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Received December 20, 1999
Revised version April 6, 2000 (4453)

STUDIA MATHEMATICA 143 (2) (2000)

Functional equations in real-analytic functions

by

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Abstract. The equation $\varphi(x) = g(x, \varphi(x))$ in spaces of real-analytic functions is considered. Connections between local and global aspects of its solvability are discussed.

1. Introduction. Given a real-analytic manifold X countable at infinity, $\dim X = m$, we consider an equation

$$\varphi(x) = g(x, \varphi(Fx)) \tag{1}$$

with real-analytic mappings

$$F : X \to X, \quad g : X \times \mathbb{R}^n \to \mathbb{R}^n$$

and an unknown real-analytic vector function

$$\varphi : X \to \mathbb{R}^n.$$

Our aim is to discuss solvability conditions for (1).

The above problem has "local" and "global" aspects. The former means the solvability in a neighborhood of a given point $x_0 \in X$, while the latter deals with the question of *whether* (1) *has a global solution* $\varphi(x)$, $x \in X$, *if it is solvable in a neighborhood of every point* $x_0 \in X$.

It turns out that at least in the case of a linear equation

$$(T\varphi)(x) \equiv \varphi(x) - A(x)\varphi(Fx) = \gamma(x) \tag{2}$$

a collection of local solutions may be used to construct a cocycle "obstructing" global solvability. This situation is similar to the *Stokes phenomena* and *Ecalle–Voronin modules* arising in classification problems of dynamical systems (see [I]).

The construction of the obstructing cocycle and its applications are the main object of the present paper (see Theorem 3.1 and results of Sections 4 and 5).

2000 *Mathematics Subject Classification*: 39B22, 39B32, 58D25.
Both authors are Gvastela Fellows, partially supported by Israel Science Foundation.

This approach allows us to study equation (2) from the linear operator point of view: under which conditions it is normally solvable, has Fredholm or semi-Fredholm property, etc.

For our purpose it is necessary to introduce a topology in the space $\mathcal{E} = \mathcal{A}(X, \mathbb{R}^n)$ of real-analytic vector functions $\varphi : X \to \mathbb{R}^n$. We endow it with the local convergence topology. Namely, a sequence $\varphi_k \in \mathcal{E}$ *converges* to $\varphi \in \mathcal{E}$ if for every $x_0 \in X$ there is a neighborhood $V \ni x_0$ such that $\varphi_k(x) \to \varphi(x)$, $x \in V$. The latter, in its turn, means that there are complex-analytic continuations $h_k(z)$ and $h(z)$ of the functions $\varphi_k(\Phi^{-1}(u))$ and $\varphi(\Phi^{-1}(u))$ in a neighborhood $\widetilde{U} \subset \mathbb{C}^m$, $\widetilde{U} \ni u_0$, such that

$$h_k(z) \to h(z) \quad (z \in \widetilde{U})$$

in the usual sense of convergence of complex-analytic functions. Here $\Phi : V \to \mathbb{R}^n$ are local coordinates in V and $u_0 = \Phi(x_0)$.

The space of *analytic germs* $\varphi : X \to \mathbb{R}^n$ at a point $x_0 \in X$ is also endowed with a convergence topology. Namely, a sequence of germs φ_k *converges* to a germ φ if there exists a common neighborhood $U \ni x_0$ and corresponding representatives $\varphi_k(x)$ and $\varphi(x)$ such that

$$\varphi_k(x) \to \varphi(x) \quad (x \in U)$$

in the aforesaid sense.

Also recall that an operator $T : \mathcal{E}_1 \to \mathcal{E}_2$ is called *normally solvable* if its image $\operatorname{Im} T$ is closed. A normally solvable operator is *Fredholm* if both subspaces $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are finite-dimensional, and *semi-Fredholm* if $\min(\dim(\operatorname{Ker} T), \dim(\operatorname{Coker} T)) < \infty$.

2. Local solvability. There is an essential distinction between the non-fixed point case, i.e., $Fx_0 \neq x_0$, and the case $Fx_0 = x_0$.

2.1. *Local solvability in a neighborhood of a non-fixed point.* Let F be a real-analytic mapping of X defined in a neighborhood of a point $x_0 \in X$, $Fx_0 \neq x_0$, and let g be a real-analytic mapping of $X \times \mathbb{R}^n$ defined in a neighborhood of $(x_0, y_0) \in X \times \mathbb{R}^n$. Assume that

$$\det F'(x_0) > 0, \quad \det \frac{\partial g}{\partial y}(x_0, y_0) \neq 0.$$

Under these assumptions the following proposition was proved in [BT1].

THEOREM 2.1. *There exist a domain $U \subset X$, containing the points x_0 and Fx_0, homeomorphic to the standard ball in $\mathbb{R}^m$, $m = \dim X$, and a real-analytic vector function*

$$\varphi : U \to \mathbb{R}^n, \quad \varphi(x_0) = g(x_0, y_0),$$

such that (1) *is fulfilled for all x from some neighborhood $\widetilde{U} \ni x_0$.*

If $\dim X > 1$, then the statement of Theorem 2.1 is true under the assumption $\det F'(x_0) \neq 0$. In the one-dimensional case, preserving the orientation by F is essential as the following example shows.

EXAMPLE 2.2. Consider the equation

$$\varphi(x) + \varphi(1-x) = \gamma(x), \quad x \in \mathbb{R}^1,$$

in a neighborhood of $x_0 = 0$. Here $g(x, y) = \gamma(x) - y$, $F(x) = 1 - x$, $\det g'_y(x, y) \equiv -1$. Assume γ to be analytic in a neighborhood of origin. If φ is an analytic solution in a neighborhood of the interval $[0,1]$, then γ should be analytic on $[0,1]$ and satisfy the condition

$$\gamma(x) = \gamma(1-x).$$

For instance, if $\gamma(x) = x$ then the equation has no analytic solutions φ defined in a neighborhood of $[0,1]$.

In fact a stronger result was obtained in [BT1]. Namely, under the assumptions of Theorem 2.1 the mapping

$$G(x, y) = (Fx, g(x, y))$$

is a local analytic diffeomorphism defined in a neighborhood of (x_0, y_0).

THEOREM 2.3. *There exists an analytic diffeomorphism of the form*

$$\Phi(x, y) = (H(x), S(x, y)), \quad H(0) = x_0, \quad H(e_1) = F(x_0),$$

mapping a neighborhood V of the segment $[0,1]e_1 \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^n$ into a neighborhood $U \subset X \times \mathbb{R}^n$ containing (x_0, y_0) and $(Fx_0, g(x_0, y_0))$ such that

$$(\Phi^{-1} \circ G \circ \Phi)(u, z) = (u + e_1, Jz)$$

for all (u, z) from some neighborhood $\widetilde{V} \subset \mathbb{R}^m \times \mathbb{R}^n$ of the origin.

Here $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$, and J is the identity if $\det g'_y(x_0, y_0) > 0$ and $J = \operatorname{diag}(-1, 1, \ldots, 1)$ otherwise.

In other words, Theorem 2.3 describes a *normal form* of the diffeomorphism G in a neighborhood of a non-fixed point. In particular, the diffeomorphism F itself turns out to be local-analytically *conjugate* to the shift:

$$(H^{-1} F H)(u) = u + e_1$$

for all u from a neighborhood of the origin. It is easy to show (see [BT1]) that Theorem 2.1 follows from Theorem 2.3.

2.2. *Local solvability in a neighborhood of a fixed point.* Assume that $X = \mathbb{R}^m$ and $Fx_0 = x_0$.

Investigations of this case go back to Poincaré, Dulac and Siegel (see [B] for modern results and references). All of them deal with the behavior of the mapping F in a neighborhood of a fixed point, in particular, with the possibility of reducing F to a normal form. This leads us to some specific functional equation, called the *Schröder equation*

$$\Phi(Fx) = H(\Phi(x)), \quad \Phi(x) = x + \varphi(x).$$

Its study involves all problems arising for the general equation (1).

