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The asymptotics of the Perron–Frobenius operator of a class of interval maps preserving infinite measures

by

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Abstract. We determine the asymptotic behaviour of the iterates of the Perron–Frobenius operator for specific interval maps with an indifferent fixed point which gives rise to an infinite invariant measure.

1. Introduction. In [T5] and under more general conditions in [Z2] a theorem is proved concerning the asymptotic behaviour of $\{\sum_{k=0}^{n-1} P^k\}_{n=1}^{\infty}$, where P is the Perron–Frobenius operator associated with interval maps with indifferent fixed points and infinite invariant measure. This theorem provides us with an abundance of uniform sets and Darling–Kac sets, which play a crucial role in deducing probabilistic laws (see [A0, A1, A2, T7]). A natural, as well as fascinating, further step is to study the asymptotic behaviour of the sequence $\{P^n\}_{n=0}^{\infty}$ itself. In view of classical local ratio limit theorems we may expect to find strong results.

The purpose of the present paper is to study this problem for a class of examples. Both the method employed and part of the examples are taken from the unpublished papers [T3, TR, T4]. To the author's knowledge, no further results of this type have been obtained in the meantime, except for the piecewise affine linear case, which in fact does not go beyond the Markov chain setting. The specifying property of the maps considered here is a kind of concavity (respectively convexity) with regard to the invariant measure. It is quite possible that the method of proof can be refined so as to be applied to more general classes. Our examples may also be helpful in establishing further results.

The content of the paper is arranged in the following way. In Section 2 we state the main result, in Section 3 we discuss some particular examples. The proof is carried out in Section 4. Sections 5 and 6 contain applications and a concluding remark.

2. The main result. We are dealing with a class of maps T of $[0,1]$ into itself with two increasing full branches and an indifferent fixed point at 0, which gives rise to an infinite invariant measure equivalent to the Lebesgue measure λ. A well known example belonging to our class is the *Lasota–Yorke map*

$$T(x) = \begin{cases} x/(1-x), & x \in [0, 1/2), \\ 2x-1, & x \in [1/2, 1], \end{cases}$$

which has invariant density $h(x) = 1/x$ (see [LY, LM, BG]). For this example the theorem stated here was obtained in [T3] by further developing the analysis in [LY]. The conditions below are chosen in such a way that the method of proof can basically be retained.

Let c denote the inner endpoint of the monotonicity intervals, and let $I_0 = (0, c)$, $I_1 = (c, 1)$. Then T is assumed to satisfy the following conditions:

(i) $T_{|I_j}$ has a C^1-extension $\overline{T}_{|I_j}$ to $\overline{I}_j$, $(\overline{T}_{|I_j})' > 0$, and $\overline{T(I_j)} = [0, 1]$ $(j = 0, 1)$.

(ii) T is convex in a neighbourhood of 0.

(iii) T admits an invariant measure μ equivalent to λ, such that the density $d\mu/d\lambda$ has a version of the form

$$h(x) = \widetilde{h}(x)/x^p, \qquad x \in (0, 1], \text{ where } p \geq 1,$$

and $\widetilde{h}$ is positive, continuous and of bounded variation on $[0, 1]$.

(iv) The function

$$\psi = \frac{h \circ T \cdot T'}{h}$$

is increasing on I_0.

The number $\alpha = 1/p$ will be called the *return index* of T. By $\mathcal{B}$ we denote the σ-field of Lebesgue measurable subsets of $[0, 1]$.

Before we state the main result, we shall make a few comments on these conditions beginning with condition (iv).

Let f_j denote the inverse of $\overline{T}_{|I_j}$ and define the function ω_j on $(0, 1]$ by

$$\omega_j = \frac{h \circ f_j \cdot f_j'}{h} \qquad (j = 0, 1).$$

Since h is invariant, Kuzmin's equation

$$\omega_0 + \omega_1 = 1$$

holds, and ω_0, ω_1 extend continuously to $[0, 1]$ with $\omega_0(0) = 1, \omega_1(0) = 0$. Correspondingly, ψ extends continuously to $[0, c]$ with $\psi(0) = 1$.

The identity

$$\omega_0 = \frac{1}{\psi \circ f_0}$$

and condition (iv) show that ω_0 is decreasing, and hence ω_1 increasing, on $[0, 1]$.

For a direct geometric interpretation consider the conjugated map

$$S = \varphi \circ T \circ \varphi^{-1} \quad \text{on } [0, \infty), \quad \text{where} \quad \varphi(x) = \int_x^1 h(t)\, dt, \ x \in (0, 1],$$

and note that $S' = \psi \circ \varphi^{-1}$ on $[\varphi(c), \infty)$. In terms of S, condition (iv) states that S is concave on $[\varphi(c), \infty)$ (and hence convex on $[0, \varphi(c))$).

The conditions (i)–(iii) determine the asymptotic behaviour of $T(x)$ for $x \to 0$, as is clarified by the following proposition.

