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Weyl's theorems and continuity of spectra in the class of p-hyponormal operators

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Abstract. We show that p-hyponormal operators obey Weyl's and a-Weyl's theorem. Also, we show that the spectrum, Weyl spectrum, Browder spectrum and approximate point spectrum are continuous functions in the class of all p-hyponormal operators.

1. Introduction. Let H be a complex infinite-dimensional separable Hilbert space and let $B(H)$ (resp. $K(H)$) denote the Banach algebra of all bounded operators (resp. the ideal of all compact operators) on H. If $A \in B(H)$, then $\sigma(A)$ denotes the spectrum of A, $\varrho(A)$ denotes the resolvent set of A and $r(A)$ denotes the spectral radius of A. The following sets are well-known semigroups of operators on H:

$$\Phi_+(H) = \{A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim \mathcal{N}(A) < \infty\},$$
$$\Phi_-(H) = \{A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim H/\mathcal{R}(A) < \infty\}.$$

The semigroup of *semi-Fredholm operators* is $\Phi(H) = \Phi_+(H) \cup \Phi_-(H)$. If A is semi-Fredholm and $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \dim H/\mathcal{R}(A)$, then we may define an *index*: $i(A) = \alpha(A) - \beta(A)$. We also consider the sets

$$\Phi_0(H) = \{A \in \Phi(H) : i(A) = 0\} \quad (\textit{Weyl operators}),$$
$$\Phi_+^-(H) = \{A \in \Phi_+(H) : i(A) \le 0\}.$$

The following spectra of $A \in B(H)$ are familiar:

$$\sigma_{\rm p}(A) = \{\lambda \in \mathbb{C} : \text{ there exists } x \in H \setminus \{0\} \text{ such that } Ax = \lambda x\},$$
$$\sigma_{\rm a}(A) = \{\lambda \in \mathbb{C} : \inf_{x\in H,\, \|x\|=1} \|(A-\lambda)x\| = 0\}$$
— the *approximate point spectrum*,
$$\sigma_{\rm w}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_0(H)\} \text{ — the } \textit{Weyl spectrum},$$

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$$\sigma_{\mathrm{b}}(A) = \bigcap\{\sigma(A+K) : AK = KA,\ K \in K(H)\}$$
— the *Browder spectrum*,

$$\sigma_{\mathrm{ea}}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_{+}^{-}(H)\}$$
— the *essential approximate point spectrum*,

$$\sigma_{\mathrm{ab}}(A) = \bigcap\{\sigma_{\mathrm{a}}(A+K) : AK = KA,\ K \in K(H)\}$$
— the *Browder essential approximate point spectrum*.

Let $\pi_{00}(A)$ be the set of all $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma(A)$ and $0 < \dim \mathcal{N}(A-\lambda) < \infty$, let $\pi_0(A)$ be the set of all *normal eigenvalues* of A, that is, of all isolated points of $\sigma(A)$ for which the corresponding spectral projection has finite-dimensional range and let $\pi_{00}^{\mathrm{a}}(A)$ be the set of all $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_{\mathrm{a}}(A)$ and $0 < \dim \mathcal{N}(A-\lambda) < \infty$.

We say that A *obeys Weyl's theorem* [2, 5] if

$$\sigma_{\mathrm{w}}(A) = \sigma(A) \setminus \pi_{00}(A),$$

and we say that A *obeys a-Weyl's theorem* [12] if

$$\sigma_{\mathrm{ea}}(A) = \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

If (τ_n) is a sequence of compact subsets of $\mathbb{C}$, then its limit inferior is

$$\liminf \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$$

and its limit superior is

$$\limsup \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}.$$

If $\liminf \tau_n = \limsup \tau_n$, then $\lim \tau_n$ is said to exist and is equal to this common limit. A mapping p, defined on $B(H)$, whose values are compact subsets of $\mathbb{C}$ is said to be *upper* (resp. *lower*) *semicontinuous* at A provided that if $A_n \to A$ in the norm topology then $\limsup p(A_n) \subset p(A)$ (resp. $p(A) \subset \liminf p(A_n)$). If p is both upper and lower semicontinuous at A, then it is said to be *continuous* at A and in this case $\lim p(A_n) = p(A)$.

We say that $A \in B(H)$ is *p-hyponormal* provided that $(A^*A)^p - (AA^*)^p \geq 0$. If $p = 1$, then A is called *hyponormal*, and if $p = 1/2$, then A is called *semi-hyponormal*. It is well known that a p-hyponormal operator is q-hyponormal for $q \leq p$. Let $\mathcal{H}(p)$ be the class of p-hyponormal operators.

2. Preliminary results. Necessary and sufficient conditions for an operator $A \in B(H)$ to obey a-Weyl's theorem have been discussed by V. Rakočević in [12]. Here we employ the conditions C_1^{a} and C_2^{a} from Baxley [2] to give conditions which are sufficient for obeying a-Weyl's theorem.

