## Contents of Volume 143, Number 1




STUDIA MATHEMATICA

*Executive Editors*: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (2000): Vols. 138–143 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl





Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman

PRINTED IN POLAND

ISSN 0039-3223




# Universal divisors in Hardy spaces


by

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**Abstract.** We study a division problem in the Hardy classes $H^p(\mathbb{B})$ of the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ which generalizes the $H^p$ corona problem, the generators being allowed to have common zeros. Precisely, if $S$ is a subset of $\mathbb{B}$, we study conditions on a $\mathbb{C}^k$-valued bounded holomorphic function $B$, with $B_{|S} = 0$, in order that for $1 \le p < \infty$ and any function $f \in H^p(\mathbb{B})$ with $f_{|S} = 0$ there is a $\mathbb{C}^k$-valued $H^p(\mathbb{B})$ holomorphic function $F$ with $f = B \cdot F$, i.e. the module generated by the components of $B$ in the Hardy class $H^p(\mathbb{B})$ is the entire module $M_S := \{f \in H^p(\mathbb{B}) : f_{|S} = 0\}$. As a special case, for $S = \emptyset$, we get the $H^p$ corona theorem.


**1. Introduction.** Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$, $H^p(\mathbb{B})$, $1 \le p \le \infty$, the Hardy classes of $\mathbb{B}$ and $S \subset \mathbb{B}$ a subset of $\mathbb{B}$.

DEFINITION 1.1. Let $B = (B_1, \ldots, B_N) \in (H^\infty(\mathbb{B}))^N$. We shall say that $B$ is a *universal divisor* (of dimension $N$) for $S$ if $B_{|S} = 0$ and for any $1 \le p < \infty$ and any function $f \in H^p(\mathbb{B})$ with $f_{|S} = 0$, there is a $\mathbb{C}^N$-valued $H^p(\mathbb{B})$ function $F$ with $f = B \cdot F := \sum_{i=1}^N B_i F_i$.

We shall say that $S$ is the *support* of $B$.

EXAMPLES. • $n = 1$, $S = \{\alpha_i \in \mathbb{D} : i \in \mathbb{N}\}$ a Blaschke sequence. Then the associated Blaschke product is a universal divisor and there is no other nonempty set $S$ which can be the support of a universal divisor.

• $n = 2$, $B = (B_1, B_2) \in H^\infty(\mathbb{B})^2$ with $|B|^2(z) := |B_1(z)|^2 + |B_2(z)|^2 \ge \delta^2 > 0$ for all $z \in \mathbb{B}$, and $S = \emptyset$. Then $B$ is a universal divisor for $\emptyset$ (the $H^p(\mathbb{B})$ corona theorem [2]), i.e.

$$\forall p \in [1, \infty[,\ \forall f \in H^p(\mathbb{B}),\ \exists F \in H^p(\mathbb{B})^2, \quad f = B \cdot F.$$

The aim of this paper is to generalize these examples. Let us give another definition, with $\Phi_a$ an automorphism of $\mathbb{B}$ interchanging $a$ and 0.

DEFINITION 1.2. Let $S$ be a sequence of points in $\mathbb{B} \subset \mathbb{C}^n$. We shall say that a $\mathbb{C}^N$-valued bounded holomorphic function $B := (B_1, \ldots, B_N)$ is *$N$-strongly defining* for $S$ if $B_{|S} = 0$ and:


2000 *Mathematics Subject Classification*: 32A35, 32A26.

(1) there are $n$ functions among the $B_i$, $\widetilde{B} := (B_1, \ldots, B_n)$ say, such that for all $a \in S$, $\widetilde{B} = M_a \cdot \Phi_a$ with an $n \times n$ matrix $M_a$ in $H^\infty(\mathbb{B})$ and satisfying:

$$|M_a|_{\mathrm{op}} \le \delta^{-1} \text{ on } \mathbb{B} \quad \text{and} \quad |M_a^{-1}|_{\mathrm{op}} \le \delta^{-1} \text{ on } \{|\Phi_a| < \delta\}.$$

(2) $\forall \varepsilon > 0, \exists \eta > 0, (z \in \bigcap_{a\in S}\{|\Phi_a| \ge \varepsilon\} \Rightarrow |B(z)| \ge \eta)$.

In that case we shall say that $S$ is *$N$-strongly defined.*

REMARK 1.3. The conditions in this definition imply that

$$S = B^{-1}(0) \quad \text{and} \quad \exists \eta > 0,\ |B^*| \ge \eta \text{ a.e. on } \partial\mathbb{B}.$$

Indeed, the condition (1) implies that the sets $\{|\Phi_a| < \delta\}$, $a \in S$, are disjoint ([3], Proposition 3.1); because $B$ is in $H^\infty(\mathbb{B})^N$, it has radial boundary values $B^*$ a.e. on $\partial\mathbb{B}$, hence if $\zeta \in \partial\mathbb{B}$ is a point where $B(r\zeta)$ admits a limit $B^*(\zeta)$ as $r \to 1$, choose points $z_n$ on the ray $\{r\zeta : r \in [0, 1[\}$ and in $\bigcap_{a\in S}\{|\Phi_a| \ge \delta\}$ to get $|B^*(\zeta)| = \lim_{n\to\infty} |B(z_n)| \ge \eta$.

Recall that a sequence $S = \{a_i : i \in \mathbb{N}\} \subset \mathbb{B}$ is *interpolating* for $H^\infty(\mathbb{B})$ (respectively for $\bigcap_{p<\infty} H^p(\mathbb{B})$) if for any $\lambda = \{\lambda_i : i \in \mathbb{N}\}$ in $l^\infty(\mathbb{N})$ there exists $f$ in $H^\infty(\mathbb{B})$ (respectively in $\bigcap_{p<\infty} H^p(\mathbb{B})$) such that $f(a_i) = \lambda_i$ for all $i$.

In [3] it is proved that $S$ interpolating for $H^\infty(\mathbb{B})$ implies that there are $n$ bounded holomorphic functions $B_i$ with the property (1) but we shall also need the other property and to get it the number of the $B_i$'s has to be increased.

In the same paper [3] it was proved that if there is a $\widetilde{B}$ with the property (1), then $S$ is an interpolating sequence for $\bigcap_{p<\infty} H^p(\mathbb{B})$.

The very first example is the automorphism $\Phi_a$ itself and, in Section 2, we get

THEOREM 1.4. *Let $f \in H^p(\mathbb{B})$ be such that $f(a) = 0$. Then there is a vector-valued function $F$ in $H^p(\mathbb{B})^n$ such that $f = \Phi_a \cdot F$ with $\|F\|_p \le C_p\|f\|_p$, the constant $C_p$ being independent of $a \in \mathbb{B}$.*

The main result of Section 2 is

THEOREM 1.5. *If $S$ is an $H^\infty(\mathbb{B})$ interpolating sequence, then there is an $(n+2)$-strongly defining function $B$ for $S$.*

We do not know, for $S$ an interpolating sequence in $\mathbb{B} \subset \mathbb{C}^n$, if there is an $n$-strongly defining function, but in order to get the fact that an interpolating sequence is the support of a universal divisor this is not necessary because we have



THEOREM 1.6. *If $B$ is an $N$-strongly defining function for $S$ in the unit ball of $\mathbb{C}^2$, then $B$ is a universal divisor for $S$ in $H^p(\mathbb{B})$.*

This is the main result of Section 3. We state and prove this theorem in $\mathbb{C}^2$. Using the recent solution of the $\overline{\partial}$-equation by Andersson and Carlsson [5], our theorem may generalize to the unit ball of $\mathbb{C}^n$.

