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On the existence for the Cauchy–Neumann problem for the Stokes system in the L_p-framework

by

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Abstract. The existence for the Cauchy–Neumann problem for the Stokes system in a bounded domain $\Omega \subset \mathbb{R}^3$ is proved in a class such that the velocity belongs to $W_r^{2,1}(\Omega \times (0,T))$, where $r > 3$. The proof is divided into three steps. First, the existence of solutions is proved in a half-space for vanishing initial data by applying the Marcinkiewicz multiplier theorem. Next, we prove the existence of weak solutions in a bounded domain and then we regularize them. Finally, the problem with nonvanishing initial data is considered.

1. Introduction. In a bounded domain Ω in $\mathbb{R}^3$ with boundary S we consider the initial-boundary value problem for the Stokes system:

$$
\begin{aligned}
&u_t - \nu \Delta u + \nabla p = F, \\
&\operatorname{div} u = G, \\
&\bar{n} \cdot \mathbb{T}(u,p)|_{S_T} = H, \\
&u|_{t=0} = u_0,
\end{aligned}
\tag{1.1}
$$

where $\mathbb{T}(u,p) = \{T(u,p)\}_{i,j=1,2,3} = \{\nu(\partial_i u_j + \partial_j u_i) - p\delta_{ij}\}$ is the stress tensor, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ the velocity vector, $p(x,t)$ the pressure, $\nu > 0$ the constant viscosity coefficient and $\bar{n}$ the exterior normal vector to S.

To solve (1.1) we have to impose the following compatibility conditions on the initial and boundary data:

$$
\begin{aligned}
&\operatorname{div} u_0(x) = G(x,0), \\
&\bar{n} \cdot \mathbb{T}(u_0, p_0)(x)|_S = H(x,0),
\end{aligned}
\tag{1.2}
$$

where p_0 is defined by $\bar{n} \cdot \mathbb{T}(u_0, p_0) \cdot \bar{n} = H(0) \cdot \bar{n}$ on S. From $(1.2)_2$ we get the initial boundary condition $p|_{t=0} = p_0$.

We prove the existence of solutions for system (1.1). The main result of this paper is the following

THEOREM 1. *Let $r > 3$, $S \in W_r^{2-1/r}$, $F \in L_r(\Omega_T)$, $G \in W_r^{1,0}(\Omega_T)$,*

$$\frac{\partial G}{\partial t} - \operatorname{div} F = \operatorname{div} B + A, \qquad A, B \in L_r(\Omega_T),$$

diam supp $A < 2\lambda$, $u_0 \in W_r^{2-2/r}(\Omega)$, $H \in W_r^{1-1/r,1/2-1/(2r)}(S_T)$.

Then there exists a unique solution of problem (1.1)–(1.2) such that:

$$u \in W_r^{2,1}(\Omega_T), \qquad p \in W_r^{1,0}(\Omega_T), \qquad p \in W_r^{1-1/r,1/2-1/(2r)}(S_T),$$

and the following estimate holds:

(1.3) $\|u\|_{W_r^{2,1}(\Omega_T)} + \|p\|_{W_r^{1,0}(\Omega_T)} + \|p\|_{W_r^{1-1/r,1/2-1/(2r)}(S_T)}$

$$\leq C(T)[\|F\|_{L_r(\Omega_T)} + \|G\|_{W_r^{1,0}(\Omega_T)} + \|B\|_{L_r(\Omega_T)}$$

$$+ \lambda\|A\|_{L_r(\Omega_T)} + \|u_0\|_{W_r^{2-2/r}(\Omega)} + \|H\|_{W_r^{1-1/r,1/2-1/(2r)}(S_T)}],$$

where $\Omega_T = \Omega \times (0,T)$, $S_T = S \times (0,T)$ and $C(T)$ is an increasing positive function of T.
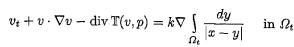
Theorem 1 can be found in [7] without proof.

The aim of the paper is to present a new approach to obtaining L_r-estimates for solutions of evolution equations. We apply our technique to the Stokes system. To prove existence of solutions to (1.1) we use the technique of regularizers. Therefore we consider problem (1.1) locally in a neighbourhood of either an interior point or a boundary point. The boundary neighbourhood problem (1.1) is transformed to a problem in the half-space $x_3 > 0$. By applying the Fourier transform with respect to time and tangent directions, problem (1.1) becomes a system of ordinary differential equations (see (3.2)) whose solutions have the form (3.5).

Solonnikov [9] calculates explicitly the inverse Fourier transform of solutions (3.5) and expresses them in the form of potentials in the half-space $x_3 > 0$. Then he estimates them in suitable norms. In our case we directly estimate the solutions of the ordinary differential equations (3.5) using the Marcinkiewicz multiplier theorem [2, 3].

Moreover, in [10] the existence of solutions to (1.1) is proved in Hölder spaces in a domain Ω which can be either bounded or unbounded.

The result of this paper has an auxiliary character. Our ultimate goal is to prove stability of an equilibrium solution to the free boundary problem for a self-gravitating incompressible fluid. First we have to prove existence of local solutions to the corresponding Navier–Stokes problem (see [4]).

Next we are going to prove existence of global-in-time solutions of the following problem:

$$v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v,p) = k\nabla \int_{\Omega_t} \frac{dy}{|x-y|} \qquad \text{in } \Omega_t,$$

(1.4)
$$\operatorname{div} v = 0 \qquad\qquad\qquad \text{in } \Omega_t,$$

$$\overline{n} \cdot \mathbb{T}(v,p) = -p_0\overline{n} \qquad\qquad \text{on } S_t = \partial\Omega_t,$$

$$v|_{t=0} = v_0, \qquad \Omega_t|_{t=0} = 0,$$

$$v \cdot \overline{n} = \text{velocity of the boundary } S_t,$$

where $v(x,t)$ is the velocity of the fluid, $p(x,t)$ the pressure, $\mathbb{T}(v,p)$ the stress tensor, k the constant of gravity, p_0 the external constant pressure.

By an *equilibrium solution* of problem (1.4) we mean the following solution of (1.4):

$$v = 0, \qquad \Omega_t = B_R,$$

(1.5)
$$\nabla p(x) = k \int_{B_R} \frac{dy}{|x-y|} \qquad \text{in } B_R,$$

$$p = p_0 \qquad\qquad\qquad \text{on } \partial B_R,$$

where B_R is a ball of radius R.

In [5] we prove stability of solutions of (1.5). For this purpose the L_p-approach is much more appropriate, because all considerations are simpler and shorter.

Summarizing, to show stability in [5] we need the result of this paper.

2. Notation. In our considerations we will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$, where $m, n \in \mathbb{R}_+ \cup \{0\}$, $r \geq 1$ and $Q_T = Q \times (0,T)$, with the norm

(2.1) $\|u\|_{W_r^{m,n}(Q_T)}^r = \int_0^T \int_Q |u(x,t)|^r \, dx\, dt$

$$+ \sum_{0 \leq |m'| \leq [|m|]} \int_0^T \int_Q |D_x^{m'} u(x,t)|^r \, dx\, dt$$

$$+ \sum_{|m'| = [|m|]} \int_0^T dt \int_Q \int_Q \frac{|D_x^{m'} u(x,t) - D_x^{m'} u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}} \, dx\, dx'$$

$$+ \sum_{0 \leq |n'| \leq [|n|]} \int_0^T \int_Q |D_t^{n'} u(x,t)|^r \, dx\, dt$$

$$+ \int_Q dx \int_0^T \int_0^T \frac{|D_t^{[n]} u(x,t) - D_t^{[n]}(x,t')|^r}{|t-t'|^{1+r(n-[n])}} \, dt\, dt',$$

where $s = \dim Q$, $[\alpha]$ is the integral part of α, $D_x^l = \partial_{x_1}^{l_1}\ldots\partial_{x_s}^{l_s}$, where $l = (l_1,\ldots,l_s)$ is a multiindex. If Q is a manifold the above norm is defined using a partition of unity.

In the case when $Q_T = \mathbb{R}^s \times \mathbb{R}$ we can apply the Fourier transform and define the Bessel-potential spaces given by the norm

$$(2.2)\qquad \|u\|_{H_r^{m,n}(\mathbb{R}^{s+1})} = \|u\|_{L_r(\mathbb{R}^{s+1})} + \|\mathcal{F}_{t,x}^{-1}[|\xi|^m\widehat{u}(\xi,\xi_0)]\|_{L_r(\mathbb{R}^{s+1})}$$
$$+ \|\mathcal{F}_{t,x}^{-1}[|\xi_0|^n\widehat{u}(\xi,\xi_0)]\|_{L_r(\mathbb{R}^{s+1})},$$

where $\widehat{u}(\xi,\xi_0)$ is the Fourier transform of $u(x,t)$,

$$\widehat{u}(\xi,\xi_0) = \int e^{-i\xi_0 t}\int e^{-i\xi\cdot x}u(x,t)\,dx\,dt \equiv \mathcal{F}_{t,x}[u](\xi,\xi_0),$$

and $\mathcal{F}^{-1}$ the inverse transformation

$$\mathcal{F}_{t,x}^{-1}[\widehat{u}](x,t) = (2\pi)^{-2(s+1)}\int e^{i\xi_0 t}\int e^{i\xi\cdot x}\widehat{u}(\xi,\xi_0)\,d\xi\,d\xi_0$$

where $\xi = (\xi_1,\ldots,\xi_s)$ and $\xi\cdot x = \xi_1 x_1 + \ldots + \xi_s x_s$.

If $m,n \in \mathbb{N}$ then $W_r^{m,n}(Q_T) = H_r^{m,n}(Q_T)$ (see [11]).

In the proof we will use the following results.