We start by defining the sets

$$\pi_0^{\mathrm{a}}(A) = \{\lambda \in \sigma_{\mathrm{a}}(A) : \lambda \text{ is an eigenvalue of } A\},$$
$$\pi_{0\mathrm{f}}^{\mathrm{a}}(A) = \{\lambda \in \sigma_{\mathrm{a}}(A) : \lambda \text{ is an eigenvalue of } A \text{ and } \dim \mathcal{N}(A-\lambda) < \infty\}.$$

LEMMA 2.1. *For every operator $A \in B(H)$,*

$$\sigma_{\mathrm{a}}(A) \setminus \pi_0^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A).$$

Proof. Let $\lambda \in \sigma_{\mathrm{a}}(A) \setminus \sigma_{\mathrm{ea}}(A)$. Then $0 < \alpha(A-\lambda)$ and $i(A-\lambda) \leq 0$ (see [11]), i.e. $0 < \alpha(A-\lambda) \leq \beta(A-\lambda) \leq \infty$. This, by the definition of the set $\pi_0^{\mathrm{a}}(A)$, implies that $\lambda \in \pi_0^{\mathrm{a}}(A)$. ∎

LEMMA 2.2. *For every operator $A \in B(H)$,*

$$\sigma_{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A).$$

Proof. In view of Lemma 2.1, it is sufficient to show that

$$\pi_0^{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A).$$

Let $\lambda \in \pi_0^{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A)$. Then $\lambda \in \sigma_{\mathrm{a}}(A)$ and $\alpha(A-\lambda) = \infty$. Let $\{x_n\}$ be a sequence in $\mathcal{N}(A-\lambda)$ such that $(x_n, x_m) = 0$ for $n \neq m$. We show that $\lambda \in \sigma_{\mathrm{a}}(A+K)$ for every $K \in K(X)$, i.e. $\lambda \in \bigcap\{\sigma_{\mathrm{a}}(A+K) : K \in K(H)\} = \sigma_{\mathrm{ea}}(A)$.

Let $K \in K(H)$ and suppose that $y = \lim Kx_n$. Then the sequence $y_n = (A+K-\lambda)x_n$, $n \in \mathbb{N}$, satisfies

$$\lim y_n = \lim (A+K-\lambda)x_n = \lim Kx_n = y.$$

Now suppose that $\lambda \notin \sigma_{\mathrm{a}}(A+K)$. Then there exists $m > 0$ such that

$$\|(A+K-\lambda)x\| \geq m\|x\| \quad \text{for every } x \in H,$$

i.e. $A+K-\lambda$ is one-one on $\mathcal{R}(A+K-\lambda)$ and

$$\|(A+K-\lambda)^{-1}y\| \leq \frac{1}{m}\|y\| \quad \text{for every } y \in \mathcal{R}(A+K-\lambda).$$

This implies, by [9, p. 190], that the range of $A+K-\lambda$ is closed. Hence, given $y = \lim(A+K-\lambda)x_n \in \overline{\mathcal{R}(A+K-\lambda)}$, there exists $x \in H$ such that $y = (A+K-\lambda)x$. Denote the restriction of $A+K-\lambda$ to $\mathcal{R}(A+K-\lambda)$ by B. Then B is a *regular* operator (i.e., the operator B^{-1} is well defined), and

$$\lim x_n = \lim B^{-1}Bx_n = \lim B^{-1}y_n = B^{-1}y = x.$$

This, however, contradicts our assumption that the sequence $\{x_n\}$ is orthogonal. Hence, $\lambda \in \sigma_{\mathrm{a}}(A+K)$. ∎

DEFINITION 2.3. An operator $A \in B(H)$ obeys *condition* C_1^{a} if for every infinite sequence $\{\lambda_n\} \subset \pi_{0\mathrm{f}}^{\mathrm{a}}(A)$ of distinct eigenvalues no sequence $\{x_n\}$ of corresponding normalized eigenvectors converges.

DEFINITION 2.4. An operator $A \in B(H)$ obeys *condition* C_2^{a} if for every $\lambda \in \pi_{00}^{\mathrm{a}}(A)$ the operator $A-\lambda$ has closed range.