COROLLARY 1.7. *Let $B := (B_1, \ldots, B_k) \in H^\infty(\mathbb{B})^k$ be such that*

$$\sum_{i=1}^{k} |B_i|^2 \ge \delta^2 > 0 \quad \text{in } \mathbb{B}.$$

*Then for any function $f \in H^p(\mathbb{B})$ there is a vector-valued function $F \in H^p(\mathbb{B})^k$ such that $f = B \cdot F$.*

In order to prove this take $S = \emptyset$ and the hypothesis is just what is needed to know that $B$ is $k$-strongly defining for $S$. Then we get the $H^p$ corona theorem for any number of generators in the unit ball of $\mathbb{C}^2$.

At this point it is worthwhile to mention the well known results of G. Henkin [6] and N. Varopoulos [11] about the corona problem in the ball $\mathbb{B}$ of $\mathbb{C}^n$.

THEOREM 1.8 (Henkin, Varopoulos). *Let $g_1, \ldots, g_N \in H^\infty(\mathbb{B})$ be such that*

$$\sum_{i=1}^{k} |g_i|^2 \ge \delta^2 > 0 \quad \text{in } \mathbb{B}.$$

*Then there exist $f_1, \ldots, f_N \in \bigcap_{p\ge1} H^p(\mathbb{B})$ solving*

$$f_1g_1 + \ldots + f_Ng_N = 1.$$

This theorem can be deduced from Corollary 1.7 because $1 \in \bigcap_{p\ge1} H^p(\mathbb{B})$ and our solutions $f_1, \ldots, f_N$ depend only on the data $f$ and on $g_1, \ldots, g_N$ but not on $p$. But this theorem does not imply the corollary.

As another corollary we get a result of C. Horowitz [8]:

COROLLARY 1.9 (C. Horowitz). *Let $\Phi$ be an interpolating Blaschke product in the unit disc $\mathbb{D}$ in $\mathbb{C}$. Then $\Phi$ is a universal divisor for $\sigma = \Phi^{-1}(0)$ and for the Bergman classes $B^p(\mathbb{D})$, for any $p \in [1, \infty[$.*

This work started when the first author was visiting the Catalan Institute of Mathematics under the European Program Picasso.

He benefited from many discussions with P. Ahern (who also gave a proof of Theorem 2.1), J. Bruna and E. Doubtsov on these topics. In particular E. Doubtsov proved the existence of an inner function $\varphi$ in $\mathbb{B}$ such that if $f \in H^p(\mathbb{B})$ and $f = \varphi g$ with $g$ holomorphic in $\mathbb{B}$, then $g \in H^p(\mathbb{B})$ (a "good" inner function in Rudin's sense); and P. Ahern, refining this result, proved that there are good inner functions $\varphi$ such that if $S := \varphi^{-1}(0)$ and

$f \in H^p(\mathbb{B})$ with $f_{|S} = 0$ then there is a $g \in H^p(\mathbb{B})$ with $f = \varphi g$, proving that there are supports of universal divisors of codimension one.

## 2. Existence of strongly defining functions

THEOREM 2.1. *Let $f \in H^p(\mathbb{B})$ be such that $f(a) = 0$. Then there is a vector-valued function $F$ in $H^p(\mathbb{B})^n$ such that $f = \Phi_a \cdot F$ with $\|F\|_p \le C\|f\|_p$, the constant $C_p$ being independent of $a \in \mathbb{B}$.*

Proof. Let $Z = \Phi_a(z)$ be a change of variables. Then the jacobian for the Lebesgue measure on $\partial\mathbb{B}$ is

$$J_a(z) = \frac{(1-|a|^2)^n}{|1-\bar{a}z|^{2n}} = P(a,z) \quad \text{(i.e. the Poisson kernel);}$$

this jacobian can be written as

$$J_a(z) = |k_a(z)|^p \quad \text{with} \quad k_a(z) := \frac{(1-|a|^2)^{n/p}}{(1-\bar{a}z)^{2n/p}},$$

the function $k_a$ being holomorphic in $\mathbb{B}$.

Now $g(z) := f \circ \Phi_a(z)$ is such that

$$g(0) = 0 \quad \text{and} \quad \int_{\partial\mathbb{B}} |g|^p J_a(z)\, dv(z) = \|f\|_p^p,$$

hence if we put $h(z) := g(z)k_a(z)$, we get $h(0) = 0$ and $\|h\|_p = \|f\|_p$.

We are now in a position to apply the theorem of Ahern and Schneider [9, p. 115]: there is a constant $C > 0$ and a vector-valued function $H$ such that

$$h(z) = z \cdot H(z) \quad \text{and} \quad \|H\|_p \le C\|h\|_p.$$

Let $G(z) := k_a(z)^{-1}H(z)$. Then

$$\int_{\partial\mathbb{B}} |G|^p J_a\, dv = \|H\|_p^p \le C^p\|h\|_p^p = C^p\|f\|_p^p.$$

Finally let $F(z) := G \circ \Phi_a(z)$. We get

$$\|F\|_p^p = \int_{\partial\mathbb{B}} |F|^p\, dv = \int_{\partial\mathbb{B}} |G \circ \Phi_a|^p\, dv = \int_{\partial\mathbb{B}} |G|^p J_a\, dv \le C^p\|f\|_p^p,$$

and

$$\begin{aligned} f(z) &= g \circ \Phi_a(z) = h \circ \Phi_a(z) k_a^{-1} \circ \Phi_a(z) = \Phi_a \cdot H \circ \Phi_a k_a^{-1} \circ \Phi_a(z) \\ &= \Phi_a(z) \cdot G \circ \Phi_a(z) = \Phi_a(z) \cdot F(z). \square \end{aligned}$$

Now we prove the main result of this section:

THEOREM 2.2. *If $S$ is an $H^\infty(\mathbb{B})$ interpolating sequence, then there is an $(n+2)$-strongly defining function $B$ for $S$.*



Proof. First, recall that there is a sequence $(\beta_i)_{i\in\mathbb{N}} \subset H^\infty(\mathbb{B})$ such that $\beta_i(a_j) = \delta_{ij}$ for each $a_j \in S$, and $\sum_{i=1}^\infty |\beta_i(z)| \le C$ for all $z \in \mathbb{B}$, where the constant $C$ only depends on the sequence $S$ [3, Section 2].

For every integer $k$, let

$$H_k := \beta_k \prod_{i\ne k} (1-\beta_i).$$

Note that $H_k \in H^\infty(\mathbb{B})$, $H_k(a_j) = \delta_{jk}$ and $\sum_{i=1}^\infty |H_i(z)| \le Ce^C$ for all $z \in \mathbb{B}$.

The $(n+2)$-strongly defining function $B$ will be

$$B_i := \sum_k H_k \Phi_k^i, \quad 1 \le i \le n,$$

$$B_{n+1} := \prod_k (1-\beta_k),$$

$$B_{n+2} := \prod_k (1-H_k),$$

where $\Phi_k^i$ is the $i$th component of $\Phi_{a_k}$. One can verify that $B \in (H^\infty(\mathbb{B}))^{n+2}$, $B_{|S} = 0$ and because the sequence $(H_k)_k$ has the same properties as the sequence $(\beta_k)_k$, the proof of [3, Section 2] shows that $\widetilde{B} = (B_1, \ldots, B_n)$ satisfies condition (1).

Let $z \in \bigcap_{a\in S}\{|\Phi_a| > \delta\}$ and assume that $|B_{n+1}(z)| < \eta$ and $|B_{n+2}(z)| < \eta$.

Since $\sum_k |\beta_k(z)| \le C$, the set $I$ of indices such that $|\beta_i(z)| > 1/2$ for all $i \in I$ is finite; one can remark that $|I| \le 2C$ for all $z \in \mathbb{B}$.