THEOREM 2.1 (Marcinkiewicz theorem, see [3]). *Suppose that a function* $\Phi : \mathbb{R}^m \to \mathbb{C}$ *is smooth enough and there exists a constant* $M > 0$ *such that for every* $x \in \mathbb{R}^m$ *we have*

$$\left|x_{j_1}\ldots x_{j_k}\left|\frac{\partial^k\Phi}{\partial x_{j_1}\ldots\partial x_{j_k}}\right|\right| \le M, \quad 0 \le k \le m, \; 1 \le j_1 < \ldots < j_k \le m.$$

Then the operator

$$Pg(x) = (2\pi)^{-m}\int\limits_{\mathbb{R}^m} dy\, e^{ixy}\Phi(y)\int\limits_{\mathbb{R}^m} e^{-iyz}g(z)\,dz$$

is bounded in $L_p(\mathbb{R}^m)$ *and*

$$\|Pg\|_{L_p(\mathbb{R}^m)} \le A_{p,m}M\|g\|_{L_p(\mathbb{R}^m)}.$$

PROPOSITION 2.2 (see [1]). *Let* $u \in W_r^{m,n}(\Omega_T)$. *If*

$$\kappa = \sum_{i=1}^{3}\left(\alpha_i + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{m} + \left(\beta + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{n} < 1$$

then the following estimate holds:

$$\|D_t^\beta D_x^\alpha u\|_{L_q(\Omega_T)} \le \varepsilon\|u\|_{W_r^{m,n}(\Omega_T)} + c(\varepsilon)\|u\|_{L_2(\Omega_T)}$$

where $q \ge r \ge 2$, $\varepsilon \in (0,1)$ *and* $c(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$.

PROPOSITION 2.3. *Let* $r \ge 2$, $u \in W_r^{2,1}(\Omega_T)$ *and* $u|_{t=0} = 0$. *Then*

$$\|u\|_{W_r^{1,1/2}(\Omega_T)} \le c(T^{1/r} + T^{1/2})\|u\|_{W_r^{2,1}(\Omega_T)}.$$

Proof. First we note that

$$(2.3)\qquad \int\limits_\Omega\int\limits_0^T\int\limits_0^T \frac{|u(x,t) - u(x,t')|^r}{|t-t'|^{1+r/2}}\,dt'\,dt\,dx$$
$$\le T^{r/2}\int\limits_\Omega\int\limits_0^T\int\limits_0^T \frac{|u(x,t) - u(x,t')|^r}{|t-t'|^{1+r}}\,dt'\,dt\,dx \le T^{r/2}\|u\|_{W_r^{0,1}(\Omega_T)}^r.$$

Next we have

$$(2.4)\qquad \|u\|_{W_r^{1,0}(\Omega_T)} \le T^{1/r}\|u\|_{L_\infty(0,T;W_r^1(\Omega))} \le cT^{1/r}\|u\|_{W_r^{2,1}(\Omega_T)},$$

where in the last inequality we use $u|_{t=0} \equiv 0$. From (2.3) and (2.4) we get the assertion.

In our considerations we will use well known results such as imbedding theorems for Sobolev spaces. All constants are denoted by c.

3. Problem in the half-space. The first step to solve problem (1.1) is to consider this system in the half-space $x_3 \ge 0$ with $u_0 = 0$ in $(1.1)_4$. To solve it we have to consider two cases. The first is when $F = 0$ in the half-space and the second when the whole space without boundary conditions is examined.

The problem (1.1) in the half-space $x_3 > 0$ with vanishing forces reads

$$(3.1)\qquad\begin{aligned}
&u_t - \nu\Delta u + \nabla p = 0,\\
&\operatorname{div} u = 0,\\
&\nu(u_{3,1} + u_{1,3})|_{x_3=0} = H_1,\\
&\nu(u_{3,2} + u_{2,3})|_{x_3=0} = H_2,\\
&(2\nu u_{3,3} - p)|_{x_3=0} = H_3,\\
&u|_{t=0} = 0.
\end{aligned}$$

To solve (3.1) we apply the Fourier transform

$$v(s,\xi',x_3) = \int\limits_0^\infty dt\, e^{-st}\int\limits_{\mathbb{R}^2} e^{-ix'\xi'}u(t,x)\,dx',$$

$$q(s,\xi',x_3) = \int\limits_0^\infty dt\, e^{-st}\int\limits_{\mathbb{R}^2} e^{-ix'\xi'}p(t,x)\,dx',$$

where $s = i\xi_0$ and $\xi' = (\xi_1,\xi_2), x' = (x_1,x_2)$.

Assuming that $H \in W_r^{1-1/r,1/2-1/(2r)}(x_3 = 0)$, $r > 3$ and $H|_{t=0} = 0$ we can extend H by zero for $t < 0$. Therefore we can look for solutions of (3.1) vanishing for $t < 0$.

After the transformation system (3.1) takes the form

$$\nu\left(-\frac{d^2}{dx_3^2}+r^2\right)v_1 + i\xi_1 q = 0,$$

$$\nu\left(-\frac{d^2}{dx_3^2}+r^2\right)v_2 + i\xi_2 q = 0,$$

$$\nu\left(-\frac{d^2}{dx_3^2}+r^2\right)v_3 + \frac{dq}{dx_3} = 0,$$

$$i\xi_1 v_1 + i\xi_2 v_2 + \frac{dv_3}{dx_3} = 0,$$

(3.2)

$$\nu\left(\frac{dv_1}{dx_3}+i\xi_1 v_3\right)\Big|_{x_3=0} = h_1,$$

$$\nu\left(\frac{dv_2}{dx_3}+i\xi_2 v_3\right)\Big|_{x_3=0} = h_2,$$

$$\left(2\nu\frac{dv_3}{dx_3}-q\right)\Big|_{x_3=0} = h_3,$$

$$v \to 0,\ q \to 0 \quad \text{as } x_3 \to \infty,$$

where $r^2 = s/\nu + |\xi'|^2$, $\arg r \in (-\pi/4, \pi/4)$. Solving $(3.2)_{1,2,3,4}$ with $(3.2)_8$ we get (see also [6])

(3.3)
$$v = \Phi(s,\xi')e^{-rx_3} + \phi(s,\xi')(i\xi_1, i\xi_2, -|\xi'|)e^{-|\xi'|x_3},$$
$$q = -s\phi(s,\xi')e^{-|\xi'|x_3},$$

where $\Phi = (\Phi_1, \Phi_2, (i\xi_1\Phi_1 + i\xi_2\Phi_2)/r)$. From boundary conditions $(3.2)_{5,6,7}$ we obtain

(3.4)
$$\nu[-r\Phi_1 - 2i\xi_1|\xi'|\phi + (i\xi_1/r)(i\xi_1\Phi_1 + i\xi_2\Phi_2)] = h_1,$$
$$\nu[-r\Phi_2 - 2i\xi_1|\xi'|\phi + (i\xi_2/r)(i\xi_1\Phi_1 + i\xi_2\Phi_2)] = h_2,$$
$$s\phi + 2\nu(|\xi'|^2\phi - i\xi_1\Phi_1 - i\xi_2\Phi_2) = h_3.$$

Solving system (3.4) and using (3.3) we get a solution of (3.2):

(3.5)
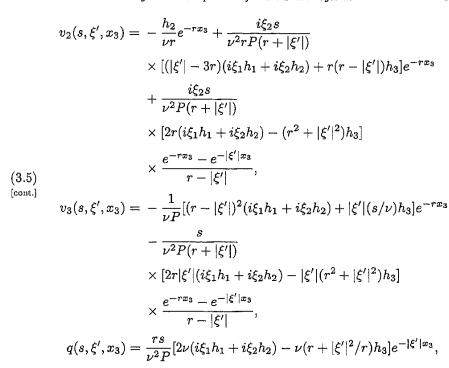$$v_1(s,\xi',x_3) = -\frac{h_1}{\nu r}e^{-rx_3} + \frac{i\xi_1 s}{\nu^2 r P(r+|\xi'|)}$$
$$\times [(|\xi'|-3r)(i\xi_1 h_1 + i\xi_2 h_2) + r(r-|\xi'|)h_3]e^{-rx_3}$$
$$+ \frac{i\xi_1 s}{\nu^2 P(r+|\xi'|)}$$
$$\times [2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3]$$
$$\times \frac{e^{-rx_3} - e^{-|\xi'|x_3}}{r-|\xi'|},$$

$$v_2(s,\xi',x_3) = -\frac{h_2}{\nu r}e^{-rx_3} + \frac{i\xi_2 s}{\nu^2 r P(r+|\xi'|)}$$
$$\times [(|\xi'|-3r)(i\xi_1 h_1 + i\xi_2 h_2) + r(r-|\xi'|)h_3]e^{-rx_3}$$
$$+ \frac{i\xi_2 s}{\nu^2 P(r+|\xi'|)}$$
$$\times [2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3]$$
$$\times \frac{e^{-rx_3} - e^{-|\xi'|x_3}}{r-|\xi'|},$$

(3.5)
[cont.]

$$v_3(s,\xi',x_3) = -\frac{1}{\nu P}[(r-|\xi'|)^2(i\xi_1 h_1 + i\xi_2 h_2) + |\xi'|(s/\nu)h_3]e^{-rx_3}$$
$$- \frac{s}{\nu^2 P(r+|\xi'|)}$$
$$\times [2r|\xi'|(i\xi_1 h_1 + i\xi_2 h_2) - |\xi'|(r^2 + |\xi'|^2)h_3]$$
$$\times \frac{e^{-rx_3} - e^{-|\xi'|x_3}}{r-|\xi'|},$$

$$q(s,\xi',x_3) = \frac{rs}{\nu^2 P}[2\nu(i\xi_1 h_1 + i\xi_2 h_2) - \nu(r+|\xi'|^2/r)h_3]e^{-|\xi'|x_3},$$

where

(3.6)
$$P = (r^2 + |\xi'|^2)^2 - 4r|\xi'|^3.$$

LEMMA 3.1. *For P defined by (3.6) the following estimates hold:*

$$|P| \ge \frac{2}{\nu}|s| \cdot |\xi'|^2, \qquad |s|^2 \le 3\nu^2|P|.$$

The proof can be found in [8].

LEMMA 3.2. *Let $h \in W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)$. Then there exists a solution of problem (3.1) such that $u(t,x) \in W_{r,\text{loc}}^{2,1}(\mathbb{D}^4)$ and $p(t,x) \in W_{r,\text{loc}}^{1,0}(\mathbb{D}^4)$, and the following estimates are valid:*

(3.7)
$$\|D_x^2 u\|_{L_r(\mathbb{D}^4)} + \|D_t u\|_{L_r(\mathbb{D}^4)} \le c\|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)},$$
$$\|D_x p\|_{L_r(\mathbb{D}^4)} \le c\|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)},$$
$$\|u\|_{W_r^{2-1/r,1-1/(2r)}(\mathbb{R}^3)} \le c\|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)},$$
$$\|p\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)} \le c\|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)},$$

where $\mathbb{D}^4 = \mathbb{R}_{x'}^2 \times [0,\infty)_{x_3} \times \mathbb{R}_t$.