THEOREM 2.5. *If an operator* $A \in B(H)$ *obeys condition* C_1^{a}, *then*

$$\sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A).$$

Proof. It is clear from Lemma 2.2 that $\sigma_{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A)$. Since $\sigma_{\mathrm{ea}}(A)$ is a closed and compact, it follows that

$$\overline{\sigma_{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A)} \subset \sigma_{\mathrm{ea}}(A).$$

Suppose now that $\lambda \in (\pi_{0\mathrm{f}}^{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A)) \setminus \overline{\sigma_{\mathrm{a}}(A) \setminus \pi_{0\mathrm{f}}^{\mathrm{a}}(A)}$. Then λ is a nonisolated eigenvalue such that $\alpha(A-\lambda) < \infty$. Hence, there exists a sequence $\{\lambda_n\} \subset \pi_{0\mathrm{f}}^{\mathrm{a}}(A)$ such that $\lambda_n \to \lambda$. Let $\{x_n\}$ be a sequence of corresponding eigenvectors for λ_n and let x be an eigenvector for λ. We will show that $\lambda \in \sigma_{\mathrm{a}}(A+K)$ for every $K \in K(H)$, i.e. $\lambda \in \sigma_{\mathrm{ea}}(A)$.

Suppose to the contrary that there exists a $K \in K(H)$ such that $\lambda \notin \sigma_{\mathrm{a}}(A+K)$. Since $A+K-\lambda$ has closed range and is one-one on $\mathcal{R}(A+K-\lambda)$, it follows that it is an invertible operator on $\mathcal{R}(A+K-\lambda)$. Let S be its inverse on $\mathcal{R}(A+K-\lambda)$.

Let $\lim x_n = y$ and $y_n = (A+K-\lambda)x_n$. Then

$$\lim y_n = \lim((A-\lambda_n)x_n + Kx_n + (\lambda-\lambda_n)x_n) = y,$$

and it follows that

$$Sy = \lim Sy_n = \lim S(A+K-\lambda)x_n = \lim x_n,$$

i.e. condition C_1^{a} does not hold. This is a contradiction. ∎

THEOREM 2.6. *If* $A \in B(H)$ *obeys condition* C_2^{a}, *then*

$$\sigma_{\mathrm{ea}}(A) \subset \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

Proof. Let $\lambda \in \sigma_{\mathrm{ea}}(A)$ and suppose that $\lambda \in \pi_{00}^{\mathrm{a}}(A)$. Since $\lambda \in \pi_{00}^{\mathrm{a}}(A)$ it follows that $A-\lambda$ has closed range and $0 < \alpha(A-\lambda)$. Hence $A-\lambda \in \Phi_+(H)$ and, since $\lambda \in \sigma_{\mathrm{ea}}(A)$, it follows that $i(A-\lambda) > 0$. By the continuity of index it thus follows that λ is an interior point of $\sigma_{\mathrm{a}}(A)$. This is a contradiction. ∎

THEOREM 2.7. *If* $A \in B(H)$ *obeys conditions* C_1^{a} *and* C_2^{a}, *then a-Weyl's theorem holds for* A, *i.e.*

$$\sigma_{\mathrm{ea}}(A) = \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

Proof. By Theorems 2.5 and 2.6. ∎

3. p-hyponormal operators.

We start with some elementary results about p-hyponormal operators. The following lemma is known (Aluthge [1], Uchiyama [14]).

LEMMA 3.1 ([1]). *Let* $A \in \mathcal{H}(p)$. *If* A^{-1} *exists, then it is p-hyponormal.*

LEMMA 3.2 ([14, Lemma 4]). *Let* $A \in \mathcal{H}(p)$ *and let* H_1 *be a closed subspace of* H. *If* A *maps* H_1 *into itself, then the restriction of* A *to* H_1 *is p-hyponormal.*

Given $A \in \mathcal{H}(p)$, $0 < p < \frac{1}{2}$, decompose A into its normal and pure parts: $A = A_{\mathrm{n}} \oplus A_{\mathrm{p}}$ $(= A_{|H_{\mathrm{n}}} \oplus A_{|H \ominus H_{\mathrm{n}}})$. Let $A_{\mathrm{p}} \in \mathcal{H}(p)$ have polar decomposition $A_{\mathrm{p}} = U_{\mathrm{p}}|A_{\mathrm{p}}|$; then $|A_{\mathrm{p}}|$ is a quasi-affinity and U_{p} is an isometry. Define $\widehat{A}_{\mathrm{p}} = |A_{\mathrm{p}}|^{1/2}U_{\mathrm{p}}|A_{\mathrm{p}}|^{1/2}$ and, letting $\widehat{A}_{\mathrm{p}}$ have the polar decomposition $\widehat{A}_{\mathrm{p}} = V_{\mathrm{p}}|\widehat{A}_{\mathrm{p}}|$, set $\widetilde{A}_{\mathrm{p}} = |\widehat{A}_{\mathrm{p}}|^{1/2}V_{\mathrm{p}}|\widehat{A}_{\mathrm{p}}|^{1/2}$. Then $\widehat{A}_{\mathrm{p}} \in \mathcal{H}(p+1/2)$, $\widetilde{A} \in \mathcal{H}(1)$, $\sigma(A_{\mathrm{p}}) = \sigma(\widehat{A}_{\mathrm{p}}) = \sigma(\widetilde{A}_{\mathrm{p}})$, and both $\widehat{A}_{\mathrm{p}}$ and $\widetilde{A}_{\mathrm{p}}$ are pure [8]. Let $\widetilde{A} = A_{\mathrm{n}} \oplus \widetilde{A}_{\mathrm{p}}$, $B = 1_{H_{\mathrm{n}}} \oplus |\widehat{A}_{\mathrm{p}}|^{1/2}|A_{\mathrm{p}}|^{1/2}$ and $C = 1_{H_{\mathrm{n}}} \oplus U_{\mathrm{p}}|A_{\mathrm{p}}|^{1/2}V_{\mathrm{p}}|\widehat{A}_{\mathrm{p}}|^{1/2}$; then B is a quasi-affinity, C^* has dense range and

