|_p \le \|u_t(\cdot,t)\|_2^{2/p}\|u_t(\cdot,t)\|_\infty^{1-2/p} \le C(1+t)^{-(1-1/p)/2-1}$$

for every $p\in[2,\infty]$.

Now, let us check the assumptions (2.11) for $p\in[2,\infty)$ and $f(x,t) = -(|u_t|^{q-1}u_t)(x,t)$. By (5.2), we have

$$\|f(\cdot,t)\|_1 = \|u_t(\cdot,t)\|_q^q \le C(1+t)^{-1-3(q-1)/2}$$

and

$$\text{(5.3)} \qquad \|f(\cdot,t)\|_p = \|u_t(\cdot,t)\|_{pq}^q \le C(1+t)^{-(1-1/p)/2-1-3(q-1)/2}.$$

The assumption on q guarantees that the inequalities (2.11) hold true for $\varepsilon = 3(q-1)/2 > 0$, and an application of Theorem 2.1 completes the proof for $p\in[2,\infty)$.

For the proof of the remaining case $p=\infty$, we should also check the assumption (2.12) which here has the form $\|f(\cdot,t)\|_2 \le C(1+t)^{-3/2-\varepsilon}$. However, this follows immediately from (5.3) with $p=2$ for $\varepsilon=(6q-7)/2$. Obviously, ε is positive because $q>2$. ∎

Proof of Theorem 2.3. Here we apply [18, Theorem 1] where, under the assumptions gathered in Theorem 2.3, it was proved that

$$\|u(\cdot,t)\|_2 \le C(1+t)^{-n/4}$$

and the energy functional E decays like

$$E(t) \equiv \|u_t(\cdot,t)\|_2^2 + \nu\|\nabla u(\cdot,t)\|_2^2 + \int_{\mathbb{R}^n} \widetilde{g}(u(x,t))\,dx \le C(1+t)^{-n/2-1}$$

for all $t\ge 0$ and C independent of t. Hence, the well known Gagliardo–Nirenberg–Sobolev inequalities lead, for $1\le\alpha\le(n+2)/(n-2)$ if $n=3$ and for $1\le\alpha<\infty$ if $n=1,2$, to

$$\|g(u(\cdot,t))\|_1 \le k_0\|u(\cdot,t)\|_{\alpha+1}^{\alpha+1} \le C\|\nabla u(\cdot,t)\|_2^{n(\alpha-1)/2}\|u(\cdot,t)\|_2^{\alpha+1-n(\alpha-1)/2} \le C(1+t)^{-n\alpha/2} = C(1+t)^{-1-\varepsilon}$$

for all $t\ge 0$, C independent of t, and positive ε provided $\alpha>2/n$.

A similar reasoning gives

$$\|g(u(\cdot,t))\|_2 \le k_0\|u(\cdot,t)\|_{2(\alpha+1)}^{\alpha+1} \le C\|\nabla u(\cdot,t)\|_2^{-n\alpha/2}\|u(\cdot,t)\|_2^{\alpha+1-n\alpha/2} \le C(1+t)^{-n/4-n\alpha/2} = C(1+t)^{-n/4-1-\varepsilon}$$

for $\varepsilon=(n\alpha-2)/2>0$ by the assumptions on α. Hence, the inequalities (2.11) with $p=2$ for $f(x,t)=-g(u(x,t))$ hold true, and Theorem 2.1 concludes the proof for $p=2$.

Let us complete the proof for other $p>2$. First, by the Gagliardo–Nirenberg–Sobolev inequalities and the decay estimates of the energy $E(t)$, we have

$$\text{(5.4)} \qquad \|u(\cdot,t)\|_q \le C\|\nabla u(\cdot,t)\|_2^{n(1-2/q)/2}\|u(\cdot,t)\|_2^{1-n(1-2/q)/2} \le C(1+t)^{-n(1-1/q)/2}$$

for a constant $C>0$, all $t\ge 0$, and every exponent q satisfying $2\le q\le 2n/(n-2)$ for $n=3$, $2\le q<\infty$ for $n=2$, and $2\le q\le\infty$ for $n=1$.

Next, by the Hölder inequality, we obtain

$$\text{(5.5)} \qquad \|u(\cdot,t)-MG_2(\cdot,t)\|_p \le C\|u(\cdot,t)-MG_2(\cdot,t)\|_2^{1-\alpha}(\|u(\cdot,t)\|_q^\alpha + \|MG_2(\cdot,t)\|_q^\alpha)$$

for $2<p<q\le 2n/(n-2)$ if $n=3$, $2<p<q<\infty$ if $n\le 2$, and $\alpha=(1/2-1/p)/(1/2-1/q)$. Using the inequality (5.4) and the relation $\|u(\cdot,t)-MG_2(\cdot,t)\|_2 = o(t^{-n/4})$ as $t\to\infty$ (examined in the first part of this proof), we deduce that the right hand side of (5.5) is bounded by

$$o(t^{-(n/4)(1-\alpha)})(1+t)^{-(n/2)(1-1/q)\alpha} = o(t^{-(n/2)(1-1/p)}) \quad \text{as } t\to\infty.$$

The case $p=\infty$ if $n=1$ is proved by an argument similar to that given at the end of the proof of Theorem 2.2, hence we skip the details. ∎

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