PROPOSITION. *Let T satisfy the conditions* (i) *and* (ii), *and assume T admits an invariant measure μ equivalent to λ, such that $d\mu/d\lambda$ has a version h which is continuous and positive on $(0, 1]$. Then T has no fixed point in I_0, and, for every $p \geq 0$, the following asymptotic relations are equivalent:*

(1) $h(x) \sim \beta/x^p$ $(x \to 0)$,

(2) $T(x) - x \sim a x^{p+1}$ $(x \to 0)$,

where β, a are positive constants.

Proof. Let $g = h \circ f_1 \cdot f_1'$, which is continuous and positive on $[0, 1]$. Assume $T(\xi) = \xi$ and $\xi \in I_0$. By Kuzmin's equation,

$$(1 - f_0'(\xi))h(\xi) = g(\xi),$$

so $f_0'(\xi) < 1$. Choose $y \in (0, \xi)$ such that $y < f_0(y)$. The interval $[y, f_0(y))$ is a wandering set and

$$\bigcup_{n=0}^{\infty} T^{-n}([y, f_0(y))) \subseteq [y, 1],$$

which contradicts $\mu([y, 1]) < \infty$.

To establish the equivalence of (1) and (2), we make use of the identity

$$h = \sum_{n=0}^{\infty} g \circ f_0^n \cdot (f_0^n)',$$

which is obtained as follows. Let $A = [c, 1]$, and let $\varphi(x) = \min\{n \geq 1 : T^n(x) \in A\}$, $x \in (0, 1]$. As T has no fixed point in I_0, we have $\varphi < \infty$ on $(0, 1] \setminus \{c\}$. Let $D_n = A \cap \{\varphi > n\}$, $n \geq 0$. Since $\mu(A) < \infty$, the formula

$$\mu(E) = \sum_{n=0}^{\infty} \mu(D_n \cap T^{-(n+1)}E), \qquad E \in \mathcal{B},$$

holds (see e.g. [A0]). Taking into account that

$$D_n = A \cap T^{-1}([0, f_0^n(1))), \qquad n \geq 1,$$

we see that the formula is equivalent to

$$h = \sum_{n=0}^{\infty} g \circ f_0^n \cdot (f_0^n)' \quad \lambda\text{-a.e. on } (0,1].$$

Arguments as in the first part of the proof of Lemma 4 in [T2] show that the right hand side is continuous on $(0,1]$. Therefore the above a.e. equality is a pointwise equality. The second part of the cited proof and a modified argument in case $f_0'(0) < 1$ then yield

$$h(x) \sim g(0) \frac{x}{x - f_0(x)} \quad (x \to 0),$$

which implies the equivalence of (1) and (2). ∎

In accordance with the notation in [T2] we put

$$w_n(T) = \mu\left(\bigcup_{k=0}^{n-1} T^{-k}([c,1]) \right), \quad n \geq 1.$$

As in Theorem 4 of [T2],

$$w_n(T) \sim \begin{cases} \widetilde{h}(0) \log n, & \alpha = 1, \\ \frac{1}{p-1} \widetilde{h}(0) (pan)^{1-\alpha}, & \alpha < 1, \end{cases}$$

where a is as in the Proposition. To verify this assertion note that

$$f_0^n(1) \sim (pan)^{-1/p} \quad (n \to \infty)$$

and

$$w_n(T) \sim \widetilde{h}(0) \int_{f_0^n(1)}^{1} \frac{dx}{x^p} \quad (n \to \infty).$$

Recall that the *Perron–Frobenius operator* $P : L_1(\lambda) \to L_1(\lambda)$ is defined by

$$\int_A Pu \, d\lambda = \int_{T^{-1}A} u \, d\lambda, \quad u \in L_1(\lambda), \, A \in \mathcal{B}.$$

In what follows, Pu always denotes the version given by

$$Pu = u \circ f_0 \cdot f_0' + u \circ f_1 \cdot f_1'.$$

As for the example in [LY], it can be seen that $P^n u \to 0$ in measure for all $u \in L_1(\lambda)$. Under suitable restrictions on u, however, proper normalization leads to non-trivial limiting behaviour.

THEOREM. *Let* $T : [0,1] \to [0,1]$ *satisfy the conditions* (i)–(iv) *with return index* α. *Then, for all Riemann-integrable functions* u *on* $[0,1]$,

$$w_n(T) P^n u \to \left(\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 u \, d\lambda \right) h \quad (n \to \infty),$$

uniformly on compact subsets of $(0,1]$.

In particular, we have the following local ratio limit theorem:

For any two Riemann-integrable functions u, v *on* $[0,1]$ *with* $v > 0$,

$$\frac{P^n u(x)}{P^n v(y)} \to \frac{\int_0^1 u \, d\lambda}{\int_0^1 v \, d\lambda} \cdot \frac{h(x)}{h(y)} \quad (n \to \infty)$$

uniformly on compact subsets of $(0,1] \times (0,1]$.