Since $1 - x \ge e^{-2x}$ for all $x \in [0, 1/2]$, we have

$$e^{-2\sum_{k\notin I}|\beta_k(z)|} \prod_{i\in I} |1-\beta_i(z)| \le |B_{n+1}(z)| < \eta, \qquad \prod_{i\in I} |1-\beta_i(z)| < \eta e^{2C}.$$

Thus there exists $i \in I$ such that $|1-\beta_i(z)| < (\eta e^{2C})^{1/|I|}$ and because $|H_k(z)| \le |1-\beta_i(z)| \cdot |\beta_k(z)| e^C$ for $k \ne i$, we have

$$\sum_{k\ne i} |H_k(z)| < Ce^C(\eta e^{2C})^{1/|I|} \le C'\eta^{1/(2C)} \le 1/2$$

for $\eta$ small enough. Therefore

$$e^{-2\sum_{k\ne i}|H_k(z)|} |1-H_i(z)| \le |B_{n+2}(z)| < \eta, \qquad |1-H_i(z)| < \eta e.$$

To conclude one can see that

$$\begin{aligned} |\widetilde{B}(z)| &\ge |H_i(z)| \cdot |\Phi_i(z)| - \sum_{k\ne i} |H_k(z)| \cdot |\Phi_k(z)| \\ &\ge (1-\eta e)\varepsilon - C'\eta^{1/(2C)} \ge \eta \end{aligned}$$

for $\eta$ small enough. ■

## 3. Universal divisors in $H^\infty(\mathbb{B})$

THEOREM 3.1. *If $B$ is an $N$-strongly defining function for $S$ in the unit ball of $\mathbb{C}^2$, then $B$ is a universal divisor for $S$ in $H^p(\mathbb{B})$, $1 \le p < \infty$.*

The proof will necessitate some preliminaries.

Let $f \in H^p(\mathbb{B})$. Then we can write

$$f = \sum_{j=1}^{N} B_j f \frac{\overline{B}_j}{|B|^2} = B \cdot H \quad \text{with} \quad H := f\left(\frac{\overline{B}_1}{|B|^2}, \ldots, \frac{\overline{B}_N}{|B|^2}\right).$$

Because $H$ is not holomorphic, we have to correct it. Unfortunately this involves $(0,2)$-forms and terms of the kind $\overline{\partial}(\overline{B}_l/|B|^2) \wedge \overline{\partial}(\overline{B}_m/|B|^2)$ which are not even integrable near a point of $S$, the common zeros of the $B_l$'s.

Hence we must modify $H$ near $S$. In order to do this recall that by Theorem 2.1,

$$\forall a \in S,\ \exists F_a \in H^p(\mathbb{B})^n, \quad f = \Phi_a \cdot F_a.$$

Let $\widetilde{B} := (B_1, B_2)$ be the interpolating part of $B$, i.e. for all $a \in S$, $\widetilde{B} = M_a \cdot \Phi_a$, with $|M_a|_{\mathrm{op}} \le \delta^{-1}$ on $\mathbb{B}$ and $|M_a^{-1}|_{\mathrm{op}} \le \delta^{-1}$ on $\{|\Phi_a| < \delta\}$. Then setting

$$G_a := (({}^t M_a^{-1} F_a)_1, ({}^t M_a^{-1} F_a)_2, 0, \ldots, 0) \tag{3.1}$$

we get $f = B \cdot G_a$ on $\{|\Phi_a| < \delta\}$. Hence we are done near the points of $S$; now we mix the two solutions in the following step.

Because $S$ is $N$-strongly defined, the sets $\{|\Phi_a| < \delta\}$, $a \in S$, are disjoint, and we may set

$$\chi_S := \sum_i \chi\left(\frac{|\Phi_i|^2}{\delta^2}\right), \quad G := \sum_i \chi\left(\frac{|\Phi_i|^2}{\delta^2}\right) G_i,$$

with the shorter notations $\Phi_i := \Phi_{a_i}$ and $G_i := G_{a_i}$ for $a_i \in S$, and with $\chi$ being a $\mathcal{C}^\infty$ function defined on $\mathbb{R}_+$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $0 \le t \le 1/2$, and $\chi(t) = 0$ for $t \ge 1$.

Then we have $f = B \cdot L$ with

$$L := G + (1 - \chi_S) H. \tag{3.2}$$

Of course if $S = \emptyset$ we just take $\chi \equiv 0$, because then $|B| \ge \eta$ everywhere in $\mathbb{B}$.

Now the vector-valued function $L = (L_1, \ldots, L_N)$ is in $\mathcal{C}^\infty(\mathbb{B})$, but still is not holomorphic and we shall have to "correct" it, but this time we shall be able to do it with the right Carleson type estimates.

**3.1.** *Notations and definitions.* In the following $\varrho(z) := |z|^2 - 1$ is the defining function of the ball.



DEFINITION 3.2. Let $\omega$ be a $\overline{\partial}$-closed $(0,1)$-form in $L^1(\mathbb{B})$ and $f \in L^1(\partial\mathbb{B})$. We say that $\overline{\partial}_b f = \omega$ if

$$\forall \varphi \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\mathbb{B}}), \overline{\partial}\varphi = 0, \quad \int_{\partial\mathbb{B}} f\varphi = \int_{\mathbb{B}} \omega \wedge \varphi. \tag{3.3}$$

Let $f \in \mathcal{C}^1(\mathbb{B})$ and $\overline{\partial} f \in L^1(\mathbb{B})$. We say that $f^* \in L^1(\partial\mathbb{B})$ is a *Stokes boundary value* of $f$ if $\overline{\partial}_b f^* = \overline{\partial} f$.

At the end we want to deal with holomorphic functions, hence we shall need the next lemma.

LEMMA 3.3. *If $f^* \in L^p(\partial\mathbb{B})$, $1 \le p \le \infty$, and $\overline{\partial}_b f^* = 0$ then $f^*$ is the (usual) boundary value of a function in the Hardy class $H^p(\mathbb{B})$.*

Proof. Using Proposition (2.2) of [10], we know that there is a function $U \in L^1(\mathbb{B})$ such that $\overline{\partial} U = 0$, i.e. $U$ is holomorphic in $\mathbb{B}$, and

$$\forall \varphi \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\mathbb{B}}), \quad \int_{\partial\mathbb{B}} f^*\varphi = \int_{\mathbb{B}} U \overline{\partial}\varphi. \tag{3.4}$$

Let us apply this with

$$\varphi(\zeta) := P(z,\zeta)\partial\varrho(\zeta) \wedge (\partial\overline{\partial}\varrho(\zeta))^{n-1}$$

for $z$ fixed in $\mathbb{B}$, where $P$ is the Poisson–Szegő kernel of $\mathbb{B}$.

A simple computation gives, with $\beta := d\zeta_1 \wedge \ldots \wedge d\zeta_n$,

$$\overline{\partial}\varphi = c_n \frac{(-\varrho(z))^n}{(1 - \overline{z} \cdot \zeta)^n} \cdot \frac{1}{(1 - z \cdot \overline{\zeta})^{n+1}} \beta \wedge \overline{\beta}$$

and we notice that

$$c_n \frac{1}{(1 - z \cdot \overline{\zeta})^{n+1}} \beta \wedge \overline{\beta}$$

is precisely the Bergman kernel of the ball and, on $\partial\mathbb{B}$, $\varphi$ is precisely the Poisson kernel. Hence using (3.4) with that $\varphi$ we get

$$\widetilde{f^*}(z) = \int_{\mathbb{B}} U(\zeta) \frac{(-\varrho(z))^n}{(1 - \overline{z} \cdot \zeta)^n} c_n \frac{1}{(1 - z \cdot \overline{\zeta})^{n+1}} \beta \wedge \overline{\beta}$$

where $\widetilde{f^*}(z)$ is the Poisson integral of $f^*$; but

$$U(\zeta) \frac{(-\varrho(z))^n}{(1 - \overline{z} \cdot \zeta)^n}$$

is a holomorphic function in $\zeta$ and from the reproducing property of the Bergman kernel we get $\widetilde{f^*}(z) = U(z)$. Hence the lemma follows. ■

Let us recall the notion of Carleson measures of order $\alpha$ as defined in [4].