$$\widetilde{A}B = BA \quad \text{and} \quad C\widetilde{A} = AC.$$

The following lemma is an easy consequence of the above.

LEMMA 3.3. *Let* $A \in \mathcal{H}(p)$. *Then* $\alpha(A-\lambda) = \alpha(\widetilde{A}-\lambda)$, $\beta(A-\lambda) \geq \beta(\widetilde{A}-\lambda)$ *and* $\sigma_{\mathrm{s}}(A) = \sigma_{\mathrm{s}}(\widetilde{A})$, *where* $\sigma_{\mathrm{s}} = \sigma$ *or* σ_{p} *or* σ_{a}.

LEMMA 3.4. *If* λ *is an isolated point of* $\sigma(A)$, $A \in \mathcal{H}(p)$, *and either* $\alpha(A-\lambda)$ *or* $\beta(A-\lambda)$ *is finite, then* $A-\lambda \in \Phi_0(H)$ *and* $\lambda \in \sigma_{\mathrm{p}}(A)$.

Proof. It is clear from Lemma 3.3 that λ is an isolated point of $\sigma(\widetilde{A})$ such that either $\alpha(\widetilde{A}-\lambda)$ or $\beta(\widetilde{A}-\lambda)$ is finite. This, by [13, Lemma XI.5.5], implies that $\lambda \in \sigma_{\mathrm{p}}(\widetilde{A}) = \sigma_{\mathrm{p}}(A)$. The eigenvalues of an $\mathcal{H}(p)$ operator being normal, we have $\alpha(A-\lambda) = \beta(A-\lambda) < \infty$, i.e. $A-\lambda \in \Phi_0(H)$. ∎

THEOREM 3.5. *If either* A *or* A^* *is in* $\mathcal{H}(p)$, *then*

$$A-\lambda \in \Phi_0(H) \Leftrightarrow \widetilde{A}-\lambda \in \Phi_0(H).$$

Proof. Suppose that $A \in \mathcal{H}(p)$ and $A-\lambda \in \Phi_0(H)$; then $\mathcal{R}(A-\lambda)$ is closed, either $\alpha(A-\lambda)$ or $\beta(A-\lambda)$ is finite and $i(A-\lambda) = 0$. The eigenvalues of an $\mathcal{H}(p)$ operator being normal, it follows that $\mathcal{N}(A-\lambda) \subseteq \mathcal{N}((A-\lambda)^*)$. Hence $\alpha(A-\lambda) = \beta(A-\lambda) < \infty$ with $\mathcal{N}(A-\lambda) = \mathcal{N}((A-\lambda)^k)$, for all $k = 1, 2, \ldots$ This, by [13, Theorem VI.4.5], implies that λ is an isolated point of $\sigma(A)$, and hence (by Lemma 3.4), $\widetilde{A}-\lambda \in \Phi_0(H)$.

Conversely, if $\widetilde{A}-\lambda \in \Phi_0(H)$, then $\lambda \in \sigma_{\mathrm{p}}(\widetilde{A}) = \sigma_{\mathrm{p}}(A)$ and $\alpha(\widetilde{A}-\lambda) = \alpha(A-\lambda) < \infty$. Hence $A-\lambda \in \Phi_0(H)$.

Since a similar argument works if $A^* \in \mathcal{H}(p)$, the proof is complete. ∎

4. Weyl's theorems.

It is shown in [5] that Weyl's theorem holds for the class $\mathcal{H}(p)$. We show that it also holds if A^* is p-hyponormal.

PROPOSITION 4.1. *If either A or A^* is a p-hyponormal operator, then A obeys Weyl's theorem, i.e.*

$$\sigma_{\mathrm{w}}(A) = \sigma(A) \setminus \pi_{00}(A).$$

Proof. If A is p-hyponormal, then by [5, Theorem 0] A obeys Weyl's theorem.