To stress the basic probabilistic implication of the Theorem, let the initial value of the iteration process $\{T^n\}$ be given by a random variable X_0 with density u, so that $X_n := T^n(X_0)$ has density $P^n u$ $(n \geq 0)$, and let u be Riemann-integrable. Furthermore, let A be a measurable set with positive measure, bounded away from 0. Integration over A yields

$$\text{Prob}(\{X_n \in A\}) \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \cdot \frac{\mu(A)}{w_n(T)} \quad (n \to \infty).$$

Thus the Theorem asserts that the conditional density of X_n given $\{X_n \in A\}$ converges uniformly to the proper limit, i.e. to $(1/\mu(A)) h 1_A$. Further consequences will be mentioned in Section 5.

3. Examples

1. *The Lasota–Yorke example and related maps.* Let first T be the Lasota –Yorke map mentioned at the beginning of the previous section:

$$T(x) = \begin{cases} x/(1-x), & x \in [0,1/2), \\ 2x - 1, & x \in [1/2,1], \end{cases}$$

where $h(x) = 1/x$. The conditions (i)–(iii) are obviously satisfied with $\alpha = 1$. Since

$$\psi(x) = \frac{1}{1-x}, \quad x \in [0,1/2],$$

condition (iv) also holds. As a result, for all Riemann-integrable functions u on $[0,1]$,

$$(\log n) P^n u \to \left(\int_0^1 u \, d\lambda \right) h \quad (n \to \infty)$$

uniformly on compact subsets of $(0,1]$.

The map T is a member of the family

$$T_q(x) = \begin{cases} \dfrac{x}{1-x}, & x \in [0,1/2), \\ \dfrac{2x-1}{q+(1-q)x}, & x \in [1/2,1], \, q > 0. \end{cases}$$

Up to a smooth conjugation, this is the class of all maps with two increasing, fractional linear and full branches, such that $T'(0) = 1$, $T'(1) > 1$. The

invariant density is given by

$$h(x) = \frac{1}{x(1 - (1-q)x)} \quad \text{(see [Sch1])}.$$

Since $\psi(x) = 1 + x/(1 - (2-q)x)$, $x \in [0, 1/2]$, our conditions are satisfied for all $q > 0$.

These examples show that convexity of both branches of T is not necessary for condition (iv) to hold.

2. *P. Manneville's example and a generalizing family.* Another simple example in our class is

$$T(x) = x + x^2 \pmod 1,$$

a map considered in [M]. As shown in [T6], $h(x) = 1/x + 1/(1+x)$ is the invariant density of T. Again it is readily verified that T satisfies our conditions (with $\alpha = 1$).

In order to obtain a family of maps covering the whole spectrum of return indices $\alpha \in (0, 1]$ we observe that $g(x) = x + x^2$ satisfies

$$\frac{g'(x)}{g(x)} = h(x),$$

which yields the identities

$$h(f_0(x))f_0'(x) = \frac{1}{x} \quad \text{and} \quad h(f_1(x))f_1'(x) = \frac{1}{1+x},$$

and thus Kuzmin's equation. Generalizing this observation, put

$$h_p(x) = \frac{1}{x^p} + \frac{1}{(1+x)^p}, \quad x \in (0, 1], \ p \geq 1,$$

define g_p as the unique C^1-function on $[0, 1]$ satisfying

$$\frac{g_p'(x)}{(g_p(x))^p} = h_p(x), \quad x \in (0, 1], \quad \text{and} \quad g_p(1) = 2,$$

and define the map T_p by

$$T_p(x) = g_p(x) \pmod 1.$$

For $p > 1$,

$$g_p(x) = x\left(1 + \left(\frac{x}{1+x}\right)^{p-1} - x^{p-1}\right)^{1/(1-p)}, \quad x \in [0, 1].$$

By construction, h_p is invariant for T_p, and $\lim_{p \to 1} T_p = T$. Note also that $p = 2$ yields the analytically appealing map

$$T_2(x) = \frac{x(1+x)}{1 + x - x^2} \pmod 1.$$

The maps T_p obviously satisfy conditions (i) and (iii). Condition (ii) holds since

$$g_p''(x) = p\frac{x^{p+1} + (1+x)^{p+1} - 1}{x^{p+1}(1+x)^{p+1}}(g_p(x))^{2p-1} > 0, \quad x \in (0, 1].$$

To see that condition (iv) holds, note that for $x \in I_0$,

$$\psi(x) = \frac{h_p(g_p(x))g_p'(x)}{h_p(x)}$$

$$= h(g_p(x))(g_p(x))^p = 1 + \left(\frac{g_p(x)}{1 + g_p(x)}\right)^p.$$

Thus our theorem applies to all T_p, $p \geq 1$. The constant a, needed to determine the normalizing sequence $\{w_n(T)\}$ for $p > 1$, is equal to 1 since

$$\lim_{x \to 0} g_p''(x)/x^{p-1} = p(p+1).$$

3. *One-sided Boole transformations.* Let T satisfy our conditions. We have already mentioned the conjugation

$$S = \varphi \circ T \circ \varphi^{-1} \quad \text{with} \quad \varphi(x) = \int_x^1 h(t)\,dt, \quad x \in (0, 1],$$

yielding a Lebesgue measure preserving map S on $[0, \infty)$ such that $S_{|[\varphi(c), \infty)}$ is concave. Our third class of examples will be constructed taking the reverse route. We start with the Boole transformation

$$B(x) = x - \frac{b}{x - a}, \quad x \in \mathbb{R} \setminus \{a\},$$

where $a \in \mathbb{R}$, $b > 0$ are fixed parameters. As is well known, B preserves Lebesgue measure λ (cf. [A0, Sch2]). We denote by d the positive zero of B and define S by

$$S(x) = \begin{cases} B(x), & x \in [d, \infty), \\ B(x + a - d), & x \in [0, d). \end{cases}$$