First of all, the *formal solvability* is necessary for (1) to be solvable in germs of analytic functions. This means the existence of a formal power series

$$\widehat{\varphi}(x) = \sum_{|J|=0,\,|I|=0}^{\infty} \varphi_I (x-x_0)^I$$

satisfying the equation

$$\widehat{\varphi}(x) = \widehat{g}(x, \widehat{\varphi}(\widehat{F}x)),$$

where $\varphi_I \in \mathbb{R}^m$, $I = (I_1, \ldots, I_m) \in \mathbb{Z}_+^m$, $|I| = I_1 + \ldots + I_m$, and

$$(x-x_0)^I = (x_1 - x_{0,1})^{I_1} \ldots (x_m - x_{0,m})^{I_m}.$$

The series

$$\widehat{F}(x) = x_0 + \sum_{|I|=1}^{\infty} F_I (x-x_0)^I$$

and

$$\widehat{g}(x,y) = \sum_{|J|=0,\,|I|=0}^{\infty} g_{I,J} (x-x_0)^I (y-y_0)^J$$

are Taylor series at x_0 and (x_0, y_0) of the mappings Fx and $g(x,y)$ respectively.

It is well known that the absence of *resonance relations*

$$\alpha_i \lambda_1^{p_1} \ldots \lambda_m^{p_m} = 1$$

is sufficient for (1) to be formally solvable. Here

$$\alpha_i \in \operatorname{spec} \frac{\partial g}{\partial y}(x_0, y_0), \quad \lambda_i \in \operatorname{spec} F'(x_0),$$

and $p_i \in \mathbb{Z}_+$.

In particular, for the Schröder equation with

$$Fx = \Lambda(x-x_0) + o(x-x_0), \quad H(x) = \Lambda(x-x_0) + o(x-x_0),$$

the corresponding resonance relations have the form

$$\lambda_j = \lambda_1^{p_1} \ldots \lambda_m^{p_m}, \quad \lambda_i \in \operatorname{spec} \Lambda, \quad \sum p_i \geq 2.$$

If there are no such relations, then F is *formally conjugate* to the linear mapping $H(x) = \Lambda x$, i.e., the corresponding Schröder equation is formally

solvable at x_0. Poincaré and Dulac proved that if F and H are formally conjugate and

(3) $$\lambda \in \operatorname{spec} F'(x_0) \Rightarrow |\lambda| < 1,$$

then F and H are local-analytically conjugate. In fact, a more general assertion is well known and may be proved by the Poincaré–Dulac method (see [S]).

THEOREM 2.4. *Let F be a local analytic diffeomorphism in a neighborhood of its fixed point $x_0 \in \mathbb{R}^m$ and let* (3) *be fulfilled. Then every formal solution $\widehat{\varphi}$ of* (1) *has a positive radius of convergence.*

If (3) does not hold, then the conclusion of Theorem 2.4 is not fulfilled as the following example shows.

EXAMPLE 2.5. A scalar equation

$$\varphi(x) - x\varphi(2x) = 1, \quad x \in \mathbb{R},$$

has no solution φ analytic in a neighborhood of $x = 0$, since its unique formal solution

$$\varphi(x) = \sum_{k=0}^{\infty} 2^{k(k-1)/2} x^k$$

diverges for $x \neq 0$.

The more general equation

(4) $$\varphi(x) - x\varphi(2x) = \gamma(x), \quad x \in \mathbb{R},$$

with an arbitrary function γ analytic at $x = 0$, is always formally solvable: its formal solution is

(5) $$\varphi(x) = \sum_{k=0}^{\infty} \varphi_m x^m$$

where

$$\varphi_m = 2^{m(m-1)/2} \sum_{k=0}^{m} 2^{-k(k-1)/2} \frac{\gamma^{(k)}(0)}{k!}.$$

Since $|\gamma^{(k)}(0)| \leq Ck!q^k$, the series

(6) $$l[\gamma] = \sum_{k=0}^{\infty} 2^{-k(k-1)/2} \frac{\gamma^{(k)}(0)}{k!}$$

converges for every analytic germ at $x = 0$. If $l[\gamma] \neq 0$ then the series (5) diverges at $x \neq 0$, and (4) has no solutions φ analytic at $x = 0$. On the other hand, if $l[\gamma] = 0$, then

$$\varphi_m = -2^{m(m-1)/2} \sum_{k=m+1}^{\infty} 2^{-k(k-1)/2} \frac{\gamma^{(k)}(0)}{k!}$$

satisfies $|\varphi_m| \le C^m$ with some C and (5) defines a solution of (4) analytic at $x=0$. Therefore, a necessary and sufficient condition for (4) to have a local analytic solution at $x=0$ is $l[\gamma]=0$ where l is a continuous linear functional on the topological space of analytic germs at $x=0$.

EXAMPLE 2.6. The equation

$$\varphi(x)-\varphi\left(\frac{x}{1+x}\right)=\gamma(x) \tag{7}$$

has a formal solution at $x=0$ if and only if $\gamma(0)=\gamma'(0)=0$. In addition, if $\gamma(0)=\gamma'(0)=0$, then the function

$$\varphi(x)=\sum_{n=0}^{\infty}\gamma\left(\frac{x}{1+nx}\right)$$

is an analytic solution of (7) in $(0,\varepsilon)$ where $\varepsilon>0$ is any number smaller than the radius of analyticity of $\gamma(x)$ at $x=0$. We will now show that this solution may not be analytic in an arbitrarily small neighborhood of $x=0$. To this end, let $\zeta=x^{-1}$ and $\gamma^*(\zeta)=\gamma(\zeta^{-1})$. Then γ^* is analytic in $\{\zeta:|\zeta|>\varepsilon^{-1}\}$, and (1) may be written in the form

$$\varphi^*(\zeta)-\varphi^*(\zeta+1)=\gamma^*(\zeta),\qquad |\zeta|>\varepsilon^{-1},$$

with $\varphi^*(\zeta)=\varphi(\zeta^{-1})$. The domain Ω^+ lying outside the circle $\{|z|\le\varepsilon^{-1}\}$ and the half-band $|\Im\zeta|\le\varepsilon^{-1},\Re\zeta\le 0$ are invariant with respect to the shift $\zeta\mapsto\zeta+1$, and since $|\gamma^*(\zeta)|\le K|\zeta|^{-2}$, the latter equation has an analytic solution

$$\varphi_+^*(\zeta)=\sum_{n=0}^{\infty}\gamma^*(\zeta+n),\qquad \zeta\in\Omega^+.$$

Similarly we find a solution

$$\varphi_-^*(\zeta)=-\sum_{k=n}^{\infty}\gamma^*(\zeta-n),\qquad \zeta\in\Omega^-,$$

where $\Omega^-=-\Omega^+$ is the mirror reflexion of Ω^+ in the imaginary axis, invariant with respect to the inverse shift $\zeta\mapsto\zeta-1$. The inverse transformation $x=\zeta^{-1}$ maps Ω^+ onto a domain D^+ bounded by three curves:

$$\{z:|z|=\varepsilon;\ \Re z\ge 0\},\qquad \{z:|z-i\varepsilon/2|=\varepsilon^2/4;\ \Re z<0\},$$
$$\{z:|z+i\varepsilon/2|=\varepsilon^2/4;\ \Re z<0\},$$

and the domain Ω^- onto $D^-=-D^+$. The domain D^+ is invariant with respect to $z\mapsto z/(1+z)$, D^- is invariant with respect to $z\mapsto z/(1-z)$, and the series

$$\varphi_+(x)=\sum_{n=0}^{\infty}\gamma\left(\frac{x}{1+nx}\right),\quad \varphi_-(x)=-\sum_{n=1}^{\infty}\gamma\left(\frac{x}{1-nx}\right)$$

corresponding to φ_+^* and φ_-^* converge to analytic solutions of (7) in D^+ and D^- respectively. These solutions are continuous in the closures of D^+ and D^- and vanish at $x=0$. Assume that φ is a solution of (7) analytic in some neighborhood of $x=0$. Then the functions $\psi_+(x)\equiv\varphi(x)-\varphi_+(x)$ and $\psi_-(x)\equiv\varphi(x)-\varphi_-(x)$ are analytic in D^+ and D^- respectively, continuous in their closures and satisfy the homogeneous equation $\psi_\pm(x)=\psi_\pm(Fx)$. Iterating these equations in the respective domains we find $\psi_+(x)\equiv\psi_+(0)$, $x\in D^+$, $\psi_-(x)\equiv\psi_-(0)$, $x\in D^-$. Hence by continuity $\psi_+(x)-\psi_-(x)\equiv 0$, $x\in D^+\cap D^-$, and we arrive at the equation

$$\sum_{n=-\infty}^{\infty}\gamma\left(\frac{x}{1+nx}\right)=0,\quad x\in D^+\cap D^-, \tag{8}$$

which is a *necessary* condition for solvability of (7). On the other hand, if (8) is fulfilled, then

$$\varphi(x)=\begin{cases}\displaystyle\sum_{n=0}^{\infty}\gamma\left(\frac{x}{1+nx}\right), & x\in D^+,\\ \displaystyle-\sum_{n=-\infty}^{-1}\gamma\left(\frac{x}{1+nx}\right), & x\in D^-,\end{cases}$$

defines an analytic function in $D^+\cup D^-$ satisfying (7), which proves that (8) is a *sufficient* condition for the local analytic solvability of (7).