First of all, a *pseudoball* $Q(a,h)$ of center $a \in \partial\mathbb{B}$ and radius $h \in ]0,1[$ is

$$Q(a,h) := \{\eta \in \mathbb{B} : |1 - \overline{a} \cdot \eta| < h\}.$$

A measure $\mu$ is *Carleson* if there is a constant $C > 0$ such that

$$|\mu|(Q(a,h)) \leq Ch^n.$$

The set of Carleson measures is denoted by $W^1(\mathbb{B})$; we denote by $W^0(\mathbb{B})$ the set of all bounded measures on $\mathbb{B}$. Now we are able to define *Carleson measures of order* $\alpha \in [0,1]$, denoted by $W^\alpha(\mathbb{B})$, as the intermediate space in the sense of complex interpolation of Banach spaces between the bounded measures and the Carleson measures defined above:

$$W^\alpha(\mathbb{B}) := [W^0(\mathbb{B}), W^1(\mathbb{B})]_\alpha.$$

In [4] it is shown that if $\mu$ is in $W^\alpha(\mathbb{B})$, $\alpha > 0$, then there exists a measure $\nu$ in $W^1(\mathbb{B})$ and a function $f$ in $L^p(|\nu|)$, $p = 1/(1-\alpha)$, such that $d\mu = f\,d\nu$.

For a $(0,k)$-form $\omega = \sum_{|I|=k} \omega_I d\bar{z}_I$, we define $|\omega|^2 := \sum_I |\omega_I|^2$.

DEFINITION 3.4. (1) The space of *Carleson* $(0,1)$-*forms of order* $\alpha$ in $\mathbb{B}$ is

$$W^\alpha_{(0,1)}(\mathbb{B}) = \left\{\omega \in \mathcal{C}^\infty_{(0,1)}(\mathbb{B}) : |\omega| + \left|\frac{\omega \wedge \overline{\partial}\varrho}{\sqrt{-\varrho}}\right| \in W^\alpha(\mathbb{B})\right\}.$$

(2) The space of *Carleson* $(0,2)$-*forms of order* $\alpha$ in $\mathbb{B}$ is

$$W^\alpha_{(0,2)}(\mathbb{B}) = \{\omega \in \mathcal{C}^\infty_{(0,2)}(\mathbb{B}) : \sqrt{-\varrho}\,|\omega| \in W^\alpha(\mathbb{B})\}.$$

DEFINITION 3.5. The space of *Carleson–Wolff* $(0,1)$-*forms of order* 1 in $\mathbb{B}$ is

$$CW_{(0,1)}(\mathbb{B}) = \{\omega \in \mathcal{C}^\infty_{(0,1)}(\mathbb{B}) : \\ -\varrho|\omega|^2 + |\omega \wedge \overline{\partial}\varrho|^2 - \varrho|\mathcal{L}\omega| + \sqrt{-\varrho}\,|\mathcal{L}\omega \wedge \overline{\partial}\varrho| \in W^1(\mathbb{B})\}$$

where $\mathcal{L}$ is any smooth $(1,0)$-vector field on $\overline{\mathbb{B}}$.

From now on, $\mathbb{B}$ will be the unit ball of $\mathbb{C}^2$.

Let $\Lambda^k(\mathbb{C}^N)$ be the exterior algebra on $\mathbb{C}^N$, let $e_i$, $i = 1,\dots,N$, be the canonical basis of $\Lambda^1(\mathbb{C}^N)$, and $e_\alpha := e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k}$, $\alpha_i \in \{1,\dots,N\}$, the associated basis of $\Lambda^k(\mathbb{C}^N)$.

Let $M^k_r$ be the set of $\mathcal{C}^\infty(\mathbb{B})$ differential forms in $\mathbb{B}$ of type $(0,r)$ with values in $\Lambda^k(\mathbb{C}^N)$ and let:

- $L^k_0$ be the set of elements of $M^k_0$ whose coefficients are functions having a Stokes boundary value in $L^p(\partial\mathbb{B})$,
- $L^k_1$ be the set of elements $\omega$ of $M^k_1$ such that $\omega = f\omega_1 + \omega_2$, $\omega_1$ has coefficients in $CW_{(0,1)}(\mathbb{B})$, $\omega_2$ has coefficients in $W^\alpha_{(0,1)}(\mathbb{B})$ and $f \in H^p(\mathbb{B})$ with $\alpha = 1 - 1/p$,
- $L^k_2$ be the set of elements of $M^k_2$ whose coefficients are in the space $W^\alpha_{(0,2)}(\mathbb{B})$ of Carleson $(0,2)$-forms with $\alpha = 1 - 1/p$.



These spaces are suitable for our purposes because we have the existence of a linear operator $\mathcal{S}$, defined later on, such that if $\omega \in L^k_i$ and $\overline{\partial}\omega = 0$ then $\overline{\partial}(\mathcal{S}\omega) = \omega$ and $\mathcal{S}\omega \in L^k_{i-1}$ for $i = 1, 2$.

REMARK 3.6. Because Carleson measures of order $\alpha$ are bounded measures and our forms are smooth, the components of an element in $L^j_i$, $i \geq 1$, are in $L^1(\mathbb{B})$ and the definition for $\overline{\partial}_b$ can be used.

The spaces $L^k_0$ are modules over $H^\infty(\mathbb{B})$ because if $B \in H^\infty(\mathbb{B})$ and $f^* \in L^p(\partial\mathbb{B})$ is a Stokes boundary value for $f$ then $B_r(z) := B(rz) \in \mathcal{C}^\infty(\overline{\mathbb{B}})$ for all $r < 1$ and

$$\forall r < 1, \quad \int_{\partial\mathbb{B}} f^* B_r\varphi = \int_{\mathbb{B}} \overline{\partial} f \wedge B_r\varphi$$

because if $\varphi \in \mathcal{C}^\infty_{2,1}(\overline{\mathbb{B}})$ and $\overline{\partial}\varphi = 0$, then the same is true for $B_r\varphi$. Using Lebesgue's dominated convergence theorem, we can let $r \to 1$ to deduce that $B_r \to B^* \in L^\infty(\partial\mathbb{B})$ a.e. on $\partial\mathbb{B}$ and

$$\forall \varphi \in \mathcal{C}^\infty_{2,1}(\overline{\mathbb{B}}), \overline{\partial}\varphi = 0, \quad \int_{\partial\mathbb{B}} f^* B^*\varphi = \int_{\mathbb{B}} \overline{\partial} f \wedge B\varphi,$$

which means that $B^* f^*$ is a Stokes boundary value of $Bf$ and $B^* f^* \in L^p(\partial\mathbb{B})$; hence the assertion of the remark follows.

**3.2.** *Koszul's complex.* Recall that $B := (B_1,\dots,B_N)$ is a vector-valued bounded holomorphic function in $\mathbb{B} \subset \mathbb{C}^2$ such that $B^{-1}(0) = S$ and $|B| \geq \eta > 0$ on $\bigcap_{a\in S}\{|\Phi_a| > \delta\}$.

We shall use the Koszul complex method, introduced in this context by Hörmander [7] to "correct" the vector $L$ defined by equation (3.2).

Let us define two linear operators acting on $M^k_r$: first,

$$R_B(\omega) = \omega \wedge \sum_{i=1}^N \frac{\overline{B}_i}{|B|^2} e_i, \quad \omega \in M^k_r \cap \{\operatorname{supp}\omega \subset \{|B| \geq \eta\}\};$$

then $R_B(\omega) \in M^{k+1}_r \cap \{\mu : \operatorname{supp} \subset \{|B| \geq \eta\}\}$.