Since $a - d$ is the negative zero of B, S is a two-to-one map from $[0, \infty)$ into itself, which is concave on $[d, \infty)$. Evidently,

$$\lambda(S^{-1}(A)) = \lambda(B^{-1}(A)) \quad \text{for all Borel subsets } A \text{ of } [0, \infty).$$

Therefore S preserves λ. We carry over the map S to $[0, 1]$ by means of the function $\varphi(x) = d(1/x - 1)$, $x \in (0, 1]$. Taking into account that $d(d - a) = b$ we obtain

$$\varphi^{-1} \circ S \circ \varphi = T_q, \quad \text{where} \quad q = \frac{b}{d^2} \quad (> 0)$$

and

$$T_q(x) = \begin{cases} \dfrac{x(1 + (q-2)x)}{1 + (q-2)x - qx^2}, & x \in [0, 1/2), \\[3mm] \dfrac{x(1-2x)}{1 - (q+2)x + qx^2}, & x \in [1/2, 1]. \end{cases}$$

Since $|\varphi'|$ is an invariant density for T_q, it is easily seen that T_q satisfies the conditions (i)–(iv) (with $p = 2$).

Note that T_4 is conjugated to $T(x) = g_2(x) \pmod 1$, where g_2 is as in the preceding class of examples. In fact, $T_4 = \phi \circ T \circ \phi^{-1}$ with $\phi = g_2/2$.

4. Proof of the Theorem. To begin with, we restate the Theorem in terms of the Perron–Frobenius operator $\widehat{T}$ with respect to the invariant measure μ. Let $L_1(\mu)$ be the set of all μ-integrable real functions v on $[0,1]$ with $v(0) = 0$, and let the version of $\widehat{T}v$ be specified by

$$\widehat{T}v = v \circ f_0\, \omega_0 + v \circ f_1\, \omega_1, \quad v \in L_1(\mu).$$

We shall use the abbreviation $v_n = \widehat{T}^n v$ $(n \geq 0)$. Since

$$\frac{1}{h} Pu = \widehat{T}\left(\frac{1}{h} u\right)$$

and h is bounded away from 0 and infinity on compact subsets of $(0,1)$, the assertion of the Theorem is equivalent to

$$w_n(T)\, v_n \to \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 v \, d\mu$$

uniformly on compact subsets of $(0,1)$ for all functions v of the form $v = (1/h)u$ where u is Riemann-integrable on $[0,1]$.

Let $\mathcal{M}$ denote the class of increasing functions v in $L_1(\mu)$ with positive integral. The elements v of $\mathcal{M}$ are non-negative on $[0,1]$ and satisfy $\lim_{x \to 0} v(x) = 0$.

The first step of the proof, which we divide into three lemmas, shows the assertion for $v \in \mathcal{M}$. The second step is an approximation procedure.
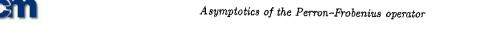
First of all, we state the crucial implication of condition (iv) (cf. [LY]).

LEMMA 1. *We have* $\widehat{T}(\mathcal{M}) \subseteq \mathcal{M}$.

Proof. Choose $v \in \mathcal{M}$ and $x, y \in [0,1]$ with $x \leq y$. Taking into account that $\omega_0 + \omega_1 = 1$ we obtain

$$v_1(y) - v_1(x) = (v(f_1(x)) - v(f_0(y)))(\omega_0(x) - \omega_0(y))$$
$$+ (v(f_0(y)) - v(f_0(x)))\omega_0(x) + (v(f_1(y)) - v(f_1(x)))\omega_1(y).$$

Since $f_1(x) \geq f_0(y)$, and v, f_0, f_1 are increasing and ω_0 is decreasing, all differences are non-negative. Hence $v_1 \in \mathcal{M}$. ∎

As an immediate consequence we see that the sequence $\{v_n(1)\}$ is decreasing for all $v \in \mathcal{M}$. In fact, for $n \geq 0$,

$$v_n(1) - v_{n+1}(1) = v_n(1) - (v_n(f_0(1))\omega_0(1) + v_n(1)\omega_1(1))$$
$$= \omega_0(1)(v_n(1) - v_n(f_0(1))),$$

which is non-negative for $v \in \mathcal{M}$. This property allows us to determine the asymptotics of $\{v_n(1)\}$ by means of the standard version of Karamata's Tauberian Theorem.