Proof. First we take the pressure $(3.5)_4$ with $h_2 = h_3 = 0$. Then

$$(3.8) \qquad q(s, \xi', x_3) = 2i\frac{\xi_1 rs}{\nu P}e^{-|\xi'|x_3}h_1(s, \xi').$$

From Lemma 3.1 one can check that for some M,

$$|s^{l_1}\xi_1^{l_2}\xi_2^{l_3}|\left|\frac{\partial^k}{\partial s^{l_1}\partial\xi_1^{l_2}\partial\xi_2^{l_3}}2i\frac{\xi_1 rs}{\nu P}\right| < M,$$

where $l_i = 0$ or 1, so that the condition of Theorem 2.1 is satisfied. To see this we note that

$$\left|Ps\partial_s\frac{1}{P}\right| = \left|s\frac{P_{,s}}{P}\right| = \left|s\frac{2(r^2 + |\xi'|^2) - 2\frac{1}{r}|\xi'|^3}{P}\right| < M,$$

$$\left|P|\xi'|\partial_{|\xi'|}\frac{1}{P}\right| = \left||\xi'|\frac{P_{|\xi'|}}{P}\right| = \left||\xi'|\frac{8|\xi'|(r^2 + |\xi'|^2) - 4\frac{1}{r}|\xi'|^4 - 12r|\xi'|^2}{P}\right|$$

$$= \left|\frac{8|\xi'|^2 sr + 4s|\xi'|^3 - 4|\xi'|s^2}{rP} - \frac{4|\xi'|P}{rP}\right| < M,$$

which gives, by Lemma 3.1, the desired estimate.

We consider the norm of derivatives of the pressure

$$(3.10) \qquad D_{x_1}p(t, x) = c\int_{\mathbb{R}} e^{st}\int_{\mathbb{R}^2}e^{ix'\xi'}2i\frac{\xi_1 rs}{\nu P}\xi_1 e^{-|\xi'|x_3}h_1(s, \xi')\,d\xi_0\,d\xi.$$

We know that

$$K^1 = \mathcal{F}_{t,x'}^{-1}[e^{-|\xi'|x_3}] = \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

and

$$(3.11) \qquad \mathcal{F}_{t,x'}^{-1}[\xi_1 e^{-|\xi'|x_3}] = \partial_{x_1}K^1 = \frac{x_3 x_1}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}.$$

We also have

$$(3.12) \qquad \int K_{,x_1}^1\,dx' = 0.$$

Hence to estimate $\|D_{x'}p\|_{L_r(\mathbb{D}^4)}$ we have to show that

$$(3.13) \qquad I_1(t, x) = \int_{\mathbb{R}^2}dy'\,K_{,y_1}^1(y', x_3)[h(t, x' - y') - h(t, x')]$$

belongs to $L_r(\mathbb{D}^4)$. By (3.12) we have

$$\mathcal{F}_{t,x'}^{-1}[\xi_1 e^{-|\xi'|x_3}h(s, \xi')] = I_1.$$

Hence we have

$$(3.14) \qquad \|I_1\|_{L_r(\mathbb{R}^3)} \leq 3\int_{\mathbb{R}^2}\frac{|y_1|x_3}{(y'^2 + x_3^2)^{5/2}}N(y')\,dy',$$

where

$$N(y') = \|h(x' - y', t) - h(x', t)\|_{L_r(\mathbb{R}_t \times \mathbb{R}_{x'}^2)}.$$

Applying the Hölder inequality to (3.14) we get

$$(3.15) \qquad \|I_1\|_{L_r(\mathbb{D}^4)}^r$$

$$\leq 3\int_0^\infty dx_3\int_{\mathbb{R}^2}dy'\,\frac{|y_1|^r x_3}{(y'^2 + x_3^2)^{r+3/2}}N^r(y')\left(\int_{\mathbb{R}^2}\frac{x_3\,dy'}{(y'^2 + x_3)^{3/2}}\right)^{r/r^*}$$

$$\leq c\int_0^\infty dx_3\int_{\mathbb{R}^2}dy'\,\frac{y_1^r x_3}{(y'^2 + x_3^2)^{r+3/2}}|y'|^\alpha\frac{N^r(y')}{|y'|^\alpha},$$

where we have used the fact that $\int x_3(y'^2 + x_3^2)^{-3/2}\,dy' = 2\pi$.

We examine the integral

$$(3.16) \qquad J_1 = \int_0^\infty dx_3\,\frac{|y'|^r x_3}{(y'^2 + x_3^2)^{r+3/2}}|y'|^\alpha.$$

Taking $w = x_3/|y'|$ we get

$$J_1 = \int_0^\infty dw\,\frac{w|y'|^{r+2+\alpha}}{|y'|^{2r+3}(1 + w^2)^{r+3/2}},$$

so J_1 does not depend on $|y'|$ if

$$r + 2 + \alpha = 2r + 3$$

or

$$(3.17) \qquad \alpha = r + 1$$

and is finite when

$$(3.18) \qquad 2r + 3 - 1 > 1,$$

which holds for our r. Therefore from (3.15) we get

$$\|I_1\|_{L_r(\mathbb{D}^4)}^r \leq c\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\frac{|h(t, x' - y') - h(t, x')|^r}{|y'|^\alpha}\,dx'\,dy'\,dt,$$

so by definition (2.1), the fact that $h \in W_r^{1-1/r, 1/2-1/(2r)}(\mathbb{R}^2 \times \mathbb{R})$ (note also that $2 + r(1 - 1/r) = 1 + r$) and (3.17) we have

$$(3.19) \qquad \|I_1\|_{L_r(\mathbb{D}^4)} \leq c\|h\|_{W_r^{1-1/r, 0}(\mathbb{R}^3)}.$$

Similar considerations can be applied in the cases $h_2 \neq 0$ and $h_3 \neq 0$. When we consider the derivation with respect to x_3 we use $\partial_{x_3}(e^{-|\xi'|x_3}) = -|\xi'|e^{-|\xi'|x_3}$, which reduces our considerations to cases with x'. Finally, we get

$$(3.20) \qquad \|D_x p\|_{L_r(\mathbb{D}^4)} \leq c\|h\|_{W_r^{1-1/r, 0}(\mathbb{R}^3)},$$

which gives $(3.7)_2$.

Now we take the velocity. First we assume that $h_2 = h_3 = 0$ and we consider the terms from $(3.5)_{1,2,3}$ with e^{-rx_3}. We take the first term; the others can be estimated in the same way. We consider

$$u_1(t,x) = \int e^{st} \int e^{ix'\xi'} \left[-\frac{1}{\nu r} e^{-rx_3} h_1(s,\xi') \right] d\xi_0 \, d\xi'$$

and

$$(3.21) \qquad D_{x_1}^2 u_1(t,x) = \int e^{st} \int e^{ix'\xi'} \left[-\frac{\xi_1}{\nu r} \xi_1 e^{-rx_3} h_1(s,\xi') \right] d\xi_0 \, d\xi'.$$

We have

$$K^2 = \mathcal{F}_{t,x'}^{-1}[e^{-rx_3}] = -\frac{x_3}{\nu^{5/2} t^{5/2}} \exp\left(-\frac{|x|^2}{4\nu t} \right),$$

$$(3.22) \qquad \mathcal{F}_{t,x'}^{-1}[\xi_1 e^{-rx_3}] = K_{,x_1}^2 = \frac{1}{2\nu^{7/2} t^{7/2}} x_3 x_1 \exp\left(-\frac{|x|^2}{4\nu t} \right),$$

and

$$(3.23) \qquad \int K_{,x_1}^2 \, dx' = 0.$$

Since $|s^{l_1} \xi_1^{l_2} \xi_2^{l_3}| \cdot |\partial_s^{l_1} \partial_{\xi_1}^{l_2} \partial_{\xi_2}^{l_3} g(\xi_1/r)| \le c$, by Theorem 2.1, acting as in the pressure case we have to consider only

$$(3.24) \quad I_2(t,x) = \int_{\mathbb{R}^2} dy' \int_{\mathbb{R}} dt' K_{,y_1}^2(t',y',x_3)[h(x'-y', t-t') - h(x', t-t')].$$

In view of the Minkowski inequality, we get

$$(3.25) \quad \|I_2\|_{L_r(\mathbb{R}^3)} \le \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dy' \, |K_{,y_1}^2| N(y')$$

$$\le c x_3 \left(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dy' \frac{1}{t^{(7/2-m)r}} \exp\left(-\frac{x_3^2}{4\nu t} \right) \exp\left(-\frac{|y'|^2}{4\nu t} \right) N^r(y') \right)^{1/r}$$
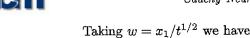
$$\times \left(\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \frac{1}{t^{mr^*}} x_1^{r^*} \exp\left(-\frac{|x|^2}{4\nu t} \right) \right)^{1/r^*},$$

where m will be specified later, $r^* = r/(r-1)$,

$$N(y') = \|h(x'-y', t-t') - h(x', t-t')\|_{L_r(\mathbb{R}_{t'} \times \mathbb{R}_{x'}^2)}$$

and for the last integral in (3.25) we have

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx' \frac{1}{t^{mr^*}} x_1^{r^*} \exp\left(-\frac{|x|^2}{4\nu t} \right)$$

$$\le c \int_{\mathbb{R}} dt \int_{\mathbb{R}} dx_1 t^{1/2} \frac{1}{t^{mr^*}} x_1^{r^*} \exp\left(-\frac{x_1^2}{4\nu t} \right) \exp\left(-\frac{x_3^2}{4\nu t} \right).$$

Taking $w = x_1/t^{1/2}$ we have

$$\int_0^\infty dt \frac{1}{t^{mr^*}} t^{1/2} \int_{\mathbb{R}} dw \, t^{1/2} w^{r^*} t^{r^*/2} \exp\left(-\frac{w^2}{4\nu} \right) \exp\left(-\frac{x_3^2}{4\nu t} \right)$$

$$= c \int_0^\infty dt \frac{t^{r^*/2+1}}{t^{r^*m}} \exp\left(-\frac{x_3^2}{4\nu t} \right).$$