The operator $d_B$ is defined by induction and linearity as follows: $d_B : M^0_r \to 0$, $d_B(e_i) = B_i$ and for $e_\alpha \in \Lambda^k$,

$$d_B(e_\alpha \wedge e_i) = B_i e_\alpha - d_B(e_\alpha) \wedge e_i \in M^{k-1}_r.$$

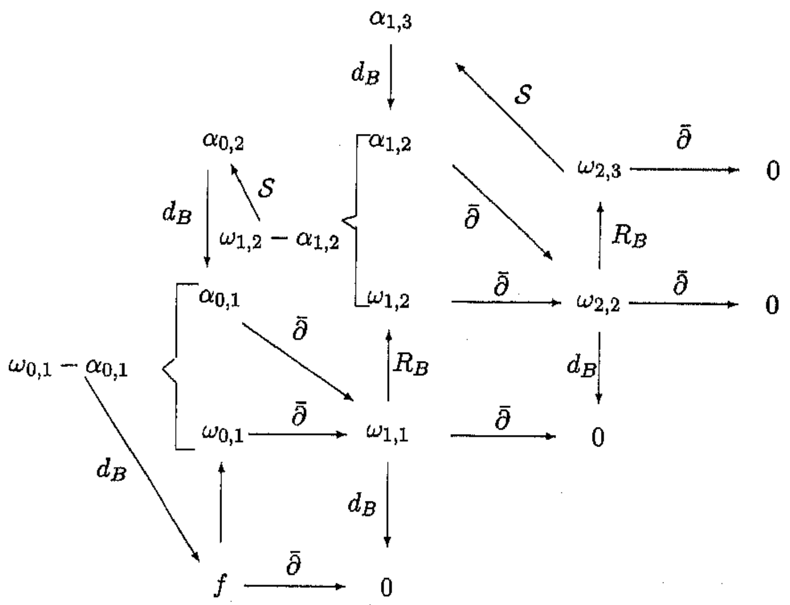
It is easily seen by induction that $d_B^2 = 0$, $\overline{\partial} d_B\omega = d_B\overline{\partial}\omega$ and

$$d_B\omega = 0 \;\Rightarrow\; d_B(R_B\omega) = \omega. \tag{3.5}$$

Hence $\alpha := R_B\omega$ is a solution to the equation $d_B\alpha = \omega$ provided that the necessary condition $d_B\omega = 0$ is fulfilled.

With $L$ defined in equation (3.2) let $\omega_{0,0} := f$ and $\omega_{0,1} := \sum_{i=1}^N L_i e_i$.

Together with the operator $\overline{\partial}$ we then have a double complex, whose elementary squares are commutative diagrams and where $\mathcal{S}$ is the operator solving the $\overline{\partial}$ equation:




To move down in this double complex we shall need results on solution of the $\overline{\partial}$ equation:

THEOREM 3.7. *There is a linear operator $\mathcal{S}$ such that:*

(1) *For every $\omega \in L_2^s$ with $\overline{\partial}\omega = 0$, $u := \mathcal{S}\omega \in L_1^s$ is such that $\overline{\partial}u = \omega$ and $d_B u \in L_1^{s-1}$.*

(2) *For every $\omega \in L_1^s$ with $\overline{\partial}\omega = 0$, $u := \mathcal{S}\omega \in L_0^s$ is such that $\overline{\partial}_b u = \omega$ and $d_B u \in L_0^{s-1}$.*

Proof. (1) is Theorem 3.5 of [3]; for a more general version see Theorem 4.1 of Andersson and Carlsson [5]. In fact the coefficients of $u$ are in the Carleson class $W^{\alpha}_{(0,1)}(\mathbb{B})$, hence $d_B u \in L_1^{s-1}$ because $B \in H^\infty(\mathbb{B})^N$.

For (2), let $\omega$ be a component of an element of $L_1^s$. Then $\omega = f\omega_1 + \omega_2$ with $\omega_1 \in CW_{(0,1)}(\mathbb{B})$, $\omega_2 \in W^{\alpha}_{(0,1)}(\mathbb{B})$ and $f \in H^p(\mathbb{B})$. By Theorems 4.1 and 4.2 of [5] there exists an operator $\mathcal{S}$ such that $\overline{\partial}_b\mathcal{S}\omega = \omega$ if $\overline{\partial}\omega = 0$ and moreover $\mathcal{S}(f\omega_1)$ and $\mathcal{S}\omega_2$ belong to $L^p(\partial\mathbb{B})$. Set $u = \mathcal{S}\omega$. The preceding fact implies that $u \in L_0^s$; that $d_B u \in L_0^{s-1}$ is a consequence of the fact that $L_0^s$ is a module over $H^\infty(\mathbb{B})$. ∎



**3.3.** *Proof of the division theorem.* Let $f$ be a holomorphic function in $H^p(\mathbb{B})$ such that $f_{|S} = 0$. We want to write $f = \sum_{i=1}^N f_i B_i$, with the $f_i$ still in $H^p(\mathbb{B})$.

For $i \geq 1$ let

$$\omega_{i,i} := \overline{\partial}\omega_{i-1,i}, \quad \omega_{i,i+1} := R_B\omega_{i,i}.$$

PROPOSITION 3.8. *The form $\omega_{i,j}$ belongs to $L_i^j$ and for $i \geq 1$ we have* $\operatorname{supp}\omega_{i,j} \subset \{|B| \geq \eta\}$.

Proof. First in $\mathbb{C}^2$ we do need not to go farther than $\omega_{2,3}$ because any $(0,2)$-form is $\overline{\partial}$-closed.

Now let us establish that $\operatorname{supp}\omega_{i,j} \subset \{|B| \geq \eta\}$ for $i \geq 1$. By definition of $\omega_{1,1}$ it is enough to show that $\operatorname{supp}\overline{\partial}L \subset \{|B| \geq \eta\}$. But

$$\overline{\partial}L = \overline{\partial}G + \overline{\partial}((1-\chi_S)H)$$

in $\bigcup_i\{|\Phi_i| < \delta/2\}$, $G$ is holomorphic and $1-\chi_S = 0$, thus $\overline{\partial}L = 0$ and the conclusion comes from the fact that $\{|B| < \eta\} \subset \bigcup_i\{|\Phi_i| < \delta/2\}$ for $\eta$ small enough by Definition 1.2(2).

For $i \geq 1$, $\omega_{i,i} = \overline{\partial}\omega_{i-1,i}$ and $\omega_{i,i+1} = R_B\omega_{i,i}$, hence $\operatorname{supp}\omega_{i,i} \subset \operatorname{supp}\omega_{i-1,i}$ and $\operatorname{supp}\omega_{i,i+1} \subset \operatorname{supp}\omega_{i,i}$ and they are all included in $\{|B| \geq \eta\}$.

For $i \geq 1$, $\omega_{i,j} \in L_i^j$ will be established in Proposition 4.16 for $i = 1$ and Proposition 4.15 for $i = 2$.

Finally $\omega_{1,1} \in L_1^1$ implies that $\omega_{0,1} \in L_0^1$ because, by Theorem 3.7(2), we have a $\beta \in L_0^1$ with $\overline{\partial}_b\beta = \omega_{1,1} = \overline{\partial}\omega_{0,1}$, which means precisely that $\omega_{0,1}$ has a Stokes boundary value in $L^p(\partial\mathbb{B})$, hence $\omega_{0,1} \in L_0^1$. ∎

Now we can play the usual diagram chasing:

LEMMA 3.9. *For $i = 0, 1, 2$ we have $d_B\omega_{i,i} = \overline{\partial}\omega_{i,i} = 0$.*

Proof. Because the $B_l$ are holomorphic, we have $d_B\overline{\partial} = \overline{\partial}d_B$, which implies that the elementary squares of the complex are commutative diagrams (by Proposition 3.8, $\operatorname{supp}\omega \subset \{|B| \geq \eta\}$ and hence everything is well defined). This proves the lemma. ∎

We can now give the proof of our main result.