LEMMA 2. *For all* $v \in \mathcal{M}$,

$$w_n(T)\, v_n(1) \to \frac{1}{\Gamma(\alpha)\,\Gamma(2-\alpha)} \int_0^1 v \, d\mu.$$

Proof. Let v be in $\mathcal{M}$, let $A = [c, 1]$, $\varphi(x) = \min\{n \geq 1 : T^n(x) \in A\}$ $(x \in (0,1])$, and

$$q_n = \frac{\mu(A \cap \{\varphi > n\})}{\mu(A)} \quad (n \geq 0).$$

Define $Q(s)$ and $V(s)$ for $s > 0$ by

$$Q(s) = \sum_{n=0}^{\infty} q_n e^{-ns}, \quad V(s) = \sum_{n=0}^{\infty} v_n(1) e^{-ns}.$$

We claim that

$$V(s)\,Q(s) \sim \left(\frac{1}{\mu(A)} \int_0^1 v \, d\mu\right)\frac{1}{s} \quad (s \to 0).$$

This could be proved by referring to the asymptotic renewal equation in [A0, A2]. The monotonicity of the functions v_n allows us, however, to take a more direct route.

Let $A_n = \bigcup_{k=0}^{n} T^{-k}A$ $(n \geq 0)$. We use the decompositions

$$A_n = \bigcup_{k=0}^{n} T^{-k}(A \cap \{\varphi > n - k\})$$

to get the identity

$$\int_A \left(\sum_{n=0}^{\infty} v_n e^{-ns}\right)\left(\sum_{n=0}^{\infty} 1_{A \cap \{\varphi > n\}} e^{-ns}\right) d\mu = \sum_{n=0}^{\infty}\left(\int_{A_n} v \, d\mu\right) e^{-ns} \quad (s > 0).$$

Therefore, for all $s > 0$,

$$V(s)\,Q(s) = \frac{1}{\mu(A)} \sum_{n=0}^{\infty}\left(\int_{A_n} v \, d\mu\right) e^{-ns} + R(s).$$

with

$$R(s) = \frac{1}{\mu(A)} \int_A \left(\sum_{n=0}^{\infty} (v_n(1) - v_n) e^{-ns} \right) \left(\sum_{n=0}^{\infty} 1_{A \cap \{\varphi > n\}} e^{-ns} \right) d\mu.$$

For $x \in A$ and $n \geq 0$,

$$0 \leq v_n(1) - v_n(x) \leq v_n(1) - v_n(f_0(1))$$
$$= \frac{1}{\omega_0(1)} (v_n(1) - v_{n+1}(1)) \quad (n \geq 0),$$

which shows that

$$0 \leq R(s) \leq \frac{v_0(1)}{\omega_0(1)} Q(s) \quad (s > 0).$$

Noting that

$$\mu(A) \sum_{n=0}^{\infty} v_n(1) \geq \sum_{n=0}^{\infty} \int_A v_n \, d\mu = \int_0^1 v \left(\sum_{n=0}^{\infty} 1_A \circ T^n \right) d\mu$$

and

$$\sum_{n=0}^{\infty} 1_A \circ T^n = \infty \quad \text{on} \quad [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(\{0\}),$$

we see that the series $\sum_{n=0}^{\infty} v_n(1)$ diverges. Therefore $\lim_{s \to 0} V(s) = \infty$, and the desired relation follows.

Since $\mu(A) \sum_{k=0}^{n-1} q_k = w_n(T)$ $(n \geq 1)$ and $\{v_n(1)\}$ is decreasing, we now obtain the assertion by applying Karamata's Tauberian Theorem (see e.g. [Fe]). ∎

Lemma 2 contains the important information that for $v \in \mathcal{M}$,

$$v_{n+1}(1) \sim v_n(1) \quad (n \to \infty),$$

which is needed to prove

LEMMA 3. *For $v \in \mathcal{M}$ and $k \geq 0$,*

$$v_n(f_0^k(1)) \sim v_n(1) \quad (n \to \infty).$$

P r o o f. Let v be in $\mathcal{M}$. Since $v_n \leq v_n(1)$ and

$$v_n(1) - v_{n+1} = (v_n(1) - v_n \circ f_0)\omega_0 + (v_n(1) - v_n \circ f_1)\omega_1$$

we have

$$0 \leq v_n(1) - v_n \circ f_0 \leq \frac{1}{\omega_0} (v_n(1) - v_{n+1}) \quad (n \geq 0).$$

This shows that

$$0 \leq 1 - \frac{v_n(f_0^{k+1}(1))}{v_n(1)} \leq \frac{1}{\omega_0(f_0^k(1))} \left(1 - \frac{v_{n+1}(f_0^k(1))}{v_n(1)} \right) \quad (n, k \geq 0),$$

and the assertion follows by induction. ∎

Now let $v \in \mathcal{M}$ and $k \geq 1$ be fixed. For $x \in [f_0^k(1), 1]$,

$$\frac{v_n(f_0^k(1))}{v_n(1)} \leq \frac{v_n(x)}{v_n(1)} \leq 1.$$

Hence

$$\frac{v_n}{v_n(1)} \to 1 \quad \text{uniformly on } [f_0^k(1), 1],$$

finishing the proof of the Theorem for $v \in \mathcal{M}$.