Taking $t = |x_3|^2/w$ we get

$$\int_0^\infty dw \, x_3^2 w^{-2} \left(\frac{x_3^2}{w} \right)^{r^*/2+1-r^*m} e^{-w/(4\nu)}$$

$$= x_3^{2+r^*+2-2r^*m} \int_0^\infty dw \frac{1}{w^{2+r^*/2+1-r^*m}} e^{-w/(4\nu)}.$$

Thus we have to assume

$$2 + r^*/2 + 1 - r^*m < 1,$$

so

$$(3.26) \qquad m > \frac{2}{r^*} + \frac{1}{2} = \frac{5}{2} - \frac{2}{r}.$$

Hence if m satisfies (3.26), then

$$(3.27) \qquad \left(\int_{\mathbb{R}_+ \times \mathbb{R}^2} dt \, dx' \frac{1}{t^{mr^*}} x_1^{r^*} \exp\left(-\frac{|x|^2}{4\nu t} \right) \right)^{1/r^*} \le c x_3^{(4+r^*-2r^*m)/r^*}.$$

We have

$$[(4+r^*-2r^*m)/r^*]r = [4/r^* + 1 - 2m]r = [5 - 4/r - 2m]r = 5r - 2mr - 4.$$

Applying (3.25) and (3.27) we get

$$(3.28) \quad \|I_2\|_{L_r(\mathbb{D}^4)}^r$$

$$\le c \int_0^\infty dx_3 \, x_3^{r+5r-2mr-4} \int_0^\infty dt \int_{\mathbb{R}^2} dy' \frac{1}{t^{(7/2-m)r}} \exp\left(-\frac{x_3^2}{4\nu t} \right)$$

$$\times \exp\left(-\frac{|y'|^2}{4\nu t} \right) |y'|^\alpha \frac{N^r(y')}{|y'|^\alpha}.$$

Hence we have to consider

$$(3.29) \quad \int_0^\infty dx_3 \, x_3^{r+5r-2mr-4}$$

$$\times \int_0^\infty dt \frac{1}{t^{(7/2-m)r}} \exp\left(-\frac{x_3^2}{4\nu t} \right) \exp\left(-\frac{|y'|^2}{4\nu t} \right) |y'|^\alpha$$

$$= c \int_0^\infty dt \frac{t^{3r-mr-2+1/2}}{t^{(7/2-m)r}} \exp\left(-\frac{|y'|^2}{t} \right) |y'|^\alpha.$$

Taking $t = |y'|^2/w$ we see that the above integral is equal to

$$\int_0^\infty dw\, \frac{|y'|^2}{w^2} \left(\frac{|y'|^2}{w}\right)^{3r-mr-3/2-(7/2-m)r} e^{-w/(4\nu)} |y'|^\alpha$$

$$= \int_0^\infty dw\, \frac{1}{w^{2+3r-mr-3/2-(7/2-m)r}} |y'|^{2+6r-2mr-3-(7-2m)r+\alpha} e^{-w/(4\nu)}.$$

The conditions for independence from $|y'|$ and for finiteness of (3.29) are

$$2 + 6r - 2mr - 3 - (7 - 2m)r + \alpha = 0$$

or

$$(3.30) \qquad\qquad \alpha = 1 + r$$

and

$$2 + 3r - mr - 3/2 - (7/2 - m)r < 1,$$
$$-r < 1,$$

which is true for m satisfying (3.26), and this implies boundedness of (3.29). Thus from (3.28), (3.30) and definition (2.1) we get

$$\|I_2\|_{L_r(\mathbb{D}^4)} \le c\|h\|_{W_r^{1-1/r,0}(\mathbb{R}^3)}$$

and in the same way as in the case of the pressure we conclude that

$$(3.31) \qquad\qquad \|D_{x'}^2 u_1\|_{L_r(\mathbb{D}^4)} \le c\|h\|_{W_r^{1-1/r,0}(\mathbb{R}^3)}.$$

To show regularity with respect to t we consider

$$D_t u_1(t,x) = c\partial_t \int e^{st} \int e^{ix'\cdot\xi'} \frac{e^{-rx_3}}{r} h_1(s,\xi')\, d\xi_0\, d\xi'.$$

Using

$$\mathcal{F}_{t,x'}^{-1}\left[\frac{e^{-rx_3}}{r}\right] = c\frac{1}{t^{3/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right)$$

and repeating the considerations for $D_{x'}^2 u_1$ we get

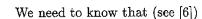$$(3.32) \qquad\qquad \|D_t u_1\|_{L_r(\mathbb{D}^4)} \le c\|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^3)}.$$

Now we take only one term with $\frac{e^{-rx_3}-e^{-|\xi'|x_3}}{r-|\xi'|}$ because others can be estimated similarly using Lemma 3.1. Choosing $h_2 = h_3 = 0$ we consider

$$u_2(t,x) = \mathcal{F}_{t,x'}^{-1}\left[\frac{i\xi_1 s}{\nu^2 P(r+|\xi'|)} [2ri\xi_1 h_1] \frac{e^{-rx_3}-e^{-|\xi'|x_3}}{r-|\xi'|}\right].$$

By Lemma 3.1,

$$\frac{i\xi_1 s}{\nu^2 P(r+|\xi'|)} 2ri\xi_1$$

satisfies the conditions of Theorem 2.1.

We need to know that (see [6])

$$K^3 = \mathcal{F}_{t,x_1,x_2}^{-1}\left[\frac{e^{-rx_3}-e^{-|\xi'|x_3}}{r-|\xi'|}\right] = -\mathcal{F}_{t,x_1,x_2}^{-1}\left[\int_0^{x_3} e^{-|\xi'|(x_3-y)} e^{-ry}\, dy\right]$$

$$= \mathcal{F}_{x_1,x_2}^{-1}\left[\int_0^{x_3} \frac{y}{t^{3/2}} e^{-y^2/(4\nu t)} e^{-|\xi'|(x_3-y)-|\xi'|^2 t}\, dy\right]$$

$$= \int_0^{x_3} \frac{y}{t^{5/2}} e^{-y^2/(4\nu t)} \int da\, \frac{x_3-y}{((x_1-a_1)^2+(x_2-a_2)^2+(x_3+z)^2)^{3/2}} e^{-|a|^2/(4\nu t)}.$$

We have

$$(3.33) \qquad D_{x_1}^2 u_2(x,t) = \mathcal{F}_{t,x'}^{-1}\left[\frac{i\xi_1 s}{\nu^2 P(r+|\xi'|)} [2ri\xi_1 h_1]\xi_1^2 \frac{e^{-rx_3}-e^{-|\xi'|x_3}}{r-|\xi'|}\right]$$

and (see [6])

$$(3.34) \qquad\qquad |D_{x'}^2 K^3| \le \frac{C}{t^{1/2}(x^2+t)^2},$$

and

$$(3.35) \qquad\qquad \int_{\mathbb{R}^2} D_{,x_i x_j}^2 K^3\, dx' = 0.$$

Therefore, similarly to the case of u_1, we introduce

$$(3.36) \quad I_3(t,x)$$
$$= \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} dy'\, D_{y_i y_j}^2 K^3(t',y',x_3)[h(x'-y',t-t') - h(x',t-t')]$$

and get

$$(3.37) \quad \|I_3\|_{L_r(\mathbb{R}^3)} \le \int_0^\infty dt \int_{\mathbb{R}^2} \frac{dy'}{t^{1/2}(y'^2+x_3^2+t)^2} N(y')$$
$$\le \left(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dy'\, \frac{1}{t^{1/2}(y'^2+x_3^2+t)^{(2-m)r}} N^r(y')\right)^{1/r}$$
$$\times \left(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dy'\, \frac{1}{t^{1/2}(y'^2+x_3^2+t)^{mr^*}}\right)^{1/r^*}.$$

For the second integral in (3.37) we have

$$\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dy'\, \frac{1}{t^{1/2}(y'^2+x_3^2+t)^{mr^*}} \le c\int_0^\infty \frac{dt}{t^{1/2}(x_3^2+t)^{mr^*-1}}.$$

If we take $w = t/x_3^2$, the last integral becomes

$$\int_0^\infty \frac{dw\, x_3^2}{x_3 w^{1/2}(1+w)^{mr^*-1} x_3^{2mr^*-2}}.$$

Hence when choosing

$$mr^* - 1 > \frac{1}{2},$$

$$(3.38) \qquad m > \frac{3}{2}\frac{r-1}{r},$$

we get

$$(3.39) \qquad \left(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dy' \frac{1}{t^{1/2}(y'^2 + x_3^2 + t)^{mr^*}}\right)^{1/r^*} \le c x_3^{(3-2mr^*)}/r^*.$$

Thus by (3.37) and (3.39) we have

$$(3.40) \quad \|I_3\|_{L_r(\mathbb{D}^4)}^r \le c \int_0^\infty dx_3\, x_3^{(3-2mr^*)(r-1)}$$

$$\times \int_{\mathbb{R}_+ \times \mathbb{R}^2} dt\, dy' \frac{1}{t^{1/2}(x^2+t)^{(2-m)r}} |y'|^\alpha \frac{N^r(y')}{|y'|^\alpha}.$$

We consider

$$(3.41) \qquad \int_0^\infty dx_3\, x_3^{(3-2mr^*)(r-1)} \int_{\mathbb{R}_+} dt \frac{1}{t^{1/2}(x^2+t)^{(2-m)r}} |y'|^\alpha,$$

so taking $w = t/x^2$ we get (for simplicity here we write $x' = y'$)

$$\int_0^\infty dx_3 \int_0^\infty dw\, x^2 \frac{1}{|x| w^{1/2}(1+w)^{(2-m)r}|x|^{2(2-m)r}} x_3^{(3-2mr^*)(r-1)}|x'|^\alpha.$$

Assuming that $(2-m)r > 1/2$ or

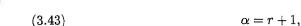$$(3.42) \qquad m < 2 - \frac{1}{2r},$$

we have to examine

$$\int_0^\infty dx_3\, x_3^{(3-2mr^*)(r-1)} |x'|^\alpha |x|^{1-(4-2m)r},$$

which after taking $w = x_3/|x'|$ is

$$\int dw\, |x'| \frac{w^{3r-3-2mr}|x'|^{3r-3-2mr+\alpha+1-(4-2m)r}}{(1+w^2)^{((4-2m)r-1)/2}}.$$