By Theorem 3.7(1), there exists $\alpha_{1,3} \in L_1^3$ such that $\overline{\partial}\alpha_{1,3} = \omega_{2,3}$ and $\alpha_{1,2} := d_B\alpha_{1,3} \in L_1^2$. Then

$$\overline{\partial}\alpha_{1,2} = \overline{\partial}d_B\alpha_{1,3} = d_B\overline{\partial}\alpha_{1,3} = d_B\omega_{2,3} = d_BR_B\omega_{2,2} = \omega_{2,2},$$

because $d_B\omega_{2,2} = 0$ by Lemma 3.9. We get

$$\overline{\partial}\alpha_{1,2} = \omega_{2,2} = \overline{\partial}\omega_{1,2}, \quad \text{hence} \quad \overline{\partial}(\omega_{1,2} - \alpha_{1,2}) = 0.$$

We already know that $\omega_{1,2}$ is in $L_1^2$ by Proposition 3.8 and that $\alpha_{1,2}$ is also in $L_1^2$, hence there is a function $\alpha_{0,2} \in L_0^2$ such that $\overline{\partial}_b\alpha_{0,2} = \omega_{1,2} - \alpha_{1,2}$ and $\alpha_{0,1} := d_B\alpha_{0,2} \in L_0^1$ by Theorem 3.7(2).

Let $\beta$ be such that $\overline{\partial}_b\alpha_{0,2} = \overline{\partial}\beta$ by definition of $\overline{\partial}_b$. Then

$$\overline{\partial}_b\alpha_{0,1} = \overline{\partial}_b d_B\alpha_{0,2} = \overline{\partial} d_B\beta = d_B\overline{\partial}\beta = d_B\overline{\partial}_b\alpha_{0,2} = d_B(\omega_{1,2}-\alpha_{1,2}) = d_B\omega_{1,2},$$

because by Remark 3.6, $d_B\alpha_{0,2}$ is a Stokes boundary value of $d_B\beta$ and $d_B\alpha_{1,2} = d_B^2\alpha_{1,3} = 0$.

But $\omega_{1,2} = R_B\omega_{1,1}$, hence $\overline{\partial}_b\alpha_{0,1} = d_BR_B\omega_{1,1} = \omega_{1,1}$, because $d_B\omega_{1,1} = 0$.

Finally we get

$$d_B(\omega_{0,1}-\alpha_{0,1}) = d_B(\omega_{0,1}-d_B\alpha_{0,2}) = d_B\omega_{0,1} = f,$$
$$\overline{\partial}_b(\omega_{0,1}-\alpha_{0,1}) = 0.$$

Putting $\widetilde{F} := \omega_{0,1}-\alpha_{0,1} \in L_0^1$, we see that the coefficients of $\widetilde{F}$ are in $H^p(\mathbb{B})$ by Lemma 3.3 and $f = d_B\widetilde{F}$. This yields the assertion of Theorem 3.1. ■

The necessary estimates used in this proof will be settled in Section 4 but before let us give an application.

**3.4.** *An application.* Let $\sigma := \{\alpha_i : i \in \mathbb{N}\} \subset \mathbb{D} \subset \mathbb{C}$. It can be viewed as a sequence $S := \{a_i = (\alpha_i, 0) : i \in \mathbb{N}\} \subset \mathbb{B} \subset \mathbb{C}^2$, and we have

PROPOSITION 3.10. *If the sequence $\sigma := \{\alpha_i : i \in \mathbb{N}\} \subset \mathbb{D}$ is $H^\infty(\mathbb{D})$ interpolating then there exists a 2-strongly defining function for $S := \{a_i = (\alpha_i, 0) : i \in \mathbb{N}\} \subset \mathbb{B}$.*

Proof. First, it is well known that in the case of the unit disc there is a sequence $(\beta_i)_{i\in\mathbb{N}}$ in $H^\infty(\mathbb{D})$ such that $\beta_i(\alpha_j) = \delta_{ij}$ for each point $\alpha_j$ of $\sigma$, and $\sum_{i=1}^\infty |\beta_i(z)| \le C$ for all $z \in \mathbb{D}$, where the constant $C$ only depends on the sequence $\sigma$ (Beurling's linear extension).

For all $i \in \mathbb{N}$ let

$$b_i(z_1) = \frac{\alpha_i - z_1}{1-\overline{\alpha}_i z_1}\cdot\frac{|\alpha_i|}{\alpha_i}$$

be the Blaschke factor associated with $\alpha_i$, and let $\beta_i$ be the functions defined above, of one variable, associated with the sequence $\sigma$.

We define the 2-strongly defining function $B = (B_1, B_2)$ by

$$B_1(z) := \prod_i b_i(z_1), \qquad B_2(z) := z_2\sum_i \beta_i(z_1)\frac{(1-|\alpha_i|^2)^{1/2}}{1-\overline{\alpha}_i z_1}.$$

As $|1-\overline{\alpha}_i z_1| \ge (1-|\alpha_i|^2)^{1/2}(1-|z_1|^2)^{1/2}$, we have $B \in (H^\infty(\mathbb{B}))^2$ and obviously $B_{|S} = 0$. Using [3, Section 2] we have $\beta_i(z_1) = \gamma_{ij}(z_1)b_j(z_1)$ and $\beta_i(z_1) - 1 = \gamma_i(z_1)b_i(z_1)$ with $|\gamma_i(z_1)| + \sum_{j\ne i}|\gamma_{ij}(z_1)| \le C$. As the sequence $\sigma$ is separated, it is well known that $B_1(z)$ can be written as

$$B_1(z) = \Phi_i^1(z)\cdot C_i(z) \quad \text{with } 0 < \gamma \le |C_i| \le 1$$



on $\{z \in \mathbb{B} : |\Phi_i(z)| < \delta_0\}$ with $\delta_0 > 0$ small enough. Moreover

$$B_2(z) = \frac{(1-|\alpha_i|^2)^{1/2}}{1-\overline{\alpha}_i z_1} z_2(1+\gamma_i(z_1)b_i(z_1)) + b_i(z_1)\sum_{j\ne i}\gamma_{ji}(z_1)\frac{(1-|\alpha_j|^2)^{1/2}}{1-\overline{\alpha}_j z_1} z_2, \tag{3.6}$$

so $B(z) = M_i(z)\cdot\Phi_i(z)$ with $|\det M_i(z)|$ bounded by constants above and below on $\{z \in \mathbb{B} : |\Phi_i(z)| < \delta_0\}$ with $\delta_0 > 0$ small enough. To establish (2) of Definition 1.2, recall that as the measure $\mu := \sum_i(1-|a_i|^2)\delta_{a_i}$ is Carleson, for each $f$ in $H^2(\mathbb{B})$ we have

$$\int_{\mathbb{B}} |f|^2\,d\mu \le C\|f\|_2^2.$$

For $f(\xi) = \frac{1}{1-\overline{z}_1\xi}(1-|z_1|^2)^{1/2}$ this implies that

$$\sum_i(1-|b_i(z_1)|^2) = \sum_i \frac{(1-|\alpha_i|^2)(1-|z_1|^2)}{|1-\overline{\alpha}_i z_1|^2} \le C \quad \forall z_1 \in \mathbb{D}.$$

If $|b_i(z_1)| \ge \gamma$ for all $i$, then $|B_1(z)| \ge e^{-C(1+1/(2\gamma))}$ because

$$\ln(1-x) \ge -x\left(1+\frac{1}{2}\cdot\frac{1}{1-x}\right) \quad \forall x \in ]0,1[.$$