Proceeding to the second step of the proof, let $\mathcal{S}$ denote the set of all functions $v \in L_1(\mu)$ for which the desired assertion holds. Then $\mathcal{S}$ is a linear subspace of $L_1(\mu)$ satisfying $\widehat{T}^{-1}(\mathcal{S}) = \mathcal{S}$. We shall use the following criterion, which is an easy consequence of the positivity of $\widehat{T}$:

If $v \in L_1(\mu)$, and for each $\varepsilon > 0$ there exist $w^{(1)}, w^{(2)} \in \mathcal{S}$ such that $w^{(1)} \leq v \leq w^{(2)}$ and $\int_0^1 (w^{(2)} - w^{(1)}) \, d\mu < \varepsilon$, then $v \in \mathcal{S}$.

We claim first that

$$\frac{1}{h} u \in \mathcal{S} \quad \text{for all } u \in \mathrm{BV}([0,1]),$$

where $\mathrm{BV}([0,1])$ denotes the set of real functions on $[0,1]$ with bounded variation. To prove this, note that for $u \in \mathrm{BV}([0,1])$ there exists a constant K such that

$$\bigvee_0^x \frac{1}{h} u \leq K x^p, \quad x \in [0,1].$$

This estimate is readily verified by means of the product rule using the representation

$$\left(\frac{1}{h} u \right)(x) = \frac{u(x)}{\widetilde{h}(x)} \cdot x^p, \quad x \in [0,1],$$

and taking into account that $\widetilde{h}$ is supposed to be of bounded variation and bounded away from 0.

Therefore the components in the canonical decomposition

$$\frac{1}{h} u = v^{(1)} - v^{(2)} \quad \text{with} \quad v^{(1)}(x) = \bigvee_0^x \frac{1}{h} u, \quad x \in [0,1],$$

are elements of $\mathcal{M}$, and the claim follows from the first step of the proof.

Now let $v = (1/h)u$ where u is Riemann-integrable on $[0,1]$, and let $\varepsilon > 0$ be given. Choose a partition of $[0,1]$ consisting of intervals J_k, $1 \leq k \leq N$, such that

$$\sum_{k=1}^{N} \beta_k \lambda(J_k) - \sum_{k=1}^{N} \alpha_k \lambda(J_k) < \varepsilon,$$

where

$$\alpha_k = \inf_{x \in J_k} u(x) \quad \text{and} \quad \beta_k = \sup_{x \in J_k} u(x).$$

Letting

$$w^{(1)} = \frac{1}{h} \sum_{k=1}^{N} \alpha_k 1_{J_k} \quad \text{and} \quad w^{(2)} = \frac{1}{h} \sum_{k=1}^{N} \beta_k 1_{J_k}$$

we have

$$w^{(1)} \le v \le w^{(2)} \quad \text{and} \quad \int_0^1 (w^{(2)} - w^{(1)}) \, d\mu < \varepsilon.$$

Evidently, $w^{(1)}$, $w^{(2)}$ are as considered before, and thus belong to $\mathcal{S}$. In view of the above mentioned criterion the proof is complete. ∎

5. Two corollaries. To indicate possible applications of the Theorem, we shall briefly discuss a mixing property considered first by E. Hopf ([H], §17) and give a simplified proof of a distributional limit theorem.

Before proceeding, we mention that a map T satisfying the conditions (i)–(iv) is conservative and ergodic with respect to λ. Conservativity is an immediate consequence of Maharam's recurrence theorem (cf. [A0]). Ergodicity can be proved as follows. Let S be defined as in Section 2, let $\widetilde{S}$ denote the map induced by S on $[0, \varphi(c))$, and let $P_{\widetilde{S}}$ be the Perron–Frobenius operator of $\widetilde{S}$ with respect to λ. The map $\widetilde{S}$ preserves λ, and is easily seen to be uniformly expanding. Exploiting the monotonicity properties of the derivatives of the branches of S, it is not difficult to verify that $\widetilde{S}$ satisfies the bounded variation condition in [R] (cf. also [BG]). Helly's theorem and an approximation argument may then be used to get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_B P_{\widetilde{S}}^k u \, d\lambda = \left(\int_0^{\varphi(c)} u \, d\lambda \right) \lambda(B)/\varphi(c)$$

for all measurable subsets B of $[0, \varphi(c))$ and all functions u in $L_1([0, \varphi(c)), \lambda)$. This proves ergodicity of $\widetilde{S}$, and thus ergodicity of T.

Our theorem clearly implies convergence in the sense of [T5]. In particular, a map T satisfying (i)–(iv) is also pointwise dual ergodic.

Strong mixing and the spreading rate. Recalling the notion of strong mixing in infinite measure spaces in the sense of [H, Kr, P, Fr], let T be a measure preserving transformation on a measure space $(M, \mathcal{B}, \mu)$ with $\mu(M) = \infty$, and let $\mathcal{R}$ be a ring of sets of finite measure generating $\mathcal{B}$ (mod 0), such that $T^{-1}(\mathcal{R}) \subseteq \mathcal{R}$. The transformation T is called *mixing with respect to* $\mathcal{R}$ (or $\mathcal{R}$-*mixing*) if there exists a sequence $\{\varrho_n\}$ of positive

numbers such that

$$\lim_{n \to \infty} \varrho_n \mu(A \cap T^{-n} B) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{R}.$$

The order of the sequence $\{\varrho_n\}$ is called the *spreading rate* of T.