To have the integral independent of $|x'|$ we need

$$1 + 3r - 3 - 2mr + \alpha + 1 - (4-2m)r = 0,$$

$$(3.43) \qquad \alpha = r + 1,$$

and to have it finite we have to impose (regularity "at 0")

$$3r - 3 - 2mr > -1$$

or

$$(3.44) \qquad m < \frac{3}{2} - \frac{1}{r},$$

and ("at ∞")

$$3r - 3 - 2mr - ((4-2m)r - 1) = -2 - r < -1.$$

Since inequalities (3.38), (3.42) and (3.44) can be satisfied for some m, (3.41) is bounded and (3.40), in view of (3.43), is estimated by

$$(3.45) \qquad \|I_3\|_{L_r(\mathbb{D}^4)} \le c\|H\|_{W_r^{1-1/r, 1/2-1/(2r)}(\mathbb{R}^3)}$$

and as before we get

$$(3.46) \qquad \|D_{x'}^2 u_2\|_{L_r(\mathbb{D}^4)} \le c\|H\|_{W_r^{1-1/r, 1/2-1/(2r)}(\mathbb{R}^3)}.$$

Similarly one can show that

$$(3.47) \qquad \|D_t u\|_{L_r(\mathbb{D}^4)} \le c\|H\|_{W_r^{1-1/r, 1/2-1/(2r)}(\mathbb{R}^3)}$$

and from (3.47) together with (3.20), (3.31), (3.32), (3.46) and from equation $(3.1)_1$ we get estimates for $\|D_{x_3}^2 u\|_{L_r}$. Hence we have

$$(3.48) \qquad u \in W_{r,\text{loc}}^{2,1}(\mathbb{D}^4),$$

which gives $(3.7)_1$ and $(3.7)_3$. From the boundary conditions we get $(3.7)_4$. Lemma 3.2 is proved.

The next step towards solving (1.1) in the half-space is to solve the following problem in the whole space:

$$(3.49) \qquad \begin{aligned} &u_{,t} - \nu\Delta u + \nabla p = f, \\ &\operatorname{div} u = 0, \\ &u|_{t=0} = 0. \end{aligned}$$

LEMMA 3.3. *Let $f \in L_r(\mathbb{R}^4)$ then there exists a unique solution of (3.49) such that $u \in W_{r,\text{loc}}^{2,1}(\mathbb{R}^4)$ and $p \in W_{r,\text{loc}}^{1,0}(\mathbb{R}^4)$ and the following estimate holds:*

$$(3.50) \qquad \|D_t u\|_{L_r(\mathbb{R}^4)} + \|D_x^2 u\|_{L_r(\mathbb{R}^4)} + \|D_x p\|_{L_r(\mathbb{R}^4)} \le c\|f\|_{L_r(\mathbb{R}^4)}.$$

Moreover if $f \in L_r^{\text{div}}(\mathbb{R}^3 \times \mathbb{R})$ then $p \equiv 0$; where

$$L_r^{\text{div}}(\mathbb{R}^3 \times \mathbb{R}) = \overline{\{f \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}) : \operatorname{div} f = 0\}}^{\|\cdot\|_{L_r}}.$$

Proof. After taking the Fourier transform system (3.49) reads

$$(s + \nu|\xi|^2)v_a + i\xi_a q = f_a, \qquad \sum_a \xi_a v_a = 0$$

or

$$AX = \begin{bmatrix} \nu r^2 & 0 & 0 & i\xi_1 \\ 0 & \nu r^2 & 0 & i\xi_2 \\ 0 & 0 & \nu r^2 & i\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ q \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \end{bmatrix} = F.$$

Thus the solution is $X = A^{-1}F$, where

$$A^{-1} = \begin{bmatrix} \frac{\xi_2^2+\xi_3^2}{\nu r^2|\xi|^2} & -\frac{\xi_1\xi_2}{\nu r^2|\xi|^2} & -\frac{\xi_1\xi_3}{\nu r^2|\xi|^2} & \frac{\xi_1}{|\xi|^2} \\ -\frac{\xi_1\xi_2}{\nu r^2|\xi|^2} & \frac{\xi_1^2+\xi_3^2}{\nu r^2|\xi|^2} & -\frac{\xi_2\xi_3}{\nu r^2|\xi|^2} & \frac{\xi_2}{|\xi|^2} \\ -\frac{\xi_1\xi_3}{\nu r^2|\xi|^2} & -\frac{\xi_2\xi_3}{\nu r^2|\xi|^2} & \frac{\xi_1^2+\xi_2^2}{\nu r^2|\xi|^2} & \frac{\xi_3}{|\xi|^2} \\ -\frac{i\xi_1}{|\xi|^2} & -\frac{i\xi_2}{|\xi|^2} & -\frac{i\xi_3}{|\xi|^2} & \frac{i\nu r^2}{|\xi|^2} \end{bmatrix}.$$

It is seen that estimate (3.50) is true and if $f \in L_r^{\mathrm{div}}$ (div $f = 0$) then $p \equiv 0$. From $(3.49)_3$ we get the assertion.

Using Lemmas 3.2 and 3.3 we deduce the main result of this section:

LEMMA 3.4. *Let $r \geq 2$, $f \in L_r(\mathbb{D}^4)$, $g \in W_r^{1,0}(\mathbb{D}^4)$, $g_t - \mathrm{div}\, f = \mathrm{div}\, B + A$, $A, B \in L_r(\mathbb{D}^4)$, $\mathrm{diam\, supp}\, A < \lambda$ and $h \in W_r^{1-1/r,1/2-1/(2r)}(x_3 = 0)$. Then there exists a unique solution of the system*

$$u_{,t} - \nu\Delta u + \nabla p = f,$$
$$\mathrm{div}\, u = g,$$
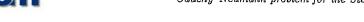$$\bar{e}_3 \cdot \mathbb{T}(u,p) = h,$$
$$u|_{t=0} = 0,$$

such that $u \in W_r^{2,1}(\mathbb{D}_T^4)$, $p \in W_r^{1,0}(\mathbb{D}_T^4)$ and $p \in W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_T^3)$ and the following estimate holds:

$$
\begin{aligned}
(3.52) \quad \|D_t u\|_{L_r(\mathbb{D}_T^4)} &+ \|D_x^2 u\|_{L_r(\mathbb{D}_T^4)} \\
&+ \|D_x p\|_{L_r(\mathbb{D}_T^4)} + \|p\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_{x'}^2 \times [0,T])} \\
&\leq c(T)[\|f\|_{L_r(\mathbb{D}_T^4)} + \|B\|_{L_r(\mathbb{D}_T^4)} \\
&\quad + \lambda\|A\|_{L_r(\mathbb{D}_T^4)} + \|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_T^3)}],
\end{aligned}
$$

where $\mathbb{D}^4 = \mathbb{R}_{x'}^2 \times [0,\infty)_{x_3} \times [0,T]_t$.

Proof. First we consider the following problem:

$$
\begin{aligned}
(3.53) \quad & \Delta w = g, \\
& w|_{x_3=0} = 0, \\
& w \to 0 \quad \text{as } x_3 \to \infty.
\end{aligned}
$$

We easily see that a solution of (3.53) exists and

$$(3.54) \qquad \|\nabla w\|_{W_r^{2,0}(\mathbb{D}^4)} \leq c\|g\|_{W_r^{1,0}(\mathbb{D}^4)}.$$

Next, we differentiate $(3.53)_1$ with respect to t to get

$$(3.55) \qquad \Delta w_t = \mathrm{div}(f+B) + A$$

with the same boundary conditions. To estimate $\|\nabla w_t\|_{L_r}$ we look for ∇w_t in the form

$$(3.56) \qquad \nabla w_t = \nabla w_t^1 + \nabla w_t^2,$$

where

$$(3.57) \qquad \Delta w_t^1 = \mathrm{div}(f+B),$$
$$(3.58) \qquad \Delta w_t^2 = A,$$

and for (3.57) and (3.58) we have the same boundary conditions as in (3.53). From (3.57) we get

$$(3.59) \qquad \|\nabla w_t^1\|_{L_r(\mathbb{D}^4)} \leq c(\|f\|_{L_r(\mathbb{D}^4)} + \|B\|_{L_r(\mathbb{D}^4)})$$

and from (3.58),

$$(3.60) \qquad \|\nabla w_t^2\|_{L_r(\mathbb{D}^4)} \leq c\lambda\|A\|_{L_r(\mathbb{D}^4)}.$$

Hence by (3.54), (3.56), (3.59) and (3.60) we obtain

$$
\begin{aligned}
(3.61) \quad \|\nabla w\|_{W_r^{2,1}(\mathbb{D}^4)} &\leq c(\|g\|_{W_r^{1,0}(\mathbb{D}^4)} \\
&\quad + \|f\|_{L_r(\mathbb{D}^4)} + \|B\|_{L_r(\mathbb{D}^4)} + \lambda\|A\|_{L_r(\mathbb{D}^4)}).
\end{aligned}
$$

If we take $u = v + \nabla w$, system (3.53) reduces to

$$
\begin{aligned}
(3.62) \quad & v_t - \nu\Delta v + \nabla p = f', \\
& \mathrm{div}\, v = 0, \\
& \bar{e}_3 \cdot \mathbb{T}(v,p) = h', \\
& v|_{t=0} = 0,
\end{aligned}
$$

where $f' = f - \nabla w + \nu\Delta\nabla w$, $h' = h - \bar{e}_3 \cdot \mathbb{T}(v,0)$ and

$$(3.63) \quad \|f'\|_{L_r} \leq \|f\|_{L_r} + c\|\nabla w\|_{W_r^{2,1}},$$
$$\|h'\|_{W_r^{1-1/r,1/2-1/(2r)}(x_3=0)} \leq \|h\|_{W_r^{1-1/r,1/2-1/(2r)}(x_3=0)} + c\|\nabla w\|_{W_r^{2,1}}.$$

Next we consider the following problem:

$$
\begin{aligned}
(3.64) \quad & \Delta p'' = \mathrm{div}\, f', \\
& p''|_{x_3=0} = 0, \\
& p'' \to 0 \quad \text{as } x_3 \to \infty.
\end{aligned}
$$

We see that $p'' \in W_{r,\mathrm{loc}}^{1,0}(\mathbb{D}^4)$ and

$$(3.65) \qquad \|\nabla p''\|_{L_r(\mathbb{D}^4)} \leq c\|f'\|_{L_r(\mathbb{D}^4)}.$$

Now we take p in the form

(3.66)
$$p = p' + p'',$$

where p'' is the solution of (3.64). Hence problem (3.62) can be reduced to

(3.67)
$$\begin{aligned}
&v_{,t} - \nu\Delta v + \nabla p' = f_0, \\
&\operatorname{div} v = 0, \\
&\overline{e}_3 \cdot \mathbb{T}(v, p') = h', \\
&v|_{t=0} = 0,
\end{aligned}$$

where $f_0 = f' - \nabla p''$ and it is easy to see that $\operatorname{div} f_0 = 0$.