Thus, $|B_1(z)| < \eta$ implies that there exists $i$ such that

$$|b_i(z_1)| < \frac{1}{2}\left(-\frac{1}{C}\ln\eta - 1\right)^{-1}$$

and by using (3.6) we have

$$|B_2(z)| \ge \left|\frac{(1-|\alpha_i|^2)^{1/2}}{1-\overline{\alpha}_i z_1} z_2\right| - C'|b_i(z_1)| \ge \sqrt{\varepsilon^2 - |b_i(z_1)|^2} - C'|b_i(z_1)|,$$

because $z \in \bigcap_{i\in\mathbb{N}}\{|\Phi_i| \ge \varepsilon\}$ and $|b_i(z_1)| = |\Phi_i^1(z)|$. Then $|B_2(z)| \ge \eta$ for $\eta > 0$ small enough. ■

COROLLARY 3.11 (C. Horowitz). *Let $\Phi$ be an interpolating Blaschke product in the unit disc $\mathbb{D}$ in $\mathbb{C}$. Then $\Phi$ is a universal divisor of $\sigma = \Phi^{-1}(0)$ for the Bergman classes $B^p(\mathbb{D})$, for any $p \in [1,\infty[$.*

Proof. Let $g \in B^p(\mathbb{D})$ with $g_{|\sigma} = 0$. Then $f(z_1, z_2) := g(z_1)$ is in $H^p(\mathbb{B})$ and of course $f_{|S} = 0$ for $S := \{a_i = (\alpha_i, 0) : i \in \mathbb{N}\}$. By the previous proposition and Theorem 3.1, there exists $F \in H^p(\mathbb{B})^2$ such that $f = B\cdot F$. Put $z_2 = 0$. Then

$$g(z_1) = f(z_1, 0) = F(z_1, 0)\prod_{i\in I} b_i(z_1)$$

where $b_i(z_1)$ is the Blaschke factor associated with $\alpha_i$. This yields the result of Horowitz because $F(z_1,0) \in B^p(\mathbb{D})$ by the subordination lemma of [1]. ∎

## 4. Estimate

**4.1.** *Special case of a point.* For what follows, we shall need more precise estimates than those in Theorem 2.1.

REMARK 4.1. On $\{|\Phi_a(z)| < \delta\}$ with $a = (r,0)$, we have $|z_2| \lesssim (1-r^2)^{1/2}$, $|1-rz_1| \simeq 1-r^2$, $|r-z_1| \lesssim 1-r^2$, the implied constants being independent of $a \in \mathbb{B}$ (this is a standard fact).

PROPOSITION 4.2. *If $a \in \mathbb{B}$, $f \in H^p(\mathbb{B})$, $f(a) = 0$, then the vector-valued function $F$ given by Theorem 2.1 satisfies*

$$\forall z \in \mathbb{B}, \quad |\Phi_a(z)| \le \delta, \quad |F(z)| \lesssim \widetilde{f}(a)$$

*with $\widetilde{f}(z) := PI(|f|)(z)$, the Poisson integral of $|f|$ at $z$, $\varrho(z) := |z|^2 - 1$, the implied constant being independent of $a \in \mathbb{B}$.*

Proof. We have

$$H_i(z) = \int_0^1 \frac{\partial h}{\partial z_i}(tz)\,dt \;\Rightarrow\; h(z) = zH(z), \text{ as soon as } h(0) = 0.$$

Applying this with $h(z) := f \circ \Phi_a(z) k_a(z)$ as in the proof of Theorem 2.1, we get

$$H_i(z) = \int_0^1 f \circ \Phi_a(tz) \frac{\partial k_a}{\partial z_i}(tz)\,dt + \int_0^1 \frac{\partial (f \circ \Phi_a)}{\partial z_i}(tz) k_a(tz)\,dt.$$

We are interested in $G(z) := k_a^{-1}(z) \cdot H(z)$, hence

$$\begin{aligned} G_i(z) &= k_a^{-1}(z) \int_0^1 f \circ \Phi_a(tz) \frac{\partial k_a}{\partial z_i}(tz)\,dt + k_a^{-1}(z) \int_0^1 \frac{\partial (f \circ \Phi_a)}{\partial z_i}(tz) k_a(tz)\,dt \\ &= I_1(z) + I_2(z). \end{aligned}$$

Now we have

$$k_a(z) := \frac{(1-|a|^2)^{2/p}}{(1-\overline{a}z)^{4/p}}, \quad \text{hence} \quad \frac{\partial k_a}{\partial z_i} = \frac{4}{p} \cdot \frac{(1-|a|^2)^{2/p}}{(1-\overline{a}z)^{4/p+1}} \overline{a}_i.$$

On $\{|z| < \delta\}$, we get

$$|k_a^{-1}(z)| \lesssim (1-|a|^2)^{-2/p} \quad \text{and} \quad \left|\frac{\partial k_a}{\partial z_i}(tz)\right| \lesssim (1-|a|^2)^{2/p};$$

hence

$$|I_1(z)| \lesssim \int_0^1 |f \circ \Phi_a(tz)|\,dt \quad \text{and} \quad |I_2(z)| \lesssim \int_0^1 \left|\frac{\partial (f \circ \Phi_a)}{\partial z_i}(tz)\right| dt.$$



We then have, using Cauchy's formula,

$$f \circ \Phi_a(tz) = \int_{\partial\mathbb{B}} \frac{f \circ \Phi_a(\zeta)}{(1-\overline{\zeta}tz)^2}\,d\sigma(\zeta),$$

hence on $\{|z| < \delta\}$,

$$|f \circ \Phi_a(tz)| \lesssim \int_{\partial\mathbb{B}} |f \circ \Phi_a(\zeta)|\,d\sigma(\zeta) = \int_{\partial\mathbb{B}} |f(\zeta)| J_a(\zeta)\,d\sigma(\zeta).$$

But $J_a(\zeta) = P(a,\zeta)$, the Poisson–Szegő kernel of the ball, and hence, setting

$$\widetilde{f}(z) := \int_{\partial\mathbb{B}} P(z,\zeta)|f(\zeta)|\,d\sigma(\zeta),$$

we get on $\{|z| < \delta\}$,

$$|f \circ \Phi_a(tz)| \lesssim \widetilde{f}(a) \quad \text{and} \quad |I_1(z)| \lesssim \widetilde{f}(a).$$

In the same way we obtain

$$\frac{\partial (f \circ \Phi_a)}{\partial z_i}(tz) = 2 \int_{\partial\mathbb{B}} t \frac{f \circ \Phi_a(\zeta)}{(1-\overline{\zeta}tz)^3} \overline{\zeta}_i\,d\sigma(\zeta),$$

and again on $\{|z| < \delta\}$,

$$\left|\frac{\partial (f \circ \Phi_a)}{\partial z_i}(tz)\right| \lesssim \widetilde{f}(a) \quad \text{and} \quad |I_2(z)| \lesssim \widetilde{f}(a);$$

hence $|G(z)| \lesssim \widetilde{f}(a)$ on $\{|z| < \delta\}$.