It follows immediately from our theorem that the transformations considered here are mixing with respect to a ring consisting of continuity sets. We state this as

COROLLARY 1. *Let* $T : [0, 1] \to [0, 1]$ *satisfy the conditions* (i)–(iv) *with return index* α, *let* $\mathcal{B}, \mu$ *be as in the previous sections, and let*

$$\mathcal{R} = \{A \in \mathcal{B} : 0 \notin \overline{A} \text{ and } \lambda(\partial A) = 0\}.$$

Then $\mathcal{R}$ *has the required properties, and* T *is* $\mathcal{R}$-*mixing with spreading rate* $\{\varrho_n\} = \{\Gamma(\alpha)\Gamma(2 - \alpha)w_n(T)\}$. ∎

The defining limiting relation still holds true even if B is assumed to be a set of finite measure.

To prove this, we note that $\{w_n(T)\widehat{T}^n u\}$ is uniformly bounded on $[0, 1]$ if u is a measurable function such that uh is bounded. For, if $|uh| \le c_0$, then

$$|u| \le c_0 \frac{1}{h} \le c_1 v, \quad \text{where} \quad v(x) = x^p, \ x \in [0, 1].$$

Since $v \in \mathcal{M}$, we have $\widehat{T}^n v \le v_n(1)$, and hence

$$w_n(T)|\widehat{T}^n u| \le c_1 w_n(T)v_n(1) \quad (n \ge 0),$$

which proves the claim in view of Lemma 2.

Now, if $A \in \mathcal{R}$ and $B \in \mathcal{B}$ with $\mu(B) < \infty$, the pointwise convergence

$$\varrho_n \widehat{T}^n 1_A \to \mu(A)$$

is dominated on B, and the relation follows by integration. ∎

The relation cannot hold for arbitrary sets A and B of finite measure. In fact, T admits weakly wandering sets of positive measure (cf. [HK]), and for these sets A obviously $\liminf_{n \to \infty} \varrho_n \mu(A \cap T^{-n} A)$ is zero. On the other hand, by means of the following construction we easily get measurable sets A of finite measure such that $\lim_{n \to \infty} \varrho_n \, \mu(A \cap T^{-n} A)$ is infinite.

Let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that the intervals $J_k = [f_0^k(1), f_0^k(1) + \beta_k]$ satisfy

$$J_k \subseteq [f_0^k(1), f_0^{k-1}(1)] \quad \text{and} \quad T(J_{k+1}) \subseteq J_k, \quad k \ge 1.$$

Let A be the union of these intervals. Then

$$A \cap T^{-n} A \supseteq \bigcup_{k > n} J_k, \quad \text{and hence} \quad \mu(A \cap T^{-n} A) \ge \sum_{k > n} \mu(J_k) \ (n \ge 0).$$

Consider, for example, the Lasota–Yorke map, choosing $\beta_1 = 1/2$ and

$$\beta_k = \frac{1}{(k+1)^2(\log(k+1))^{1+\varepsilon}}, \quad k \geq 2, \quad \text{where } 0 < \varepsilon < 1.$$

This sequence satisfies the required conditions, and the estimates

$$k\beta_k \leq \mu(J_k) \leq (k+1)\beta_k \quad (k \geq 1)$$

show that $\mu(A) < \infty$ and $\mu(A \cap T^{-n}A) \geq \kappa/(\log n)^\varepsilon$ $(n \geq 2)$ for some constant $\kappa > 0$. Therefore A has the desired property.

Arc-sine laws: A more direct approach. In [T7] the Dynkin–Lamperti arc-sine laws are studied for pointwise dual ergodic transformations. For maps as considered here, our theorem allows us to give a more direct proof of the central implication. We refer to [T7] for the full version of the limit theorem.

Let $T : [0,1] \to [0,1]$ satisfy the conditions (i)–(iv), and let A be a fixed measurable set of positive measure which is bounded away from 0. For $n \geq 1$ and $x \in \bigcup_{k=0}^{n} T^{-k}A$ let $Z_n(x)$ denote the time of the last visit of the orbit of x to the set A during the time interval $[0,n]$, i.e.,

$$Z_n(x) = \max\{k \in \{0,1,\ldots,n\} : T^k(x) \in A\}.$$

Extend Z_n to $[0,1]$ by $Z_n(x) = 0$ for $x \in [0,1] \setminus \bigcup_{k=0}^{n} T^{-k}A$. As recalled in the following corollary, the sequence $\{(1/n)Z_n\}$ exhibits beautiful limiting behaviour with respect to distributional convergence.