Now suppose that A^* is p-hyponormal. Then $\widetilde{A}^*$ is hyponormal, and since $\sigma(\widetilde{A}^*) = \sigma(\widetilde{A})^*$, $\sigma_{\mathrm{w}}(\widetilde{A}^*) = \sigma_{\mathrm{w}}(\widetilde{A})^*$ and $\pi_{00}(\widetilde{A}^*) = \pi_{00}(\widetilde{A})^*$, where $S^* = \{\overline{\lambda} : \lambda \in S\}$ for $S \subset \mathbb{C}$, we see that $\widetilde{A}$ obeys Weyl's theorem. Now, since $\sigma(A) = \sigma(\widetilde{A})$ and $\pi_{00}(A) = \pi_{00}(\widetilde{A})$, Theorem 3.5 implies that

$$\sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{w}}(\widetilde{A}) = \sigma(\widetilde{A}) \setminus \pi_{00}(\widetilde{A}) = \sigma(A) \setminus \pi_{00}(A).\ \square$$

By [12], a-Weyl's theorem holds in the class $\mathcal{H}(1)$; the following theorem shows that this is also true for $\mathcal{H}(p)$ operators.

THEOREM 4.2. (i) *If A^* is a p-hyponormal operator, then A obeys a-Weyl's theorem, i.e.*

$$\sigma_{\mathrm{ea}}(A) = \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

(ii) *If A is a p-hyponormal operator such that the points of $\pi_{00}^{\mathrm{a}}(A)$ are also isolated in $\sigma(A)$, then A obeys a-Weyl's theorem.*

Proof. (i) If A^* is p-hyponormal, then since $\sigma(A) = \sigma_{\mathrm{a}}(A)$ [4, Corollary 6], it follows that $\pi_{00}(A) = \pi_{00}^{\mathrm{a}}(A)$. Now, since A obeys Weyl's theorem (Proposition 4.1) we have

$$\sigma_{\mathrm{ea}}(A) \subset \sigma_{\mathrm{w}}(A) = \sigma(A) \setminus \pi_{00}(A) = \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

Let $\{\lambda_n\}$ be an infinite sequence of different points in $\pi_{0\mathrm{f}}^{\mathrm{a}}(A) = \pi_{0\mathrm{f}}(A)$ and let $\{x_n\}$ be a sequence of corresponding normalized eigenvectors. Then $\{\overline{\lambda}_n\}$ is a sequence of eigenvalues of T^* with same sequence of eigenvectors $\{x_n\}$ (see [4]). By [4, Corollary 5], $\{x_n\}$ has no convergent subsequence, i.e. A obeys condition C_1^{a}. Since

$$\sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A)$$

by Theorem 2.5, we conclude that a-Weyl's theorem holds for A.

(ii) Let A be p-hyponormal and let $\lambda \in \pi_{00}^{\mathrm{a}}(A)$. Then λ is an isolated point of $\sigma(A)$ and $\alpha(A-\lambda) < \infty$, and so, by Lemma 3.4, the operator $A-\lambda$ has closed range. Consequently, A obeys condition C_2^{a} , and it follows from Theorem 2.6 that

$$\sigma_{\mathrm{ea}}(A) \subset \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A).$$

Let $\{\lambda_n\}$ be an infinite sequence of distinct point in $\pi_{0\mathrm{f}}^{\mathrm{a}}(A)$. Then, by [4, Corollary 5], $(x_n, x_m) = 0$, $n \neq m$, for a sequence of normalized eigenvectors $\{x_n\}$ corresponding to $\{\lambda_n\}$. Thus $\{x_n\}$ does not converge, A obeys condition C_1^{a}, and it follows from Theorem 2.5 that

$$\sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A) \subset \sigma_{\mathrm{ea}}(A).$$

Hence, A obeys a-Weyl's theorem. ∎

We remark here that the hypothesis in (ii) that the points of $\pi_{00}^{\mathrm{a}}(A)$ are also isolated in $\sigma(A)$ is in general not satisfied. (We are grateful to Dr. Young Min Han for pointing this out.) Consider for example the hyponormal operator A which is the direct sum of the 1-dimensional zero operator and the unilateral shift. Then $0 \in \pi_{00}^{\mathrm{a}}(A)$, but 0 is not an isolated point of $\sigma(A)$.

THEOREM 4.3. *If A or A^* is in $\mathcal{H}(p)$, then $\sigma_{\mathrm{ea}}(A) = \sigma_{\mathrm{ea}}(\widetilde{A})$.*

Proof. Since A (resp. A^*) $\in \mathcal{H}(p)$ implies that $\widetilde{A}$ (resp. $\widetilde{A}^*$) $\in \mathcal{H}(1)$, by Lemma 3.3 and Theorem 4.2 we have

$$\sigma_{\mathrm{ea}}(\widetilde{A}) = \sigma_{\mathrm{a}}(\widetilde{A}) \setminus \pi_{00}^{\mathrm{a}}(\widetilde{A}) = \sigma_{\mathrm{a}}(A) \setminus \pi_{00}^{\mathrm{a}}(A) = \sigma_{\mathrm{ea}}(A).\ \square$$

Recall that the spectrum is a continuous function in the class of hyponormal operators. Our next result says that the same is true for the class $\mathcal{H}(p)$; the proof depends on the Berberian extension theorem, which we now state.