To express these conditions in terms of Taylor coefficients of γ, we note that (8) means that the function

$$\sum_{n=-\infty}^{\infty}\gamma\left(\frac{1}{z+n}\right),\quad \gamma(0)=\gamma'(0)=0,$$

being analytic and 1-periodic in $\{z:|\Im z|>\varepsilon^{-1}\}$ is identically zero. Therefore all its Fourier coefficients vanish, i.e.,

$$\int_{-\infty}^{\infty}\gamma\left(\frac{1}{t+ic}\right)e^{2ik\pi t}\,dt=0,\quad c>\varepsilon^{-1},\ k\in\mathbb{Z}.$$

Substituting here the Taylor expansion of $\gamma(z)$ we find that (7) has a solution if and only if $\gamma(z)$ is annihilated by the system of functionals

$$\left\{\gamma(0),\gamma'(0),\sum_{n=2}^{\infty}\frac{\gamma^{(n)}(0)}{n!}\frac{(2\pi ik)^{n-1}}{(n-1)!}:k\in\mathbb{Z}\right\}.$$

Conditions of *formal* solvability of general functional equations always have the form of some algebraic relations between derivatives of $\gamma(x)$ at x_0, finitely many for every such relation. In the case of a linear equation these relations are defined by linear functionals. As Examples 2.5 and 2.6 show, vanishing of these functionals (i.e., formal solvability) is not sufficient for solvability in *analytic germs*: there may exist linear functionals depending on all derivatives of $\gamma(x)$ obstructing solvability. Both types of functionals in Examples 2.5 and 2.6 are continuous in the space of germs, and since their vanishing is sufficient for local solvability, the solvability is normal.

Another type of obstacle to solvability is related to the so-called "small denominators".

EXAMPLE 2.7. The equation

$$\varphi(\xi,\eta)-b\varphi(\lambda\xi,\mu\eta)=\gamma(\xi,\eta),\quad (\xi,\eta)\in\mathbb{R}^2,$$

is formally solvable for an arbitrary analytic germ $\gamma(\xi,\eta)$ at $(0,0)$ if the nonresonance conditions

$$1-b\lambda^p\mu^q\neq 0,\quad p,q\in\mathbb{N}_+\cup\{0\},$$

are fulfilled. Since under these conditions every monomial $\xi^p\eta^q$ belongs to the image of T, the latter is dense. Moreover, if

$$|1-b\lambda^p\mu^q|\geq\varepsilon^{p+q},\quad p,q\in\mathbb{N}_+\cup\{0\},$$

with some $\varepsilon>0$, then the formal solution converges in some neighborhood of the origin and defines a local analytic solution. On the other hand, such a solution may not exist if pathologically small denominators are present. For example, if $\mu=b=e$, $\lambda=e^{-\theta}$, where θ satisfies $|\theta-p/q|\leq 1/q!$ for infinitely many integers $p,q>0$, then the function $\gamma(\xi,\eta)=(1-\xi)^{-1}(1-\eta)^{-1}$ does not belong to the image of T, since the formal solution does not converge at all. In the present situation of small denominators *the image of T is dense but not closed.*

Examples of the above type were known as far back as Euler. But solvability conditions and description of corresponding functionals for general equations of the form (2) in a neighborhood of a fixed point are not known yet.

3. Global obstacles. Suppose that the manifold X is covered by open sets U_α invariant with respect to F:

$$X=\bigcup_\alpha U_\alpha,\quad FU_\alpha\subset U_\alpha.$$

Then we can restrict the linear equation (2) to every U_α and consider the equations

$$T\varphi_\alpha=\gamma_\alpha,\quad \gamma_\alpha=\gamma|_{U_\alpha}\in\mathcal{A}(U_\alpha),$$

with respect to a function φ_α analytic on U_α. In what follows *we consider only those analytic functions γ on X for which every such restricted equation has a solution $\varphi_\alpha\in\mathcal{A}(U_\alpha)$*. We denote by $\mathcal{A}(X,\{U_\alpha\},T)$ the subset of all functions $\gamma\in\mathcal{A}(X)$ for which $\gamma\in\operatorname{Im}T|_{U_\alpha}$ for every α.

To every set $\{\varphi_\alpha\}$ of "local" solutions there corresponds the 1-*cocycle* $c_{\alpha\beta}(\gamma)=(\varphi_\beta-\varphi_\alpha)|_{U_\alpha\cap U_\beta}$. If $\{\widetilde{\varphi}_\alpha\}$ is another set of solutions corresponding to the same function γ, then $\{\widetilde{\varphi}_\alpha\}$ generates another cocycle $\{\widetilde{c}_{\alpha\beta}(\gamma)=(\widetilde{\varphi}_\beta-\widetilde{\varphi}_\alpha)|_{U_\alpha\cap U_\beta}\}$. It is evident that

$$Tc_{\alpha\beta}(\gamma)|_{U_\alpha\cap U_\beta}=T\widetilde{c}_{\alpha\beta}(\gamma)|_{U_\alpha\cap U_\beta}=0,\tag{9}$$

and $\varphi_\alpha-\widetilde{\varphi}_\alpha\in\operatorname{Ker}T|_{U_\alpha}$ for all α's. Denote by E the space of all cocycles $\{c_{\alpha\beta}\}$ satisfying (9), endowed with topology of the direct product

$$E=\prod_{\alpha,\beta}\mathcal{A}(U_\alpha\cap U_\beta)$$

where, as before, $\mathcal{A}(V)$ is the space of analytic vector functions on V. Let E_0 be the linear subset of E formed by the cocycles $c_{\alpha\beta}=\psi_\beta-\psi_\alpha$ where $\psi_\alpha\in\mathcal{A}(U_\alpha)$ are solutions of local homogeneous equations

$$T\psi_\alpha=0.$$

To each function $\gamma\in\mathcal{A}(X,\{U_\alpha\},T)$ there corresponds the equivalence class $[\gamma]$ from the quotient space E/E_0. We endow E_0 with the induced topology of E and note that, generally speaking, E_0 is not closed in E. As a result, E/E_0 may not be Hausdorff.

THEOREM 3.1. *Let F be an analytic mapping, $\{U_\alpha\}$ be an open covering of X by F-invariant subsets, and $A(x)$ and $\gamma(x)$ be analytic matrix and vector functions on X.*

(i) *Equation* (2) *has a solution $\varphi(x)$ analytic on X if and only if $[\gamma]=0$.*

(ii) *For every class $c\in E/E_0$ there exists a function γ analytic on X such that $[\gamma]=c$.*

Proof. (i) Let γ be given and let $T\varphi(x)=\gamma(x)$ for some $\varphi\in\mathcal{A}(X)$. Then the family $\{\varphi_\alpha(x)\}_{x\in U_\alpha}$ with $\varphi_\alpha(x)=\varphi(x)|_{x\in U_\alpha}$ generates the zero cocycle from $[\gamma]$, and hence $[\gamma]=0$. On other hand, if $\gamma\in\mathcal{A}(X,\{U_\alpha\},T)$ is such that $T\varphi_\alpha(x)=\gamma(x)$, $x\in U_\alpha$, for every α, and $[\gamma]=0$, then there exists a family $\{\psi_\alpha\}$ of analytic solutions of $T\psi_\alpha=0$ such that

$$\varphi_\beta(x)-\varphi_\alpha(x)=\psi_\beta(x)-\psi_\alpha(x),\quad x\in U_\alpha\cap U_\beta.$$

Now

$$\varphi(x)\equiv\varphi_\alpha(x)-\psi_\alpha(x),\quad x\in U_\alpha,$$

is a well defined function from $\mathcal{A}(X)$ satisfying (2).