COROLLARY 2. *Let T have return index α with $\alpha < 1$, and let ζ_α denote a random variable with density*

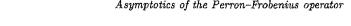$$f_{\zeta_\alpha}(x) = \frac{\sin \pi \alpha}{\pi} \cdot \frac{1}{x^{1-\alpha}(1-x)^\alpha}, \quad x \in (0,1).$$

Then

$$\frac{1}{n}Z_n \to \zeta_\alpha \quad \text{in distribution}$$

with respect to any λ-absolutely continuous probability on $[0,1]$. For $\alpha = 1$ the assertion holds with $\zeta_1 \equiv 1$.

P r o o f. Let first ν be a probability distribution on $[0,1]$ with a Riemann-integrable density u, and let $0 < x < y < 1$. Denoting by φ the first return time of A we have

$$\nu(\{x \leq (1/n)Z_n \leq y\}) = \sum_{nx \leq k \leq ny} \nu(T^{-k}(A \cap \{\varphi > n - k\}))$$

$$= \sum_{nx \leq k \leq ny} \int_0^1 P^k u \, 1_{A \cap \{\varphi > n - k\}} \, d\lambda.$$

Assume $\alpha < 1$. If we use the formula $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin \pi \alpha$ and put $b = \tilde{h}(0)a^{1-\alpha}/p^\alpha$, our theorem takes the form

$$P^n u \sim \frac{\sin \pi \alpha}{\pi b} \cdot \frac{1}{n^{1-\alpha}}h \quad (n \to \infty) \quad \text{uniformly on } [\varepsilon, 1] \text{ for each } \varepsilon > 0.$$

The asymptotic renewal equation technique in [A0], §3.8, yields

$$\sum_{k=0}^{n} \mu(A \cap \{\varphi > k\}) \sim w_n(T) \quad (n \to \infty).$$

Since

$$w_n(T) \sim \frac{b}{1-\alpha}n^{1-\alpha} \quad (n \to \infty)$$

and the sequence $\{\mu(A \cap \{\varphi > n\})\}$ is decreasing, it follows that

$$\mu(A \cap \{\varphi > n\}) \sim b/n^\alpha \quad (n \to \infty).$$

Therefore,

$$\lim_{n \to \infty} \nu(\{x \leq (1/n)Z_n \leq y\}) = \frac{\sin \pi \alpha}{\pi b} \lim_{n \to \infty} \sum_{nx \leq k \leq ny} \frac{1}{k^{1-\alpha}}\mu(A \cap \{\varphi > n - k\})$$

$$= \frac{\sin \pi \alpha}{\pi} \lim_{n \to \infty} \sum_{nx \leq k \leq ny} \frac{1}{k^{1-\alpha}(n-k)^\alpha}$$

$$= \frac{\sin \pi \alpha}{\pi} \int_x^y \frac{dt}{t^{1-\alpha}(1-t)^\alpha}.$$

As every probability density on $[0,1]$ can be approximated in $L_1(\lambda)$ by Riemann-integrable densities, the assertion of the corollary is proved for $\alpha < 1$. The case $\alpha = 1$ can be treated in a similar way. ∎

6. Concluding remark. We conclude our considerations presenting a family of examples which clarify that convexity of both $T_{|I_0}$ and $T_{|I_1}$ is not sufficient for condition (iv) to hold.

Define T_q for $q > -2$ by

$$T_q(x) = \begin{cases} \dfrac{x(1 + 2(q+1)x)}{1+qx}, & x \in [0, 1/2), \\[2ex] \dfrac{x(2x-1)}{1+q(1-x)}, & x \in [1/2, 1]. \end{cases}$$

This family interpolates the Lasota–Yorke map $(q = -1)$ and P. Manneville's example $T(x) = x + x^2 \pmod 1$, which is conjugated to T_0 via the function $\phi(x) = (x + x^2)/2$ $(T_0 = \phi \circ T \circ \phi^{-1})$.

We claim that

$$h(x) = 1/x \text{ is invariant for all } q > -2.$$

Let $q > -2$, $q \neq -1$, be fixed, and let

$$t(x) = \frac{x(1 + 2(q + 1)x)}{1 + qx} \qquad (x \in \mathbb{R}, \; qx \neq -1),$$

so that

$$T_{|I_0}(x) = t(x) \quad \text{and} \quad T_{|I_1}(x) = t\left(-\frac{x}{q + 1}\right).$$

For $y \in [0, 1]$ the solutions of the equation $t(x) = y$ are therefore

$$y_0 = f_0(y) \quad \text{and} \quad y_1 = -\frac{1}{q + 1} f_1(y).$$

Thus, by Vieta's theorem,

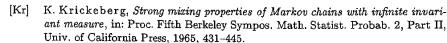$$y_0 y_1 = -\frac{y}{2(q + 1)}, \quad \text{i.e.} \quad f_0(y) f_1(y) = \frac{y}{2},$$

and logarithmic differentiation yields the assertion.

It is readily verified that the conditions (i)–(iv) are satisfied if $q \leq 1$. If $q > 1$, condition (iv) does not hold, although both branches of T_q are convex for these parameters.

As noted at the beginning, the purpose of this paper is to point out by means of examples that the question under consideration indeed leads to the expected results. It may be conjectured that our theorem is true under conditions as general as those in [Z2].

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