THEOREM 4.4 ([3]). *There exists a Hilbert space $H^0 \supset H$ and an isometric order preserving $*$-isomorphism $B(H) \ni A \mapsto A^0 \in B(H^0)$ such that*

$$\sigma(A) = \sigma(A^0) \quad \textit{and} \quad \sigma_{\mathrm{a}}(A) = \sigma_{\mathrm{a}}(A^0) = \sigma_{\mathrm{p}}(A^0).$$

THEOREM 4.5. *Let A_n or A_n^* be p-hyponormal, for all $n = 1, 2, \ldots$, and let the sequence $\{A_n\}$ converge in norm to A. Then $\lim \sigma(A_n) = \sigma(A)$, i.e. the spectrum is a continuous function in the class of all p-hyponormal operators.*

Proof. We start by proving that the Berberian extension A_m^0 of $A_m \in \mathcal{H}(p)$ is similar to an $\mathcal{H}(1)$ operator. Let $A_m^0 = T_m$. Then $T_m \in \mathcal{H}(p)$, and either $0 \in \sigma_{\mathrm{p}}(T_m)$ or $0 \notin \sigma_{\mathrm{p}}(T_m)$.

If $0 \in \sigma_{\mathrm{p}}(T_m)$, then 0 is in the joint spectrum of T_m and there exists a decomposition $T_m = 0 \oplus T_{m1}$, on $H^0 = H_0 \oplus H_1$ say, such that $0 \notin \sigma_{\mathrm{p}}(T_{m1})$. We claim that $0 \notin \sigma(|T_{m1}|)$. Suppose to the contrary that $0 \in \sigma(|T_{m1}|) = \sigma_\pi(|T_{m1}|)$. Then there exists a sequence $\{x_r\}$ of unit vectors such that $|T_{m1}|x_r \to 0$ as $r \to \infty$. But then $T_{m1}x_r \to 0$ as $r \to \infty$, i.e., $0 \in \sigma_\pi(T_{m1}) = \sigma_{\mathrm{p}}(T_{m1})$. This contradiction proves our claim.

Let T_{m1} have the polar decomposition $T_{m1} = U_{m1}|T_{m1}|$. Define $\widehat{T}_{m1} = |T_{m1}|^{1/2}U_{m1}|T_{m1}|^{1/2}$. Then $\widehat{T}_{m1} \in \mathcal{H}(p+1/2)$, with $\sigma(\widehat{T}_{m1}) = \sigma(T_{m1})$ and $\sigma_{\mathrm{p}}(\widehat{T}_{m1}) = \sigma_{\mathrm{p}}(T_{m1})$ ([8]). In particular, $0 \notin \sigma(|\widehat{T}_{m1}|)$. Let $\widehat{T}_{m1}$ have the polar decomposition $\widehat{T}_{m1} = V_{m1}|\widehat{T}_{m1}|$, and define $\widetilde{T}_{m1} = |\widehat{T}_{m1}|^{1/2}V_{m1}|\widehat{T}_{m1}|^{1/2}$. Then $\widetilde{T}_{m1} \in \mathcal{H}(1)$ with $\sigma(\widetilde{T}_{m1}) = \sigma(T_{m1})$ ([8]). Now let X_m be the invertible operator $X_m = 1_{|H_0} \oplus |\widehat{T}_{m1}|^{1/2}|T_{m1}|^{1/2}$, and let $\widetilde{T}_m \in \mathcal{H}(1)$ be the operator $\widetilde{T}_m = 0 \oplus \widetilde{T}_{m1}$. Then $\widetilde{T}_m = X_m T_m X_m^{-1}$.

In the case $0 \notin \sigma_{\mathrm{p}}(T_m)$, an argument similar to the one above shows that $0 \notin \sigma(|T_m|)$ and $0 \notin \sigma(|\widehat{T}_m|)$, where (upon letting $T_m = U_m|T_m|$) $\widehat{T}_m$ is defined by $\widehat{T}_m = |T_m|^{1/2}U_m|T_m|^{1/2}$. Let $\widehat{T}_m = V_m|\widehat{T}_m|$ and $\widetilde{A}_m = |\widehat{T}_m|^{1/2}V_m|\widehat{T}_m|^{1/2}$. Then $\widetilde{T}_m \in \mathcal{H}(1)$ and $\widetilde{T}_m = X_m T_m X_m^{-1}$, where $X_m = |\widehat{T}_m|^{1/2}|T_m|^{1/2}$.