(ii) Let $c \in E/E_0$, and let $\varphi_{\alpha\beta} \in \mathcal{A}(U_{\alpha\beta})$ be a representative of c. Consider a "real" Cousin problem

$$\varphi_\beta(x) - \varphi_\alpha(x) = \varphi_{\alpha\beta}(x), \quad \varphi_\alpha \in \mathcal{A}(U_\alpha). \tag{10}$$

It is well known (see [G], [W]) that there exists a real-analytic imbedding $\Phi : X \to \mathbb{R}^N$. The submanifold $\Phi X \subset \mathbb{R}^N$ is covered by the family $\{\Phi U_\alpha\} \subset \Phi X$ and every function $\varphi_{\alpha\beta}(x)$, $x \in U_\alpha$, may be transferred to a real-analytic function $\widetilde{\varphi}_{\alpha\beta}(y)$ on $\{\Phi U_\alpha \cap \Phi U_\beta\} \subset \mathbb{R}^N$. Now it is evident that there exists a family $\{\widetilde{U}_\alpha\}$ of complex neighborhoods of $\{\Phi U_\alpha\}$ such that $\bigcup_\alpha \widetilde{U}_\alpha$ is a Stein manifold (cf. [H]), and for every pair (α, β) the function $\varphi_{\alpha\beta}$ has an analytic extension to $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$. According to the well known theorem (cf. [H]), there exists a solution $\{\widetilde{\varphi}_\alpha(x)\}$ of the Cousin problem

$$\widetilde{\varphi}_\beta(z) - \widetilde{\varphi}_\alpha(z) = \widetilde{\varphi}_{\alpha\beta}(z), \quad z \in \widetilde{U}_\alpha \cap \widetilde{U}_\beta.$$

The set of functions

$$\varphi_\alpha(x) = \widetilde{\varphi}_\alpha(\Phi x), \quad x \in U_\alpha,$$

forms a solution of problem (10).

Starting with $\{\varphi_\alpha(x)\}$ we set

$$\gamma_\alpha(x) = T\varphi_\alpha(x), \quad x \in U_\alpha. \tag{11}$$

If $x \in U_\alpha \cap U_\beta$, then, according to the definition of E,

$$\gamma_\beta(x) - \gamma_\alpha(x) = T(\varphi_\beta(x) - \varphi_\alpha(x)) = T\varphi_{\alpha\beta}(x) = 0$$

and (11) defines a function $\gamma \in \mathcal{A}(X, \{U_\alpha\}, T)$. According to the above construction, $[\gamma] = c$, which completes the proof.

For $\gamma \in \mathcal{A}(X, \{U_\alpha\}, T)$ set $\Theta(\gamma) = [\gamma] \in E/E_0$. The operator $\Theta : \mathcal{A}(X, \{U_\alpha\}, T) \to E/E_0$ is linear and Theorem 3.1(i) states that $\operatorname{Ker}\Theta = \operatorname{Im} T$. Generally speaking, Θ is not continuous, and, as we will show later (see Section 5.2), $\operatorname{Ker}\Theta$ may not be a closed subspace. Nevertheless, if Θ is continuous, then statement (i) implies that $\operatorname{Im} T$ is closed, or in other words, that (2) is normally solvable.

4. Equations in a single variable.

In this section we consider equation (1) on the real line $\mathbb{R}^1$ and on the circle $\mathbb{T}^1$. We assume that F is an analytic diffeomorphism and denote by $\operatorname{Fix}(F)$ the set of all its fixed points. If $F : \mathbb{R}^1 \to \mathbb{R}^1$ has no fixed points at all, it preserves orientation and, as we will see later, F is analytically conjugate to the shift $x \mapsto x + 1$. If there exists only one fixed point, then F may either preserve or reverse orientation on $\mathbb{R}^1$. It easily follows from the Poincaré–Dulac Theorem that if the fixed point is of hyperbolic type, then $F(x)$ is conjugate to a linear transformation $x \mapsto ax + b$, $|a| \neq 1$. In the case of more than one fixed point F preserves orientation, i.e., $F'(x) > 0$, $x \in \mathbb{R}^1$.

4.1. *Equations with a fixed-point free diffeomorphism*

THEOREM 4.1. *Let* $\operatorname{Fix}(F) = \emptyset$ *and let*

$$G(x, y) = (Fx, g(x, y))$$

be an analytic diffeomorphism of $\mathbb{R}^1 \times \mathbb{R}^n$. *Then equation* (1) *has an analytic solution* $\varphi : \mathbb{R}^1 \to \mathbb{R}^n$.

In particular, for the linear equation (2) we have $g(x, y) = A(x)y + \gamma(x)$, and it generates an analytic diffeomorphism $G(x, y)$ if $\det A(x) \neq 0$. For such $A(x)$ the operator T is surjective in the space $\mathcal{A}(\mathbb{R}^1)$ and so $\operatorname{Coker} T = \{0\}$.

Proof of Theorem 4.1. Without loss of generality we can assume $Fx > x$. Let $F(0) = a$ and let $\varphi(x)$ be a solution of (1) in a neighborhood of $[0, a]$ guaranteed by Theorem 2.1. Then the relation

$$\varphi(x) = g(x, \varphi(Fx)), \quad x \in U,$$

gives a step-by-step continuation of φ from $[0, a]$ to $[0, \infty)$. Further, we have

$$G^{-1}(x, y) = (F^{-1}x, h(x, y)), \quad g(x, h(Fx, y)) = y,$$

and the relation

$$\varphi(x) = h(x, \varphi(F^{-1}x))$$

gives a step-by-step continuation of φ from $[F^{-1}(0), 0]$ to $(-\infty, 0]$. Both these continuations define a global solution of (1).

THEOREM 4.2. *Let* $\operatorname{Fix}(F) = \emptyset$ *and let*

$$\det A(x) \neq 0 \quad (x \in \mathbb{R}^1).$$

Then the space $\operatorname{Ker} T$ *is infinite-dimensional.*

Proof. For definiteness, assume that $\det A(x) > 0$. It follows from Theorem 2.3 that there exists a global diffeomorphism $\Phi(x, y) = (H(x), S(x)y)$ which conjugates the mapping

$$G(x, y) = (Fx, A^{-1}(x)y)$$

of $\mathbb{R}^1 \times \mathbb{R}^n$ onto itself to the simplest mapping

$$G_0(x, y) = (x + 1, y).$$

The homogeneous equation corresponding to G_0 has the form

$$\psi(x) - \psi(x + 1) = 0.$$

It is satisfied by every 1-periodic analytic vector function ψ. As a result the function $\varphi(x) = S(H^{-1}x)\psi(H^{-1}x)$ is a solution of the equation $T\varphi = 0$.

Using Theorem 4.2 it is easy to show that the conclusions of Theorems 4.1 and 4.2 remain true if $\det A(x)$ has a finite number of roots. The following example shows that neither of these assertions holds for $A(x)$ arbitrary.

EXAMPLE 4.3. The scalar equation

$$(T\varphi)(x) \equiv \varphi(x) - (\sin 2\pi x)\varphi(x+1) = 1$$

has no analytic solution on the whole line and the corresponding homogeneous equation has no analytic solutions except the trivial one. Indeed, if ψ is such a solution then

$$\psi(x) = (\sin^k 2\pi x)\psi(x+k).$$

Hence, ψ has a root of infinite order at $x = 0$. Being analytic, it is identically zero. It is easy to see that the function

$$\varphi(x) = \frac{1}{1-\sin 2\pi x}$$

is a unique meromorphic solution of the nonhomogeneous equation. Therefore the latter has no analytic solution.

If $\gamma \in \mathcal{A}(\mathbb{R}^1)$ is a function such that $\gamma(x) = O(x^{-2})$ as $\Re x \to \infty$ with $|\Im x| \le 1$, then the function

$$\varphi(x) = \sum_{k=0}^{\infty} (\sin^k 2\pi x)\gamma(x+k)$$

is a real-analytic solution of the nonhomogeneous equation $(T\varphi)(x) = \gamma(x)$. Since every $\gamma \in \mathcal{A}(\mathbb{R}^1)$ may be approximated in $\mathcal{A}(\mathbb{R}^1)$ by a sequence $\{\gamma_n\}$ satisfying the above decay condition, the image of T is dense in $\mathcal{A}(\mathbb{R}^1)$. Thus we have an operator (2) for which $\dim(\operatorname{Ker} T) = \dim(\operatorname{Coker} T) = 0$ and $\operatorname{Im} T$ is not closed in $\mathcal{A}(\mathbb{R}^1)$.