From Lemmas 3.2 and 3.3 (the case with $\operatorname{div} f = 0$) we get a solution of (3.56) which satisfies

(3.68) $\|D_t v\|_{L_r(\mathbb{D}_T^4)} + \|D_x^2 v\|_{L_r(\mathbb{D}_T^4)} + \|D_x p'\|_{L_r(\mathbb{D}_T^4)}$
$$+ \|p'\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_{x'}^2 \times [0,T])}$$
$$\leq c(T)[\|f_0\|_{L_r(\mathbb{D}_T^4)} + \|h'\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_T^3)}].$$

The constant $c(T)$ depends on T because we take the trace of the solution from Lemma 3.3 and the whole norm depends on T by (3.50).

By (3.61), (3.63), (3.65), (3.66) and (3.68) we get (3.52).

4. Problem in a bounded domain. In this section we prove the existence of solutions of the following problem in the bounded domain Ω_T:

(4.1)
$$\begin{aligned}
&u_t - \nu\Delta u + \nabla p = f, \\
&\operatorname{div} u = 0, \\
&\overline{n} \cdot \mathbb{T}(u, p)|_{S_T} = 0, \\
&u|_{t=0} = 0.
\end{aligned}$$

We restrict our considerations to the case when
$$f \in L_r^{\operatorname{div}}(\Omega_T) = \overline{\{f \in C^\infty(\Omega_T) : \operatorname{div} f = 0\}}^{\|\cdot\|_{L_r}}.$$
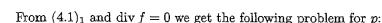
LEMMA 4.1. *If $f \in L_2^{\operatorname{div}}(\Omega_T)$ then there exists a unique solution of (4.1) such that $u \in W_2^{2,1}(\Omega_T)$ and $p \in W_2^{1,1/4}(\Omega_T) \cap W_2^{1/2,1/4}(S_T)$, and the following estimate holds:*

(4.2) $\|u\|_{W_2^{2,1}(\Omega_T)} + \|p\|_{W_2^{1,1/4}(\Omega_T)} + \|p\|_{W_2^{1/2,1/4}(S_T)} \leq c(T)\|f\|_{L_2(\Omega_T)}.$

Proof. In [10] we have existence of solutions for (4.1) and the estimate

(4.3) $\|u\|_{W_2^{2,1}(\Omega_T)} + \|p\|_{W_2^{1,0}(\Omega_T)} \leq c\|f\|_{L_2(\Omega_T)}.$

By the trace theorem and boundary condition $(4.1)_3$ we get $p \in W_2^{1/2,1/4}(S_T)$.

From $(4.1)_1$ and $\operatorname{div} f = 0$ we get the following problem for p:

(4.4)
$$\begin{aligned}
&\Delta p = 0, \\
&p|_{S_T} \in W_2^{1/2,1/4}(S_T).
\end{aligned}$$

We see that the solution p of (4.4) belongs to $W_2^{1,1/4}(\Omega_T)$. Thus by (4.3) and (4.4) we get (4.2).

Now we want to show regularity of the weak solution to problem (4.1). We introduce two collections of open sets: $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that $\overline{\omega^{(k)}} \subset \Omega^{(k)} \subset \Omega$, $\bigcup_k \omega^{(k)} = \Omega$ and $\bigcup_k \Omega^{(k)} = \Omega$ with $k \in \mathcal{M} \cup \mathcal{N}$ where $\overline{\Omega^{(k)}} \cap S = \emptyset$ if $k \in \mathcal{M}$ and $\overline{\omega^{(k)}} \cap S \neq \emptyset$ if $k \in \mathcal{N}$.

We assume that
$$\sup_k \operatorname{diam} \Omega^{(k)} \leq 2\lambda$$

for some λ small enough. Let $\zeta^{(k)}$ be a smooth function such that $0 \leq \zeta^{(k)} \leq 1$ and $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$ and $|D_x^\nu \zeta^{(k)}(x)| \leq c/\lambda^{|\nu|}$ and $1 \leq \sum_k (\zeta^{(k)})^2 \leq N_0$. We will omit (k) if it causes no confusion.

By $\xi^{(k)}$ we denote a center (a point inside) of $\omega^{(k)}$ for $k \in \mathcal{M}$ and a center of $\omega^{(k)} \cap S$ for $k \in \mathcal{N}$.

Let us consider a local coordinate system $y = (y_1, y_2, y_3)$ with center at $\xi^{(k)}$. If $k \in \mathcal{N}$ then the part $\widetilde{S^{(k)}} = S \cap \overline{\Omega^{(k)}}$ of the boundary is described by $y_3 = F(y_1, y_2)$. We choose the coordinates such that $F(0) = 0$ and $\nabla F(0) = 0$. From $S \in W_r^{2-1/r}$ we see that $F \in W_r^{2-1/r}$. Extending F to $\overline{F}$ in such a way that

$$\overline{F}(y_1, y_2, 0) = F(y_1, y_2) \quad \text{and} \quad \overline{F} \in W_r^2$$

we have

$$\overline{F} \in C^{1+\alpha} \quad \text{with } \alpha < 1 - 3/r, \quad |\nabla \overline{F}| \leq c\lambda^\alpha.$$

Now we can transform $\Omega^{(k)}$ into the half-space by the transformation

$$z = \Phi_k(y) = (\operatorname{Id} - \overline{F})(y).$$

Let $y = Y_k(x)$ be a transformation to the local coordinates y which consists of translations and rotations.

In our considerations we need some smallness argument, hence we define

(4.5)
$$\beta = \delta + c(\delta)T^a + \lambda^\alpha + \frac{T^{1/k} + T^{1/2}}{\lambda} + \lambda^{\varepsilon/k},$$

where $0 < \alpha < 1 - 3/r$, $\varepsilon, k, a, \delta$ and $c(\delta)$ will be defined by (4.12), (4.19) and (4.23).

Let us introduce the variables $U = u\zeta$ and $P = p\zeta$. Assume that $\zeta = \zeta^{(l)}$ and $l \in \mathcal{M}$. Then the equations (4.1) take the form

$$(4.6) \quad \begin{aligned} & U_t - \nu\Delta U + \nabla P = -2\nu\nabla\zeta \cdot \nabla u - \nu u\Delta\zeta + p\nabla\zeta + f\zeta \equiv f', \\ & \operatorname{div} U = u \cdot \nabla\zeta \equiv g', \\ & u_0|_{t=0} = 0. \end{aligned}$$

Now we obtain a condition on new functions f', g', A', B', where A' and B' are to be defined. For this purpose we consider

$$(4.7) \quad g'_t - \operatorname{div} f' = \zeta(g - \operatorname{div} f) + \nabla\zeta \cdot (u_t - f) + \nabla[2\nu\nabla\zeta \cdot \nabla u + \nu u\Delta\zeta - p\nabla\zeta]$$

and since

$$\nabla\zeta \cdot (u_t - f) = \nabla\zeta \cdot (\nu\Delta u - \nabla p) = \nabla \cdot [\nu(\nabla\zeta \times \operatorname{rot} u) - p\nabla\zeta] + p\Delta\zeta,$$

we have

$$(4.8) \quad g'_t - \operatorname{div} f' = \operatorname{div} B' + A',$$

where

$$(4.9) \quad \begin{aligned} & B' = \zeta B + 2\nu\nabla\zeta \cdot \nabla u + \nu u\Delta\zeta - 2p\nabla\zeta + \nu(\nabla\zeta \times \operatorname{rot} u), \\ & A' = \zeta A - B \cdot \nabla\zeta + p\Delta\zeta. \end{aligned}$$

Since we are looking for $u \in W_r^{2,1}$ and $\nabla p \in L_r$ we apply Lemma 3.4, thus we have to examine the norms $\|A'\|_{L_r}$ and $\|B'\|_{L_r}$.

By Lemma 3.4 we see that the solutions of (4.6) satisfy the following estimate:

$$(4.10) \quad \begin{aligned} & \|U\|_{W_k^{2,1}(\mathbb{D}_T^4)} + \|P\|_{W_k^{1,0}(\mathbb{D}_T^4)} \\ & \leq C(T)(\|f'\|_{L_k(\mathbb{D}_T^4)} + \|g'\|_{W_k^{1,0}(\mathbb{D}_T^4)} + \lambda\|A'\|_{L_k(\mathbb{D}_T^4)} + \|B'\|_{L_k(\mathbb{D}_T^4)}), \end{aligned}$$

where $k > 2$.