We note that $\|A_m - A\| \to 0$ implies $\|T_m - T\| = \|A_m^0 - A^0\| \to 0$ as $m \to \infty$; hence, given $\varepsilon > 0$, there exists a natural number m_0 such that

$$\|\,|T_m|^2\| \le \|T_m^*\| \cdot \|T_m - T\| + \|T\| \cdot \|T_m^* - T^*\| + \|\,|T|^2\| \le \|\,|T|^2\| + \varepsilon$$

and

$$\|\,|\widehat{T}_m|^2\| = \|\,|T_{mn} \oplus \widehat{T}_{mp}|^2\| \le \|T\|^2 + \varepsilon$$

for all $m > m_0$. In particular, $|T_m|$, $|\widehat{T}_m|$ and X_m are uniformly bounded.

Since the spectrum of an operator is upper semicontinuous [10], we have to show that $\sigma(A) \subset \liminf \sigma(A_m)$. Suppose that the contrary holds. Then $\sigma(T) \not\subseteq \liminf \sigma(T_m)$, and given $\varepsilon > 0$ we can find a natural number m_1 and a sequence $\{\lambda_m\} \subset \mathbb{C}$ such that $\lambda_m \in \sigma(T) \backslash \sigma(T_m)$ for all $m \ge m_1$. Let $\lambda \in \mathbb{C}$ be a point of accumulation of $\{\lambda_m\}$. Then there exists a natural number m_2 such that $\lambda \in \sigma(T) \setminus \sigma(T_m)$ for all $m \ge m_2$. Since $\sigma(T_m) = \sigma(\widetilde{T}_m)$, this implies that $\widetilde{T}_m - \lambda$ is regular for all $m \ge m_2$. The operator $\widetilde{T}_m - \lambda$ being hyponormal,

$$\|(\widetilde{T}_m - \lambda)^{-1}\| = r(\widetilde{T}_m - \lambda)^{-1} = \max\{1/|\lambda - \mu| : \mu \in \sigma(\widetilde{T}_m)\},$$

i.e. $(\widetilde{T}_m - \lambda)^{-1}$ is uniformly bounded for all $m \ge m_2$. We have

$$\begin{aligned}
&\|1_{|H^0} - (T_m - \lambda)^{-1}(T - \lambda)\| \\
&\quad = \|1_{|H^0} - X_m(\widetilde{T}_m - \lambda)^{-1}X_m^{-1}(T - \lambda)\| \\
&\quad = \|X_m(\widetilde{T}_m - \lambda)^{-1}X_m^{-1}[(X_m^{-1}(\widetilde{T}_m - \lambda)X_m - (T - \lambda)]\| \\
&\quad \le \|X_m(\widetilde{T}_m - \lambda)^{-1}X_m^{-1}\| \cdot \|T_m - T\| \to 0
\end{aligned}$$

as $m \to \infty$, i.e. $T - \lambda$ is invertible. This contradiction implies that we must have

$$\sigma(A) = \sigma(T) \subset \liminf \sigma(T_m) = \liminf \sigma(A_m).$$

The proof in the case $A_m^* \in \mathcal{H}(p)$ is similar. (We note that $\|(\widetilde{T}_m - \lambda)^{-1}\| = r(\widetilde{T}_m - \lambda)^{-1}$ if $\widetilde{T}_m^*$ is hyponormal.) ∎

Corollary 4.6. *Let A_n or A_n^* be p-hyponormal, for every $n \in \mathbb{N}$, and let the sequence $\{A_n\}$ converge in norm to A. Then*

$$\lim \sigma_{\mathrm{w}}(A_n) = \sigma_{\mathrm{w}}(A) \quad \textit{and} \quad \lim \sigma_{\mathrm{b}}(A_n) = \sigma_{\mathrm{b}}(A),$$

i.e. the Weyl spectrum and Browder spectrum are continuous functions in the class of all p-hyponormal operators.

Proof. Since the spectrum is continuous, the assertion follows from [7, Theorems 2.2 and 2.3]. ∎

Corollary 4.7. *Let A_n^* be p-hyponormal, for every $n \in \mathbb{N}$, and let the sequence $\{A_n\}$ converge in norm to A. Then $\lim \sigma_{\mathrm{a}}(A_n) = \sigma_{\mathrm{a}}(A)$.*

Proof. Since A_n^* are p-hyponormal operators, [3, Corollary 6] shows that $\sigma(A_n) = \sigma_{\mathrm{a}}(A_n)$. Theorem 4.5 now implies that $\lim \sigma_{\mathrm{a}}(A_n) = \sigma_{\mathrm{a}}(A)$. ∎

Theorem 4.8. *If $\{A_n^*\}$ is a sequence in $\mathcal{H}(p)$ such that $A_n \to A \in B(H)$, then $\lim \sigma_{\mathrm{ea}}(A_n) = \sigma_{\mathrm{ea}}(A)$.*