4.2. *The case of a single hyperbolic fixed point.* We assume here that F has only one fixed point x_0 and that $|F'(x_0)| < 1$.

THEOREM 4.4. *Every formal solution of* (1) *at x_0 has a positive convergence radius and extends to an analytic solution on $\mathbb{R}^1$. The operator T from* (2) *is a Fredholm operator.*

Proof. The first assertion immediately follows from Theorem 2.4 and the possibility of extending a local solution onto the entire real line using (1) and the relation $F^n x \to x_0$ $(n \to \infty)$. The linear equation is normally solvable since formal solvability implies the existence of a global solution.

To prove that $\dim(\operatorname{Ker} T) < \infty$ choose $m > 0$ such that $\|A(x_0)\| \cdot |F'(x_0)|^m < 1$. If φ_1 and φ_2 are two solutions of the homogeneous equation such that

$$\varphi_1(x) - \varphi_2(x) = O(|x - x_0|^m),$$

then $\varphi_1 = \varphi_2$. Hence, $\dim(\operatorname{Ker} T)$ does not exceed the dimension of the space of all vector polynomials of degree $\le m$, and $\dim(\operatorname{Ker} T) \le n(m+1)$.

It remains to show that $\dim(\operatorname{Coker} T) < \infty$. To this end we note that the formal solvability condition may be stated as a set of linear restrictions imposed on derivatives of γ at the fixed point. There are only a finite number of such conditions and they are determined by the resonance relations

$$\alpha_i \lambda^k = 1, \quad \alpha_i \in \operatorname{spec} A(x_0), \quad k \ge 0. \tag{13}$$

Since $|\lambda| < 1$, there are no more than pm independent relations of this type. So $\dim(\operatorname{Coker} T) < \infty$ and T is Fredholm.

Note that there exists an explicit formula for the solution of the linear equation in this case. Namely, let m be as above. Since (2) is formally solvable, there exists a vector polynomial φ_0 such that

$$\gamma_0(x) \equiv \gamma(x) - (T\varphi_0)(x) = O(|x - x_0|^m) \quad (x \to x_0).$$

The series

$$\varphi(x) = \varphi_0(x) + \sum_{k \ge 0} A(x) \dots A(F^k x)\gamma_0(F^{k+1}x)$$

defines an analytic solution φ on the real line. In addition, φ_0 depends on $\gamma(x_0), \dots, \gamma^{(m-1)}(x_0)$ only and may be chosen continuously depending on γ.

We will see that if F has more than one fixed point, then $\dim(\operatorname{Coker} T) = \infty$.

4.3. *Equations with* $\#\operatorname{Fix}(F) \ge 2$. Let now a real-analytic diffeomorphism $F : \mathbb{R}^1 \to \mathbb{R}^1$ have several fixed points $x_1 < \dots < x_n$.

THEOREM 4.5. *If all fixed points of F are of hyperbolic type and $A(x)$ is non-singular for all $x \in \mathbb{R}^1$, then* (2) *is normally solvable in $\mathcal{A}(\mathbb{R}^1)$. In addition, the subspace* $\operatorname{Ker} T$ *is finite-dimensional and* $\dim(\operatorname{Coker} T) = \infty$.

Proof. The system of open intervals

$$U_k = (x_{k-1}, x_{k+1}), \quad k = 1, \dots, n; \ x_0 = -\infty, \ x_{n+1} = \infty,$$

forms an open covering of $\mathbb{R}^1$ by sets which are invariant with respect to F. Given $\gamma \in \mathcal{A}(\mathbb{R}^1)$, assume that (2) is formally solvable at every fixed point x_k. Then for every $m > 0$ there exist polynomials φ_k whose degrees depend on m but not on k and γ and such that

$$\gamma_k(x) \equiv \gamma(x) - T\varphi_k(x) = O(|x - x_k|^m), \quad x \in U_k, \ k = 1, \dots, n.$$

Following the proof of Theorem 4.4 we choose m sufficiently large and obtain a solution of (2) in $\mathcal{A}(U_k)$ in the form

$$\Phi_k(x) = \varphi_k(x) + \gamma_k(x) + \sum_{s \ge 0} A(x) \dots A(F^s x)\gamma_k(F^{s+1}x), \ F'(x_k) < 1, \tag{14}$$

and

$$\Phi_k(x) = \varphi_k(x) - \sum_{s \ge 1} A^{-1}(F^{-1}x) \dots A^{-1}(F^{-s}x)\gamma_k(F^{-s}x), \quad F'(x_k) > 1. \tag{15}$$

The differences $c_{k,\gamma}(x) = \Phi_{k+1}(x) - \Phi_k(x)$, $k = 1, \ldots, n-1$, are solutions of the homogeneous equation (2) on the sets $U_{k+1} \cap U_k = [x_k, x_{k+1}]$, and form a 1-cocycle corresponding to the covering $\{U_k\}$. According to Theorem 3.1, equation (2) is solvable if and only if

$$c_{k,\gamma}(x) = \psi_{k+1}(x) - \psi_k(x)$$

where $\{\psi_k\}_{k=1}^n$ is a set of the solutions of the homogeneous equations $T\psi_k(x) = 0$, $x \in U_k$, $k = 1, \ldots, n$. Since the functions φ_k and γ_k in (14) and (15) depend continuously on γ, the linear operator transforming every function $\gamma \in \mathcal{A}(\mathbb{R}^1, \{U_k\}, T)$ into $\{c_{k,\gamma}\}$ is continuous. For every k the space of solutions of $T\psi_k = 0$ is finite-dimensional, implying that E_0 is finite-dimensional and hence a closed subspace in E. We conclude that the operator $\Theta : \mathcal{A}(\mathbb{R}^1, \{U_k\}, T) \to E/E_0$ defined in Section 3 is continuous and the space $\operatorname{Im} T = \operatorname{Ker} \Theta$ is closed in $\mathcal{A}(\mathbb{R}^1)$.

It is evident that $\dim(\operatorname{Ker} T) \le \min \dim(\operatorname{Ker} T)|_{U_k} < \infty$. To prove the last statement of Theorem 4.5, we note that in the present situation the space E of cocycles is formed by all collections $C = \{c_k\}_{k=1}^n$ of functions analytic in the intervals (x_k, x_{k+1}) and satisfying $Tc_k(x) = 0$. Since F has no fixed point in $U_k \cap U_{k+1}$ for every k, according to Section 4.1, the space of solutions to $T\psi(x)|_{U_k} = 0$ is infinite-dimensional. On the other hand, the space E_0 is finite-dimensional and therefore the quotient space E/E_0 is infinite-dimensional. Since the correspondence $\gamma \mapsto [\gamma]$ generates a linear one-to-one mapping of $\mathcal{A}(\mathbb{R}^1, \{U_k\}, T)$ onto E/E_0, we conclude that $\dim(\operatorname{Coker} T) = \infty$, which completes the proof.

Example 4.6. Consider a *cohomological equation*

$$\varphi(x) - \varphi(Fx) = \gamma(x), \quad x \in \mathbb{R}^1, \tag{16}$$

where F is as in Theorem 4.5. It has a local solution $\varphi_k \in \mathcal{A}(U_k)$ if and only if $\gamma(x_k) = 0$, $k = 1, \ldots, n$, and the solution is given by the series

$$\varphi_k(x) = \begin{cases} \sum_{m=0}^{\infty} \gamma(F^m x), & x \in U_k,\ F'(x_k) < 1, \\ -\sum_{m=1}^{\infty} \gamma(F^{-m} x), & x \in U_k,\ F'(x_k) > 1. \end{cases}$$

Therefore

$$c_{k,\gamma}(x) = \sum_{m=-\infty}^{\infty} \gamma(F^m x), \quad x \in U_k \cap U_{k+1},$$

and according to Theorem 3.1, equation (16) has a global solution $\varphi \in \mathcal{A}(\mathbb{R}^1)$

if and only if

$$\sum_{m=-\infty}^{\infty} \gamma(F^m x) \equiv \text{const}, \quad x \in U_k \cap U_{k+1}.$$

4.4. *Equations on the circle.* Let now $X = \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$. It is known (see [N]) that if the rotation number of F is rational, then its set of periodic points is non-empty and all these points are of the same period p.

A periodic point x_0 is called *hyperbolic* if $|(\underbrace{F \ldots F}_{p})'(x_0)| \ne 1$.