By Proposition 2.3 we have

$$(4.11) \quad \begin{aligned} & \|f'\|_{L_k(\Omega_T)} \\ & \leq c\left(\|f\|_{L_k(\Omega_T)} + \frac{T^{1/k} + T^{1/2}}{\lambda}\|u\|_{W_k^{2,1}(\Omega_T)} + \frac{1}{\lambda}\|p\|_{L_k(\Omega_T)}\right). \end{aligned}$$

From the imbedding theorem [1, Chap. 18] we see that

$$(4.12) \quad \|p\|_{L_k(\Omega_T)} \leq c\|p\|_{W_2^{1,1/4}(\Omega_T)},$$

since $(1/2 - 1/k)3 + (1/2 - 1/k)2 < 1$. This implies that now we can consider

only the case $2 < k < 10/3$. Then by Proposition 2.3 we obtain

$$\|g'\|_{W_k^{1,0}} \leq c\frac{T^{1/k} + T^{1/2}}{\lambda}\|u\|_{W_k^{2,1}},$$

$$(4.13) \quad \begin{aligned} & \|A'\|_{L_k} \leq c\left(\|A\|_{L_k} + \frac{1}{\lambda}\|B\|_{L_k} + \frac{1}{\lambda^2}\|p\|_{W_2^{1,1/4}(\Omega_T)}\right), \\ & \|B'\|_{L_k} \leq c\left(\|B\|_{L_k} + \frac{T^{1/k} + T^{1/2}}{\lambda}\|u\|_{W_k^{2,1}} + \frac{1}{\lambda}\|p\|_{W_2^{1,1/4}(\Omega_T)}\right). \end{aligned}$$

By (4.11)–(4.13) inequality (4.10) reads

$$(4.14) \quad \begin{aligned} & \|U\|_{W_k^{2,1}(\mathbb{D}_T^4)} + \|P\|_{W_k^{1,0}(\mathbb{D}_T^4)} \\ & \leq C(T)\bigg(\|f\|_{L_k(\mathbb{D}_T^4)} + \frac{T^{1/k} + T^{1/2}}{\lambda}\|u\|_{W_k^{2,1}(\mathbb{D}_T^4)} \\ & \quad + \lambda\|A\|_{L_k(\Omega_T)} + \|B\|_{L_k(\Omega_T)} + \|p\|_{W_2^{1,1/4}(\Omega_T)}\bigg). \end{aligned}$$

We would like to have a similar estimate when $k \in \mathcal{N}$. We write system (4.1) in z-coordinates:

$$(4.15) \quad \begin{aligned} & U_t - \nu\Delta_z U + \nabla_z P \\ & \quad = f' + L_1(\partial_x - \nabla\overline{F} \cdot \partial_z)(U, P) - L_1(\partial_z)(U, P) \equiv f'', \\ & \operatorname{div}_z U = g' + \nabla\overline{F} \cdot \partial_z U \equiv g'', \\ & \overline{e}_{z_3} \cdot \mathbb{T}(U, P) = \Phi(\overline{n} \cdot (\eta_{,i}u_j + \eta_{,j}u_i)) + \nabla\overline{F} \cdot \mathbb{T}(U, 0) \equiv h'', \end{aligned}$$

where $L_1(\partial_x)(u, p) = u_t - \nu\Delta_x u + \nabla_x p$.

To apply Lemma 3.4 we need new A'' and B'' which satisfy

$$(4.16) \quad \partial_t g'' - \operatorname{div} f'' = \operatorname{div} B'' + A''.$$

First we note that (the second term of the r.h.s. of $(4.15)_2$)

$$(4.17) \quad \overline{F}_{i,j}U_{j,i} = \partial_{z_i}(\overline{F}_{i,j}U_j) - U_j\overline{F}_{i,ji}.$$

To obtain the new B'' we have to consider

$$(4.18) \quad \begin{aligned} & \Delta b = U_{j,t}\overline{F}_{i,ji}, \\ & b|_{z_3=0} = 0, \\ & b \to 0 \quad \text{as } z_3 \to \infty. \end{aligned}$$

We see that

$$\|U_{j,t}\overline{F}_{i,ji}\|_{L_l(\mathbb{R}_+^3)} \leq c\|U_t\|_{L_{k'}(\mathbb{R}_+^3)},$$

where $l = rk'/(r + k')$, hence solving (4.18) we get

$$\|\nabla b\|_{W_l^1(\mathbb{R}_+^3)} \leq c\|U_t\|_{L_{k'}(\mathbb{R}_+^3)}.$$

Since $3l/(3-l) > k' + \varepsilon = k$, by Proposition 2.2 we have

$$(4.19) \qquad \|\nabla b\|_{L_k(0,T;L_{k'+\varepsilon}(\mathbb{R}^3_+))} \leq c\|U_t\|_{L_k(0,T;L_{k'}(\mathbb{R}^3_+))}$$
$$\leq c\lambda^{3\varepsilon/((k'+\varepsilon)k)}\|U_t\|_{L_k(\mathbb{D}^4_T)}.$$

Then we define A'' and B'' by

$$(4.20) \qquad \begin{aligned} B'' &= B' - (L_1(\partial_z - \nabla\overline{F}\cdot\partial_z)(U,P) - L_1(\partial_z)(U,P)) + \overline{F}_{i,j}U_j - \nabla b, \\ A'' &= A'. \end{aligned}$$

We also need estimates for components of B'', hence we examine

$$(4.21) \qquad \|L_1(\partial_z - \nabla\overline{F}\cdot\partial_z)(U,P) - L_1(\partial_z)(U,P)\|_{L_k(\mathbb{D}^4_T)}$$
$$\leq c\|\,|\nabla^2\overline{F}|\cdot|\nabla U|\,\|_{L_k(\mathbb{D}^4_T)} + c\|\,|\nabla\overline{F}|\cdot|\nabla^2 U|\,\|_{L_k(\mathbb{D}^4_T)} + c\|\,|\nabla\overline{F}|\cdot|P|\,\|_{L_k(\mathbb{D}^4_T)}.$$

To estimate the first term of the r.h.s. of (4.21) we note that

$$(4.22) \qquad \|\,|\nabla^2\overline{F}|\cdot|\nabla U|\,\|_{L_k(\mathbb{R}^3_+)} \leq \|\nabla^2\overline{F}\|_{L_r(\mathbb{R}^3_+)}\|\nabla U\|_{L_l(\mathbb{R}^3_+)},$$

where $l = kr/(r-k)$, and we have

$$\|\nabla U\|_{L_k(0,T;L_l(\mathbb{R}^3_+))} \leq (\delta + c(\delta)T^a)\|U\|_{W^{2,1}_r},$$

where $\delta, a > 0$ and $c(\delta)$ tends to infinity as $\delta \to 0$.

By (4.22) we get

$$(4.23) \qquad \|\,|\nabla^2\overline{F}|\cdot|\nabla U|\,\|_{L_k(\mathbb{D}^4_T)} \leq c\|\nabla U\|_{L_k(0,T;L_l(\mathbb{R}^3_+))}$$
$$\leq c(\delta + c(\delta)T^a)\|U\|_{W^{2,1}_k}.$$

By (4.19) with $k' + \varepsilon = k$, (4.22) and (4.23) we obtain

$$\|f''\|_{L_k} \leq \|f'\|_{L_k} + c(\delta + c(\delta)T^a + \lambda^\alpha)\|U\|_{W^{2,1}_k} + c\lambda^\alpha\|P\|_{W^{1,0}_k},$$
$$\|g''\|_{W^{1,0}_k} \leq \|g'\|_{W^{1,0}_k} + c(\delta + c(\delta)T^a + \lambda^\alpha)\|U\|_{W^{2,1}_k},$$

$$(4.24) \qquad \|h''\|_{W^{1-1/k,1/2-1/(2k)}_k} \leq c\left(\frac{T^{1/k} + T^{1/2}}{\lambda} + \lambda^\alpha\right)\|U\|_{W^{2,1}_k},$$

$$\|B''\|_{L_k} \leq \|B'\|_{L_k} + c(\delta + c(\delta)T^a + \lambda^{\varepsilon/k})\|U\|_{W^{2,1}_k} + \lambda^\alpha\|P\|_{W^{1,0}_k},$$
$$\|A''\|_{L_k} \leq \|A'\|_{L_k}.$$

Then by Lemma 3.4 and (4.24) we obtain

$$(4.25) \qquad \|U\|_{W^{2,1}_k} + \|P\|_{W^{1,0}_k} + \|P\|_{W^{1-1/k,1/2-1/(2k)}_k(z_3=0)}$$
$$\leq c\bigg(\|f'\|_{L_k} + \|g'\|_{W^{1,0}_k} + \|B'\|_{L_k} + \lambda\|A'\|_{L_k}$$
$$+ \left(\delta + c(\delta)T^a + \lambda^\alpha + \frac{T^{1/k} + T^{1/2}}{\lambda} + \lambda^{\varepsilon/k}\right)(\|U\|_{W^{2,1}_k} + \|P\|_{W^{1,0}_k})\bigg).$$

If we take β so small that

$$c\beta \equiv c\left(\delta + c(\delta)T^a + \lambda^\alpha + \frac{T^{1/2} + T^{1/k}}{\lambda} + \lambda^{\varepsilon/k}\right) < \frac{1}{2},$$

then from (4.25) we get

$$(4.26) \qquad \|U\|_{W^{2,1}_k} + \|P\|_{W^{1,0}_k} + \|P\|_{W^{1-1/k,1/2-1/(2k)}_k(z_3=0)}$$
$$\leq c(\|f'\|_{L_k} + \|g'\|_{W^{1,0}_k} + \|B'\|_{L_k} + \lambda\|A'\|_{L_k}).$$

The above considerations lead to the following lemma.

LEMMA 4.2. *If $A, B \in L_r(\Omega_T)$ and $f \in L_r^{\mathrm{div}}(\Omega_T)$ then the weak solution (u,p) of (4.1) satisfies*

$$u \in W^{2,1}_r(\Omega_T) \quad \text{and} \quad p \in W^{1,0}_r(\Omega_T) \cap W^{1-1/r,1/2-1/(2r)}_r(\Omega_T),$$

and the following estimate holds:

$$(4.27) \qquad \|u\|_{W^{2,1}_r(\Omega_T)} + \|p\|_{W^{1,0}_r(\Omega_T)} + \|p\|_{W^{1-1/r,1/2-1/(2r)}_r(\Omega_T)}$$
$$\leq c(\|f\|_{L_r(\Omega_T)} + \|B\|_{L_r(\Omega_T)} + \lambda\|A\|_{L_r(\Omega_T)})$$

if β is small enough.

Proof. First we note that $dx = J_{\Phi^{-1}}\,dz$ and $|J_{\Phi^{-1}} - 1| \leq c\lambda^\alpha$. Thus if we denote $U^{(l)} = \zeta^{(l)}u$ and $P^{(l)} = \zeta^{(l)}p$ we get

$$\|u\|_{W^{2,1}_k(\Omega_T)} \leq \sum_{l\in\mathcal{M}\cup\mathcal{N}} \|U^{(l)}\|_{W^{2,1}_k(\Omega_T)},$$

$$(4.28) \qquad \|p\|_{W^{1,0}_k(\Omega_T)} \leq \sum_{l\in\mathcal{M}\cup\mathcal{N}} \|P^{(l)}\|_{W^{1,0}_k(\Omega_T)},$$

$$\|p\|_{W^{1-1/k,1/2-1/(2k)}_k(S_T)} \leq \sum_{l\in\mathcal{N}} \|P^{(l)}\|_{W^{1-1/k,1/2-1/(2k)}_k(S_T)}.$$

Then by (4.11)–(4.14), (4.26) and properties of functions $\zeta^{(l)}$ we obtain

$$(4.29) \qquad \|u\|_{W^{2,1}_k(\Omega_T)} + \|p\|_{W^{1,0}_k(\Omega_T)} + \|p\|_{W^{1-1/k,1/2-1/(2k)}_k(S_T)}$$
$$\leq cN_0\bigg(\|f\|_{L_k} + \|B\|_{L_k} + \lambda\|A\|_{L_k} + \frac{T^{1/r} + T^{1/2}}{\lambda}\|u\|_{W^{2,1}_k(\Omega_T)} + \|p\|_{W^{1,1/4}_2(\Omega_T)}\bigg)$$

and if $(T^{1/r} + T^{1/2})cN_0/\lambda < 1/2$ we get

$$(4.30) \qquad \|u\|_{W^{2,1}_k(\Omega_T)} + \|p\|_{W^{1,0}_k(\Omega_T)} + \|p\|_{W^{1-1/k,1/2-1/(2k)}_k(S_T)}$$
$$\leq c(\|f\|_{L_k} + \|B\|_{L_k} + \lambda\|A\|_{L_k} + \|p\|_{W^{1,1/4}_2(\Omega_T)}).$$

Recalling (4.12) we have (4.30) only for $2 < k < 10/3$.