Proof. Since σ_{ea} is upper semicontinuous [6, Theorem 2.1], we have to show that $\sigma_{\mathrm{ea}}(A) \subset \liminf \sigma_{\mathrm{ea}}(A_n)$. Suppose the contrary; then there exists $\varepsilon > 0$ and, for every $n \in \mathbb{N}$, a $\lambda_n \in \sigma_{\mathrm{ea}}(A)$ such that $\lambda_n \notin (\sigma_{\mathrm{ea}}(A_n))_\varepsilon$. Also, we can suppose that $\lambda_n \to \lambda \in \sigma_{\mathrm{ea}}(A)$. Then there exists $n_0 \in \mathbb{N}$ such that $|\lambda - \lambda_n| < \varepsilon/2$ for every $n > n_0$. Now, for $n > n_0$ we have

$$d(\lambda, \sigma_{\mathrm{ea}}(A_n)) \ge d(\lambda_n, \sigma_{\mathrm{ea}}(A_n)) - |\lambda_n - \lambda| \ge \varepsilon/2,$$

i.e. $\lambda \notin \sigma_{\mathrm{ea}}(A_n)$ for every $n > n_0$. The operator A_n^* being p-hyponormal, $\widetilde{A}_n^*$ is hyponormal and it follows that

$$\begin{aligned}
\beta(A_n - \lambda) &= \alpha(A_n - \lambda)^* = \alpha(\widetilde{A}_n - \lambda)^* = \beta(\widetilde{A}_n - \lambda), \\
\alpha(A_n - \lambda) &= \beta(A_n - \lambda)^* \ge \beta(\widetilde{A}_n - \lambda)^* = \alpha(\widetilde{A}_n - \lambda).
\end{aligned}$$

Since $\widetilde{A}_n^* \in \mathcal{H}(1)$, we also have

$$\mathcal{N}(\widetilde{A}_n - \alpha)^* \subseteq \mathcal{N}(\widetilde{A}_n - \alpha).$$

Thus

$$i(A_n - \lambda) = \alpha(A_n - \lambda) - \beta(A_n - \lambda) \ge \alpha(\widetilde{A}_n - \lambda) - \beta(\widetilde{A}_n - \lambda) \ge 0.$$

Since $\lambda \notin \sigma_{\mathrm{ea}}(A_m)$ implies that $i(A_n - \lambda) \le 0$ (with $\alpha(A_n - \lambda) < \infty$), we must have $i(A_n - \lambda) = 0$ with $\alpha(A_n - \lambda) = \beta(A_n - \lambda) < \infty$. By the continuity of the index we now conclude that $i(A - \lambda) = 0$ and $\alpha(A - \lambda) < \infty$, i.e. $\lambda \notin \sigma_{\mathrm{ea}}(A)$. This contradiction proves the result. ∎

Corollary 4.9. *If $\{A_n^*\}$ is a sequence in $\mathcal{H}(p)$ such that $A_n \to A \in B(H)$, then $\lim \sigma_{\mathrm{ab}}(A_n) = \sigma_{\mathrm{ab}}(A)$.*

Proof. By Theorem 4.8, Theorem 4.2 and [6, Corollary 2.7]. ∎

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Banach principle in the space of τ-measurable operators

by

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Abstract. We establish a non-commutative analog of the classical Banach Principle on the almost everywhere convergence of sequences of measurable functions. The result is stated in terms of quasi-uniform (or almost uniform) convergence of sequences of measurable (with respect to a trace) operators affiliated with a semifinite von Neumann algebra. Then we discuss possible applications of this result.

Introduction. The study of measurable operators associated with a von Neumann algebra (vNA) and different types of the almost everywhere convergence for sequences of measurable operators goes back to the celebrated paper of I. Segal [Se]. Since then this branch of the theory of operator algebras has been explored in many different directions. One of them is the so-called non-commutative ergodic theory, which treats the almost everywhere (or norm) convergence of the Cesàro averages along the trajectory (under some kind of contraction in a non-commutative L^p-space) of an operator in L^p. This study was initiated by a number of authors, among whom we mention Lance [La] and Yeadon [Ye]. In the classical ergodic theory, one of the most powerful tools in dealing with the almost everywhere convergence of ergodic averages is the well-known Banach Principle on the convergence of sequences of measurable functions generated by a sequence of linear maps in an L^p-space. This principle is often applied in proofs concerning the almost everywhere convergence of weighted averages, averages along subsequences, moving averages, etc.

In this paper, using the notion of τ-measurable operator, we establish a non-commutative analog of the Banach Principle. Since we do not assume the finiteness of the trace, the result is stated for the quasi-uniform convergence. The proof of the main result of this paper, Theorem 2, can be easily modified for different types of the "almost everywhere" convergences in vNA, in particular, for the bilateral almost uniform (b.a.u.) convergence

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