Theorem 4.7. *Let $F : \mathbb{T}^1 \to \mathbb{T}^1$ be an analytic diffeomorphism with rational rotation number and with hyperbolic periodic points, and suppose an analytic matrix function $A(x)$ is non-singular on $\mathbb{T}^1$. Then equation* (2) *is normally solvable in $\mathcal{A}(\mathbb{T}^1)$, the subspace* $\operatorname{Ker} T$ *is finite-dimensional and* $\dim(\operatorname{Coker} T) = \infty$.

Proof. Since all periodic points $\{z_1, \ldots, z_q\}$ are hyperbolic, their number is even. Let $p \ge 1$ be the least period for all points $z_1, \ldots, z_q$. Then $z_1, \ldots, z_q$ are hyperbolic fixed points for F^p. Denote by U_j, $z_j \in U_j$, the set of corresponding F^p-invariant arcs. The case $p = 1$ may be treated exactly as in Section 4.3, and without loss of generality we assume that $p > 1$.

If $q = 2$, then $p = 2$, $U_1 \cap U_2 = \mathbb{T}^1 \setminus \{z_1, z_2\}$ is a union of two non-intersecting arcs V_1 and V_2, and F interchanges V_1 and V_2. If $q > 2$, then $F(U_k \cap U_j) \cap (U_k \cap U_j) = \emptyset$.

Let $L\varphi(x) = A(x)\varphi(Fx)$. The equation $(I - L^p)\varphi = \gamma$ is of the type considered in Section 4.3. Repeating the arguments from the proof of Theorem 4.5 for the operator $T_p = I - L^p$ we find that $\dim(\operatorname{Ker} T_p) < \infty$ and hence $\dim(\operatorname{Ker} T) < \infty$, and $\operatorname{Im} T_p$ is a closed subspace. Using Lemma 2.1 from [BB] we conclude that $\operatorname{Im} T$ is also closed.

To prove that $\dim(\operatorname{Coker} T) = \infty$, we notice that if $\gamma \in \operatorname{Im} T$, then $R\gamma = \gamma + \ldots + L^{p-1}\gamma \in \operatorname{Im} T_p$. Let us find an infinite system of linearly independent functions $\gamma \in \mathcal{A}(\mathbb{T}^1)$ such that $R\gamma \notin \operatorname{Im} T_p$.

Since F is hyperbolic at $z_1, \ldots, z_q$, every subspace $\operatorname{Ker} T_p|_{U_j}$, $j = 1, \ldots, q$, is finite-dimensional, and hence the space E_0 generated by T_p according to Section 4.3 is also finite-dimensional.

Let γ vanish with all its derivatives up to sufficiently high order $m > 0$ at $z_1, \ldots, z_q$. Then the function

$$H_\gamma(x) = \sum_{k=-\infty}^{\infty} L^{pk} R\gamma(x)$$

is well defined and analytic on $\mathbb{T}^1 \setminus \{z_1, \ldots, z_q\}$. According to Theorem 3.1(i), $R\gamma \in \operatorname{Im} T_p$ if and only if $H_\gamma \in E_0$. On the other hand, if $h(x)$, $x \in U_1 \cap U_2$, is a solution of the homogeneous equation $T_p h(x) = 0$, then according to

Theorem 3.1(ii), there exists $\Gamma \in \mathcal{A}(\mathbb{T}^1)$ such that the class $[\Gamma] \in E/E_0$ coincides with the class generated by the cocycle

$$c_{\alpha,\beta}(x) = \begin{cases} h(x), & \alpha = 1,\ \beta = 2;\ x \in U_1 \cap U_2, \\ 0, & \alpha \neq 1 \text{ or } \beta \neq 2;\ x \notin U_1 \cap U_2, \end{cases}$$

if $q > 2$, and is equal to

$$c_{1,2}(x) = \begin{cases} h(x), & x \in V_1, \\ 0, & x \notin V_2, \end{cases}$$

if $q = 2$. Since $\Gamma \in \mathcal{A}(\mathbb{R}^1, \{U_k\}, T_p)$, the equation $T_p\psi(x) = \Gamma(x)|_{U_k}$ has a solution $\psi \in \mathcal{A}(U_k)$ for every k. Hence the equation $T_p\psi(x) = \Gamma(x)$ is formally solvable at every point $z_1, \ldots, z_q$. Therefore for each m there exists a function $\psi_0 \in \mathcal{A}(\mathbb{T}^1)$ such that all derivatives of $\gamma(x) \equiv -T_p\psi_0(x) + \Gamma(x)$ up to order m vanish. It is evident that $\gamma \in \mathcal{A}(\mathbb{R}^1, \{U_k\}, T_p)$ and $[\Gamma] = [\gamma]$. If m is sufficiently large, then the series

$$\sum_{k=-\infty}^{\infty} L^{pk}\gamma(x)$$

converges and its sum is $\{c_{\alpha,\beta}(x)\} + \widetilde{c}(x)$ where $\widetilde{c} \in E_0$. Now we have

$$\begin{aligned} H_\gamma(x) &= \sum_{k=-\infty}^{\infty} L^{pk} R\gamma(x) \\ &= R \sum_{k=-\infty}^{\infty} L^{pk}\gamma(x) = h(x) + R\widetilde{c}_{12}(x), \qquad x \in U_1 \cap U_2. \end{aligned}$$

The set $\{R\widetilde{c}_{12}(x)\}$, $x \in U_1 \cap U_2$ (or $x \in V_1$ if $q = 2$) is a finite-dimensional subspace in $\mathcal{A}(U_1 \cap U_2)$ (or $\mathcal{A}(V_1)$, respectively), while h is an arbitrary function from $\operatorname{Ker} T_p|_{U_1 \cap U_2}$ (or from $\operatorname{Ker} T_p|_{V_1}$) which is infinite-dimensional according to Theorem 4.2. Therefore indeed there exists an infinite system of linearly independent functions $\gamma \in \mathcal{A}(\mathbb{T}^1)$ for which $R\gamma \notin \operatorname{Im} T_p$. Hence dimension of the quotient space $R\mathcal{A}(\mathbb{T}^1)/\operatorname{Im} T_p$ is infinite and $\dim(\operatorname{Coker} T) = \dim(\mathcal{A}(\mathbb{T}^1)/\operatorname{Im} T) = \infty$.

5. Some multi-dimensional examples. Here we illustrate relations between global solvability of (2) and its local solvability on covering sets in the case of several variables. For simplicity we consider the cohomological equation

$$(T\varphi)(x) \equiv \varphi(x) - \varphi(Fx) = \gamma(x) \quad (x \in X). \tag{17}$$

Properties of the cohomological equation essentially depend on "dynamical singularities" of F. For instance, let φ be a solution of the *Abel equation*

$$\varphi(x) - \varphi(Fx) = 1.$$

Then, iterating, we get

$$\varphi(x) - \varphi(F^n x) = n \quad (n \in \mathbb{Z}).$$

Hence, if there is a periodic point, $F^n x_0 = x_0$, then the Abel equation has no solutions. Moreover, it is easy to see that if F has a non-wandering point then there are no continuous solutions. Recall that $x_0 \in X$ is called *non-wandering* (see [N]) if for any neighborhood $U \ni x_0$ there is $n \neq 0$ such that $F^n(U) \cap U \neq \emptyset$. As shown below in Section 5.2, the absence of non-wandering points is not sufficient for the Abel equation to be solvable. A dynamical criterion of solvability of the Abel equation was found in [BL]. To formulate it let us call a compact subset $K \subset X$ *non-wandering* if for any neighborhood $U \supset K$ there exists $n \neq 0$ such that $F^n(U) \cap U \neq \emptyset$. On the other hand, a compact set K is called *wandering* if there exists a neighborhood $U \supset K$ such that $F^n(U) \cap U = \emptyset$ for all $n \neq 0$.

THEOREM 5.1 (see [BL]). *The Abel equation has a solution $\varphi \in \mathcal{A}(X)$ if and only if all compact subsets are wandering.*

5.1. *Multidimensional shift.* Let us start with the shift $Fx = x + e$, where $e \in \mathbb{R}^n$, $e \neq 0$, and $\gamma \in \mathcal{A}(\mathbb{R}^n)$. If $n = 1$, then according to a classical result of Picard, equation (17) has a solution φ analytic on the closed interval $[0, e]$, and the equation itself permits one to extend φ to an analytic solution on the entire axis $\mathbb{R}^1$. If $n > 1$, then the Picard theorem applies to (17) with respect to one of variables, and since the dependence of γ on the remaining variables is analytic, (17) is solvable in $\mathcal{A}(\mathbb{R}^n)$ for every $\gamma \in \mathcal{A}(\mathbb{R}^n)$. It is evident that the space of solutions to the homogeneous equation (17) is infinite-dimensional. We refer the reader to [BT2] for another approach to the solvability of difference equations more general than (17).