By (4.1) and (4.30) we also obtain

$$\begin{aligned} \Delta p &= 0, \\ p &\in W_k^{1-1/k,1/2-1/(2k)}(S_T), \end{aligned} \tag{4.31}$$

which gives $p \in W_k^{1,1/2-1/(2k)}(\Omega_T)$.

We put $k_1 = 5/2 < 10/3$. Then by (4.30) and (4.31) we have

$$\begin{aligned} (4.32) \quad &\|u\|_{W_{k_1}^{2,1}(\Omega_T)} + \|p\|_{W_{k_1}^{1,1/2-1/(2k_1)}(\Omega_T)} + \|p\|_{W_{k_1}^{1-1/k_1,1/2-1/(2k_1)}(S_T)} \\ &\qquad \le c(\|f\|_{L_{k_1}} + \|B\|_{L_{k_1}} + \lambda\|A\|_{L_{k_1}} + \|p\|_{W_{k_0}^{1,1/2-1/(2k_0)}(\Omega_T)}), \end{aligned}$$

where $k_0 = 2$.

Now we repeat the above considerations having estimate (4.32). Imbedding (4.12) can be replaced by

$$\|p\|_{L_k} \le c\|p\|_{W_{k_1}^{1,1/2-1/(2k_1)}(\Omega_T)}. \tag{4.33}$$

Since $k_1 = 5/2$, we have the following condition on the new k:

$$3\left(\frac{2}{5} - \frac{1}{k}\right) + \frac{10}{3}\left(\frac{2}{5} - \frac{1}{k}\right) < 1,$$

which gives $5/2 < k < 30/7$, and we put $k_2 = \min\{r, 4\}$. Then we obtain estimate (4.32) with $k_1 = k_2$ and $k_0 = k_1$.

If $r > 4$ we again repeat the above considerations and obtain, having $p \in W_{k_2}^{1,1/2-1/(2k_2)}(\Omega_T)$, a new k which has to satisfy $4 < k < 16$ and we put $k_3 = \min\{r, 15\}$, and if $r > 15$ we repeat all considerations and obtain (4.27), which follows from (4.32) with $k_1 = r$ and $k_0 = k_3$, because $W_{15}^{1,1/2-1/30} \subset L_\infty$. In the last step $\beta(r) > 0$ (see (4.5)), hence $T(r), \lambda(r) > 0$. The proof of Lemma 4.2 is finished.
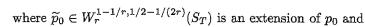
5. Proof of Theorem 1. Our aim is to show existence of a solution of problem (1.1):

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= F, \\ \operatorname{div} u &= G, \\ \bar{n} \cdot \mathbb{T}(u,p)|_{S_T} &= H, \\ u|_{t=0} &= u_0. \end{aligned}$$

By $(1.2)_2$ and the assumptions of Theorem 1, $p_0 = p|_{t=0} \in W_r^{1-3/r}(S)$. We take $\bar{u}_0 \in W_r^{2-1/r}(\Omega_T)$ and $\bar{p}_0 \in W_r^{1,0}(\Omega_T) \cap W_r^{1-1/r,1/2-1/(2r)}(S_T)$ such that

$$\begin{aligned} \Delta\bar{p}_0 &= 0, \\ \bar{p}_0|s &= \tilde{p}_0, \end{aligned}$$

where $\tilde{p}_0 \in W_r^{1-1/r,1/2-1/(2r)}(S_T)$ is an extension of p_0 and

$$\bar{u}_0(x,0) = u_0(x) \quad \text{and} \quad \|\bar{u}_0\|_{W_r^{2,1}(\Omega_T)} \le c\|u_0\|_{W_r^{2-2/r}(\Omega)},$$

$$\begin{aligned} (5.1) \quad &\bar{p}_0(x,0) = p_0(x) \quad \text{and} \quad \|\bar{p}_0\|_{W_r^{1-1/r,1/2-1/(2r)}(S_T)} \le c\|p_0\|_{W_r^{1-3/r}(S)}, \\ &\|\bar{p}_0\|_{W_r^{1,0}(\Omega_T)} \le c\|p_0\|_{W_r^{1-3/r}(S)} \\ &\qquad\qquad \le c\|u_0\|_{W_r^{2-2/r}(\Omega)} + c\|H\|_{W_r^{1-1/r,1/2-1/(2r)}(S_T)}. \end{aligned}$$

If we assume that the solution of (1.1) has the form

$$u = u' + \bar{u}_0, \quad p = p' + \bar{p}_0, \tag{5.2}$$

then problem (1.1) reduces to

$$\begin{aligned} u_t' - \nu\Delta u' + \nabla p' &= F', \\ \operatorname{div} u &= G', \\ \bar{n} \cdot \mathbb{T}(u',p')|_{S_T} &= H', \\ u|_{t=0} &= 0, \end{aligned} \tag{5.3}$$

where (using (5.1))

$$\begin{aligned} (5.4) \quad F' &= F - (\bar{u}_{0t} - \nu\Delta\bar{u}_0) - \nabla\bar{p}_0, \\ G' &= G - \operatorname{div}\bar{u}_0, \\ H' &= H - \bar{n} \cdot \mathbb{T}(\bar{u}_0, \tilde{p}_0), \\ \|F'\|_{L_r} &\le \|F\|_{L_r} + c\|u_0\|_{W_r^{2-2/r}}, \\ \|G'\|_{W_r^{1,0}} &\le \|G\|_{W_r^{1,0}} + c\|u_0\|_{W_r^{2-2/r}}, \\ \|H'\|_{W_r^{1-1/r,1/2-1/(2r)}} &\le \|H\|_{W_r^{1-1/r,1/2-1/(2r)}} + c\|u_0\|_{W_r^{2-2/r}}, \end{aligned}$$

where $G_t' - \operatorname{div} f' = \operatorname{div} B' + A$ and $B' = B - \bar{u}_0$. By (1.2) we see that $G'|_{t=0} = 0$, $H'|_{t=0} = 0$ and $p'|_{t=0} = 0$. Taking $w = \nabla\Phi$, where

$$\begin{aligned} \Delta\Phi &= G', \\ \Phi|_{S_T} &= 0, \end{aligned}$$

we have $w = \nabla\Phi \in W_r^{2,0}(\Omega_T)$, but from $\Delta\Phi_t = \operatorname{div}(B + f) + A$, as in the proof of Lemma 3.4 (see (3.53)–(3.61)), we get

$$\|w_t\|_{L_r} \le c(\|f\|_{L_r} + \|B'\|_{L_r} + \lambda\|A\|_{L_r}).$$

So $w \in W_r^{2,1}$ and

$$\|w\|_{W_r^{2,1}(\Omega_T)} \le c(\|g\|_{W_r^{1,0}(\Omega_T)} + \|f\|_{L_r} + \|B'\|_{L_r} + \lambda\|A\|_{L_r}) \tag{5.5}$$

and from $G'|_{t=0} = 0$ we have $w|_{t=0} = 0$. Thus we can take

$$u' = u'' + w, \tag{5.6}$$

and reduce problem (5.3) to

$$u_t'' - \nu\Delta u'' + \nabla p' = F'',$$

(5.7)
$$\mathrm{div}\, u'' = 0,$$

$$\overline{n} \cdot \mathbb{T}(u'', p')|_{S_T} = H'',$$

$$u|_{t=0} = 0,$$

where, by (5.5),

$$F'' = F' - (w_t - \nu\Delta w), \quad \|F''\|_{L_r} \le \|F'\|_{L_r} + c\|w\|_{W_r^{2,1}},$$

$$H'' = H' - \overline{n} \cdot \mathbb{T}(w, 0),$$

$$\|H''\|_{W_r^{1-1/r,1/2-1/(2r)}} \le \|H'\|_{W_r^{1-1/r,1/2-1/(2r)}} + c\|w\|_{W_r^{2,1}}.$$

To reduce (5.7) to (4.1) we have to solve the following problem:

(5.9)
$$\Delta p'' = \mathrm{div}\, F'',$$

$$p''|_{S_t} = H'',$$

which gives

(5.10) $$\qquad \|\nabla p''\|_{L_r} \le c(\|F''\|_{L_r} + \|H''\|_{W_r^{1-1/r,1/2-1/(2r)}}).$$

Putting $p' = p'' + p'''$ from (5.7) we get

$$u_t'' - \nu\Delta u'' + \nabla p''' = F''',$$

(5.12)
$$\mathrm{div}\, u'' = 0,$$

$$\overline{n} \cdot \mathbb{T}(u'', p''') = 0,$$

$$u''|_{t=0} = 0,$$

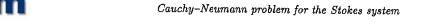where $F''' = F'' - \nabla p''$ and $\mathrm{div}\, F''' = 0$.

By Lemma 4.2, (5.1), (5.5), (5.9) and (5.10), we obtain a solution of system (5.12) which in view of (5.2), (5.6) and (5.11) gives a solution of system (1.1) given by

(5.13) $$\qquad u = \overline{u}_0 + u'' + w \quad \text{and} \quad p = p'' + p''' + \overline{p}_0,$$

which satisfies (1.2) for $T \le T_0$. Now it is enough to continue the solution to the intervals $[T_0, 2T_0], [2T_0, 3T_0], \dots$ This proves (1.3) and concludes our considerations.

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