5.2. *A hyperbolic mapping of a cut plane.* The translation $x \mapsto x + e$ is the simplest diffeomorphism of $\mathbb{R}^n$ without non-wandering points. The following example shows that general diffeomorphisms with such properties can lead us to more complicated situations.

Let $X = \mathbb{R}^2/(-\infty, 0]$ and

$$F(\xi, \eta) = (\lambda\xi, \mu\eta), \quad (\xi, \eta) \in X,\ 0 < \lambda < 1 < \mu.$$

It is easy to see that X is real-analytically diffeomorphic to $\mathbb{R}^2$, and F is non-wandering points free.

Moreover, as we show below, there exists an open F-invariant covering $X = U_1 \cup U_2 \cup U_3$ such that the restriction $T|_{U_k}$ is surjective in $\mathcal{A}(U_k)$.

Nevertheless the Abel equation

$$\varphi(\xi, \eta) - \varphi(\lambda\xi, \mu\eta) = 1, \quad (\xi, \eta) \in X, \tag{18}$$

has no continuous solution $\varphi(\xi, \eta)$. Indeed, if $\varphi(\xi, \eta)$ were such a solution, then

$$\varphi(\xi,\eta)-\varphi(\lambda^n\xi,\mu^n\eta)=n,\quad n=0,1,2,\ldots,$$

or

$$\varphi(\xi,\mu^{-n}\eta)-\varphi(\lambda^n\xi,\eta)=n.$$

For $\xi,\eta>0$, we have $(\xi,\mu^{-n}\eta)\in X$, $(\lambda^n\xi,\eta)\in X$, producing a contradiction

$$\varphi(\xi,0)-\varphi(0,\eta)=\infty$$

as $n\to\infty$. In other words, the operator $T\varphi(\xi,\eta)=\varphi(\xi,\eta)-\varphi(\lambda\xi,\mu\eta)$ is not surjective in $\mathcal{A}(X)$.

From the point of view of Theorem 5.1 the absence of solutions means the existence of non-wandering compact subsets $K\subset X$. Here is an example of such a subset:

$$K=\{x=(\xi,\eta):\xi+\eta=1,\ \xi,\eta\ge 0\}.$$

On the other hand, the open sets

$$U_1=\{(\xi,\eta):\eta>0\},\quad U_2=\{(\xi,\eta):\eta<0\},\quad U_3=\{(\xi,\eta):\xi>0\}$$

form an invariant covering of X. Since $F|_{U_k}$, $k=1,2,3$, is analytically conjugate to a shift in $\mathbb{R}^2$, according to Section 5.1 the operator T is surjective in $\mathcal{A}(U_k)$. In particular, the following functions are solutions to (18) in $\mathcal{A}(U_k)$:

$$\varphi_k(\xi,\eta)=\begin{cases}-\dfrac{\ln\eta}{\ln\mu}, & (\xi,\eta)\in U_1,\\[2mm] -\dfrac{\ln|\eta|}{\ln\mu}, & (\xi,\eta)\in U_2,\\[2mm] -\dfrac{\ln\xi}{\ln\lambda}, & (\xi,\eta)\in U_3,\end{cases}\qquad (k=1,2,3).$$

Now we can explain why the Abel equation has no solution in $\mathcal{A}(X)$ from the point of view of Theorem 3.1. The functions $\varphi_k(\xi,\eta)$ generate the cocycle

$$(19)\qquad c_{k,j}(\xi\eta)=\begin{cases}0 & (\xi,\eta)\in U_1\cap U_2=\emptyset,\\[1mm] \dfrac{\ln\eta}{\ln\lambda}-\dfrac{\ln\xi}{\ln\mu}, & (\xi,\eta)\in U_1\cap U_3,\\[2mm] \dfrac{\ln|\eta|}{\ln\lambda}-\dfrac{\ln\xi}{\ln\mu}, & (\xi,\eta)\in U_2\cap U_3,\end{cases}\qquad k\ne j.$$

To describe the set E_0, consider the equation

$$(20)\qquad \varphi(\xi,\eta)-\varphi(\lambda\xi,\mu\eta)=0,\quad (\xi,\eta)\in U_3.$$

The mapping

$$\Phi(\xi,\eta)=(\ln\xi,\xi^{1/\alpha}\eta),\qquad \lambda^{1/\alpha}\mu=1,$$

is a real-analytic diffeomorphism of U_3 onto $\mathbb{R}^2$ which transforms (20) into the equation

$$h(u,v)-h(u+\ln\lambda,v)=0$$

where $\varphi(\xi,\eta)=h(\ln\xi,\xi^{1/\alpha}\eta)$. Therefore the general solution of (20) has the form

$$\varphi(\xi,\eta)=h\left(\frac{\ln\xi}{\ln\lambda},\xi^{1/\alpha}\eta\right)$$

where $h(u,v)$ is an arbitrary real-analytic function, 1-periodic with respect to u. The same arguments applied to (20) in U_1 and U_2 show that the space E_0 corresponding to the covering $\{U_k\}$ is formed by all cocycles $c=\{c_{kj}(\xi,\eta)\}$ with

$$c_{kj}(\xi,\eta)=\begin{cases}0, & (\xi,\eta)\in U_1\cap U_2=\emptyset,\\[1mm] h_3\left(\dfrac{\ln\xi}{\ln\lambda},\xi^{1/\alpha}\eta\right)-h_1\left(\dfrac{\ln\eta}{\ln\mu},\xi\eta^{\alpha}\right), & (\xi,\eta)\in U_1\cap U_3,\\[2mm] h_3\left(\dfrac{\ln\xi}{\ln\lambda},\xi^{1/\alpha}\eta\right)-h_2\left(\dfrac{\ln|\eta|}{\ln\mu},\xi|\eta|^{\alpha}\right), & (\xi,\eta)\in U_2\cap U_3,\end{cases}$$

where $h_i(u,v)$, $i=1,2,3$, are functions analytic in $\mathbb{R}^2$, 1-periodic with respect to u.

According to Theorem 3.1, if (1) has a solution in $\mathcal{A}(X)$, then the cocycle (19) belongs to E_0. This means that there exist functions $h_i(u,v)\in\mathcal{A}(\mathbb{R}^2)$, $i=1,2,3$, such that $h_i(u+1,v)=h_i(u,v)$ and

$$\frac{\ln\xi}{\ln\lambda}-\frac{\ln\eta}{\ln\mu}=h_3\left(\frac{\ln\xi}{\ln\lambda},\xi^{1/\alpha}\eta\right)-h_1\left(\frac{\ln\eta}{\ln\mu},\xi\eta^{\alpha}\right),\qquad (\xi,\eta)\in U_1\cap U_3,$$

$$\frac{\ln\xi}{\ln\lambda}-\frac{\ln|\eta|}{\ln\mu}=h_3\left(\frac{\ln\xi}{\ln\lambda},\xi^{1/\alpha}\eta\right)-h_2\left(\frac{\ln|\eta|}{\ln\mu},\xi|\eta|^{\alpha}\right),\quad (\xi,\eta)\in U_2\cap U_3.$$

However, if $\eta=1$, $\xi\to+0$ then the first equation yields $\ln\xi/\ln\lambda=O(1)$, which is impossible. Hence $\{c_{ik}(\xi,\eta)\}\notin E_0$ in complete accordance with Theorem 3.1, and, once more, $1\notin\operatorname{Im}T$.

If, in addition, we assume that there are no resonances, i.e., $\lambda^p\mu^q\ne 1$, $p\ge 0$, $q\ge 0$, $p+q>0$, then

$$T(\xi^p\eta^q)=(1-\lambda^p\mu^q)\xi^p\eta^q,$$

and therefore $\operatorname{Im}T$ contains all polynomials in ξ,η vanishing at $(0,0)$. According to the Runge Theorem (cf. [H]) there exists a sequence of polynomials $tP_m(t)$, converging to 1 in $\mathcal{A}(\mathbb{R}_+)$, $\mathbb{R}_+=(0,\infty)$, as $m\to\infty$. Then the sequence $(\xi^2+\eta^2)P_m(\xi^2+\eta^2)\in\operatorname{Im}T$ converges to 1 in $\mathcal{A}(X)$. Therefore, $\overline{\operatorname{Im}T}$ contains all polynomials, and $\overline{\operatorname{Im}T}=\mathcal{A}(X)$. In other words, the image of T is dense but not closed in $\mathcal{A}(X)$. *Hence, the subspace* $\operatorname{Ker}\Theta$ *constructed in Section* 3 *is not closed either.*

If, on the other hand, $0 < \lambda, \mu < 1$ (or $\lambda, \mu > 1$), then all compact sets in X are wandering and according to Theorem 5.1 there exists an analytic solution of the Abel equation (18). In the present situation we can give a constructive description of the solution. To this end we note that the function

$$h(x) = \int_0^1 \ln(\lambda^{2s}\xi^2 + \mu^{2s}\eta^2)\,ds, \quad x = (\xi, \eta),$$

is analytic on X and the function

$$\sigma_{\xi,\eta}(u) \equiv h(\lambda^u\xi, \mu^u\eta) = \int_u^{u+1} \ln(\lambda^{2s}\xi^2 + \mu^{2s}\eta^2)\,ds \tag{21}$$

is an analytic diffeomorphism of $\mathbb{R}^1$. Let $\varphi(\xi, \eta)$ be the pre-image of 0, i.e., a solution to the equation $\sigma_{\xi,\eta}(\varphi) = 0$. It is evident that $\varphi(\xi, \eta)$ is analytic with respect to $(\xi, \eta) \in X$. Since $\varphi(\lambda^t\xi, \mu^t\eta)$ is the unique solution to $h(\lambda^{u+t}\xi, \mu^{u+t}\eta) = 0$, we obtain

$$\varphi(F^t(x)) + t = \varphi(x),$$

with $F^t(x) = (\lambda^t\xi, \mu^t\eta)$, and $\varphi(\xi, \eta)$ is a solution to the Abel equation (18).

In addition, if $0 < \lambda, \mu < 1$ (or $1 < \lambda, \mu$), then the operator $T\varphi(\xi, \eta) = \varphi(\xi, \eta) - \varphi(\lambda\xi, \mu\eta)$ is surjective in $\mathcal{A}(X)$. To prove this, let

$$Y = \{x = (\xi, \eta) \in X : \varphi(\xi, \eta) = 0\},$$

or according to (21),

$$Y = \Big\{(\xi, \eta) \in X : \int_0^1 \ln(\lambda^{2s}\xi^2 + \mu^{2s}\eta^2)\,ds = 0\Big\}.$$

It is evident that Y is a real-analytic curve in X which is diffeomorphic to $\mathbb{R}^1$. If $\Phi : X \to Y \times \mathbb{R}^1$ is defined by

$$\Phi(x) = (F^{\varphi(x)}(x), \varphi(x)),$$

then

$$\Phi(F(x)) = (F^{\varphi(x)}(x), \varphi(x) - 1).$$

In other words, Φ conjugates F to the shift $(y, s) \mapsto (y, s - 1)$ on $Y \times \mathbb{R}^1$. According to Section 5.1 the cohomological equation

$$\varphi(\xi, \eta) - \varphi(\lambda\xi, \mu\eta) = \gamma(\xi, \eta), \quad (\xi, \eta) \in X,$$

has a real-analytic solution $\varphi(\xi, \eta)$ for an arbitrary real-analytic function $\gamma(\xi, \eta)$.

5.3. *Source-sink on the sphere.* Let X be the standard sphere $S^n \subset \mathbb{R}^{n+1}$, and $F : S^n \to S^n$ be a diffeomorphism with two fixed points z_1 and z_2, the former being a source and the latter a sink. This means that $F'(z)$ is a matrix function on S^n with spectrum outside the unit disc at $z = z_1$ and inside the disc at $z = z_2$. Two sets $U_1 = S^n \setminus \{z_2\}$ and $U_2 = S^n \setminus \{z_1\}$ form an invariant covering of S^n with exactly one fixed point each. If $\gamma \in \mathcal{A}(X)$ and (17) has a solution φ in $\mathcal{A}(X)$, then $\gamma(z_1) = \gamma(z_2) = 0$. On the other hand, if this condition is satisfied, then the series

$$\sum_{k=0}^{\infty} \gamma(F^k z), \quad z \in U_2; \qquad -\sum_{k=-\infty}^{-1} \gamma(F^k z), \quad z \in U_1,$$

define analytic solutions to (17) in $\mathcal{A}(U_2)$ and $\mathcal{A}(U_1)$, respectively. The solutions of homogeneous equations (17) in $\mathcal{A}(U_2)$ and $\mathcal{A}(U_1)$ are constant, and the following statement is valid.

THEOREM 5.2. *The cohomological equation* (17) *has a solution* $\varphi \in \mathcal{A}(S^n)$ *if and only if* $\gamma(z_1) = \gamma(z_2) = 0$ *and*

$$\sum_{k=-\infty}^{\infty} \gamma(F^k z) = \text{const} \quad (z \neq z_1, z_2).$$

The operator T is semi-Fredholm: its image $\operatorname{Im} T$ *is closed, its kernel* $\operatorname{Ker} T$ *is one-dimensional and* $\dim(\operatorname{Coker} T) = \infty$.

Proof. The solvability criterion follows from Theorem 3.1, and it implies that the subspace $\operatorname{Im} T$ is closed. Evidently,

$$\operatorname{Ker} T = \{\gamma \in \mathcal{A}(S^n) : \gamma = \text{const}\}.$$

According to Theorem 3.1(ii) to prove the last claim of Theorem 5.2 it is sufficient to check that

$$\dim(\operatorname{Ker} T|_{S^n \setminus \{z_1, z_2\}}) = \infty.$$

Indeed, let $\varphi_0 \in \mathcal{A}(S^n \setminus \{z_1, z_2\})$ be a solution of the Abel equation, existing by Theorem 5.1. Then for any 1-periodic function $\tau \in \mathcal{A}(\mathbb{R}^1)$ the function $\psi(z) \equiv \tau(\varphi_0(z))$ is an analytic solution of the homogeneous equation on $S^n \setminus \{z_1, z_2\}$.

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Received December 27, 1999
Revised version April 10, 2000 (4455)

STUDIA MATHEMATICA 143 (2) (2000)

Selfsimilar profiles in large time asymptotics of solutions to damped wave equations

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Abstract. Large time behavior of solutions to the generalized damped wave equation $u_{tt} + Au_t + \nu Bu + F(x,t,u,u_t,\nabla u) = 0$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ is studied. First, we consider the linear nonhomogeneous equation, i.e. with $F = F(x,t)$ independent of u. We impose conditions on the operators A and B, on F, as well as on the initial data which lead to the selfsimilar large time asymptotics of solutions. Next, this abstract result is applied to the equation where $Au_t = u_t$, $Bu = -\Delta u$, and the nonlinear term is either $|u_t|^{q-1}u_t$ or $|u|^{\alpha-1}u$. In this case, the asymptotic profile of solutions is given by a multiple of the Gauss–Weierstrass kernel. Our method of proof does not require the smallness assumption on the initial conditions.

1. Introduction. The goal of this paper is to study the large time behavior of solutions to the initial value problem for the generalized semilinear wave equation with a dissipative term

$$u_{tt} + Au_t + \nu Bu + F(x,t,u,u_t,\nabla u) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

In the equation above, the pseudodifferential operators A and B are defined via the Fourier transform by the formulae

$$\widehat{Av}(\xi) = |\xi|^a\widehat{v}(\xi) \quad \text{and} \quad \widehat{Bv}(\xi) = |\xi|^b\widehat{v}(\xi) \tag{1.3}$$

for some real constants a and b satisfying $0 \le 2a < b$. Moreover, $\nu > 0$ is a fixed constant, and assumptions on the nonlinear term are specified in Section 2 below.

Our main purpose is to find conditions on the operators A and B, on the nonlinearity F, as well as on the initial data u_0 and u_1, which lead to the *selfsimilar large time behavior* of solutions. In the first step of our considerations, using the Fourier transform we solve the linear equation

2000 *Mathematics Subject Classification*: 35B40, 35L15, 35L30.

Key words and phrases: the Cauchy problem, generalized wave equation with damping, large time behavior of solutions, selfsimilar solutions.