Coming back to $F$, we get $F(z) := G \circ \Phi_a(z)$, hence $|F(z)| \lesssim \widetilde{f}(a)$ on $\{|\Phi_a(z)| < \delta\}$, proving the proposition. ∎

LEMMA 4.3. *Let $a \in \mathbb{B}$, $f \in H^p(\mathbb{B})$ and $\widetilde{f}$ be the Poisson integral of $|f|$. On the set $\{|\Phi_a(z)| < \delta < 1\}$, we have $\widetilde{f}(a) \lesssim \widetilde{f}(z)$.*

Proof. As usual we can set $a = (r,0)$. We have

$$\frac{(1-|z|^2)^2}{|1-\overline{\zeta}z|^4} - \frac{(1-r^2)^2}{|1-\overline{\zeta}_1 r|^4} = \frac{(1-|z|^2)^2}{|1-\overline{\zeta}z|^4}\left(1 - \frac{(1-r^2)^2}{(1-|z|^2)^2} \cdot \frac{|1-\overline{\zeta}z|^4}{|1-\overline{\zeta}_1 r|^4}\right).$$

But on $\{|\Phi_a(z)| < \delta < 1\}$, we have $1-|z|^2 \simeq 1-r^2$, hence

$$\frac{(1-r^2)^2}{(1-|z|^2)^2} \lesssim 1.$$

Moreover

$$1-\overline{\zeta}z = 1-\overline{\zeta}_1 r + \overline{\zeta}_1 r - \overline{\zeta}z = 1-\overline{\zeta}_1 r + \overline{\zeta}_1 r - \overline{\zeta}_1 z_1 - \overline{\zeta}_2 z_2,$$

hence

$$|1-\overline{\zeta}z| \le |1-\overline{\zeta}_1 r| + |\overline{\zeta}_1| \cdot |r-z_1| + |\overline{\zeta}_2| \cdot |z_2|,$$

and

$$\frac{|1-\overline{\zeta}z|}{|1-\overline{\zeta}_1 r|} \le 1 + \frac{|r-z_1|}{|1-\overline{\zeta}_1 r|} + \frac{|z_2|}{|1-\overline{\zeta}_1 r|}.$$

But $|1-\overline{\zeta}_1 r| \ge \frac{1}{2}(1-r^2)$, hence finally

$$\frac{|1-\overline{\zeta}z|}{|1-\overline{\zeta}_1 r|} \lesssim 1.$$

So we have proved that $|P(a,\zeta) - P(z,\zeta)| \lesssim P(z,\zeta)$ provided that $|\Phi_a(z)| < \delta < 1$.

Now

$$|\widetilde{f}(a) - \widetilde{f}(z)| \le \int_{\partial\mathbb{B}} |P(a,\zeta) - P(z,\zeta)| \cdot |f(\zeta)|\, d\sigma(\zeta) \lesssim \widetilde{f}(z),$$

proving the lemma. ∎

REMARK 4.4. By the symmetry between $a$ and $z$ in the previous lemma, under the same hypothesis, we get $\widetilde{f}(a) \simeq \widetilde{f}(z)$ on $|\Phi_a(z)| < \delta < 1$.

REMARK 4.5. We only need the previous results in $\mathbb{C}^2$ but the proofs clearly extend to $\mathbb{C}^n$.

LEMMA 4.6. *Let $a \in \mathbb{B}$ and $M_a$ be a holomorphic matrix with $|M_a(z)|_{\rm op} \le C$ for all $z \in \mathbb{B}$, such that $|M_a(z)^{-1}|_{\rm op} \le C$ in $|\Phi_a(z)| < \delta < 1$. Moreover, let $f \in H^p(\mathbb{B})$ with $f(a) = 0$ and $F_a$ be a divisor of $f$, $f = \Phi_a \cdot F_a$, given by Theorem 2.1. Finally let $H_a(z) := {}^tM_a(z)^{-1}F_a(z)$ and $G_a(z) = ((H_a)_1, (H_a)_2, 0, \ldots, 0)$, as in equation (3.1). Then $|G_a| \lesssim \widetilde{f}$ on $|\Phi_a(z)| < \delta < 1$, the implied constant depending only on $C$ and $\delta$, but not on $a \in \mathbb{B}$.*

Proof. Clearly it is enough to prove this for $H_a$. We have $|H_a(z)| \le |{}^tM_a(z)^{-1}|_{\rm op}|F_a(z)|$; but $|F_a(z)| \lesssim \widetilde{f}(a) \lesssim \widetilde{f}(z)$ by Proposition 4.2 and Lemma 4.3, which proves the lemma. ∎

**4.2.** *Estimation of the forms involved in the Koszul complex*

**4.2.1.** *Expression of the forms.* Let us recall some notations:

$$\chi_i := \chi\Big(\frac{|\Phi_{a_i}|^2}{\delta^2}\Big), \quad \chi_i' := \frac{1}{\delta^2}\chi'\Big(\frac{|\Phi_{a_i}|^2}{\delta^2}\Big).$$

We showed in equation (3.1) that there exist holomorphic maps defined on $\{|\Phi_{a_i}| < \delta\}$ by

$$G_i = (({}^tM_i^{-1}F_i)_1, ({}^tM_i^{-1}F_i)_2, 0, \ldots, 0) = (G_i^1, \ldots, G_i^N) \tag{4.1}$$

with $F_i$ a map in $(H^p(\mathbb{B}))^2$ given by division of $f$ at the point $a_i \in S$, and $M_i$ a $2\times 2$ matrix with coefficients in $H^\infty(\mathbb{B})$, uniformly bounded on $\mathbb{B}$ and with $M_i^{-1}$ uniformly bounded on $\{|\Phi_{a_i}| < \delta\}$.



Since the Koszul complex comes from equation (3.2), we have

$$\omega_{0,1} = \sum_{j=1}^{N} \Big(\sum_i \chi_i G_i^j + \Big(1 - \sum_i \chi_i\Big)\frac{\overline{B}_j}{|B|^2} f\Big) e_j, \tag{4.2}$$

and $\omega_{1,1} := \overline{\partial}\omega_{0,1}$, $\omega_{1,2} := R_B(\omega_{1,1})$, so that

$$\begin{aligned}\omega_{1,2} = \sum_{j,k=1}^{N} \Big\{ &\Big(\sum_i \chi_i'(\Phi_{a_i}^1 \overline{\partial\Phi}_{a_i}^1 + \Phi_{a_i}^2 \overline{\partial\Phi}_{a_i}^2) G_i^j\Big) \\ &+ \Big(1 - \sum_i \chi_i\Big) f \overline{\partial}\Big(\frac{\overline{B}_j}{|B|^2}\Big) \\ &- \Big(\sum_i \chi_i'(\Phi_{a_i}^1 \overline{\partial\Phi}_{a_i}^1 + \Phi_{a_i}^2 \overline{\partial\Phi}_{a_i}^2)\Big) f \frac{\overline{B}_j}{|B|^2} \Big\} \frac{\overline{B}_k}{|B|^2} e_j \wedge e_k\end{aligned} \tag{4.3}$$

and $\omega_{2,2} := \overline{\partial}\omega_{1,2}$; moreover $\omega_{2,3} := R_B(\omega_{2,2})$, and so

$$\begin{aligned}\omega_{2,3} = \sum_{j,k,l=1}^{N} \Big\{ &-2\Big(\sum_i \chi_i'(\Phi_{a_i}^1 \overline{\partial\Phi}_{a_i}^1 + \Phi_{a_i}^2 \overline{\partial\Phi}_{a_i}^2)\Big) \\ &\times f \frac{\overline{B}_k}{|B|^2} \wedge \overline{\partial}\Big(\frac{\overline{B}_j}{|B|^2}\Big) \\ &+ \overline{\partial}\Big(\frac{\overline{B}_k}{|B|^2}\Big) \wedge \Big(\sum_i \chi_i'(\Phi_{a_i}^1 \overline{\partial\Phi}_{a_i}^1 + \Phi_{a_i}^2 \overline{\partial\Phi}_{a_i}^2) G_i^j \\ &+ \Big(1 - \sum_i \chi_i\Big) f \overline{\partial}\Big(\frac{\overline{B}_j}{|B|^2}\Big) \\ &- \Big(\sum_i \chi_i'(\Phi_{a_i}^1 \overline{\partial\Phi}_{a_i}^1 + \Phi_{a_i}^2 \overline{\partial\Phi}_{a_i}^2)\Big) f \frac{\overline{B}_j}{|B|^2}\Big) \Big\} \frac{\overline{B}_l}{|B|^2} e_j \wedge e_k \wedge e_l.\end{aligned} \tag{4.4}$$

**4.3.2.** *Lemmas for the majorization of the forms.* Set

$$|\partial B| := \sum_{j=1}^{N} |\partial B_j|, \quad |\partial B \wedge \partial\varrho| := \sum_{j=1}^{N} |\partial B_j \wedge \partial\varrho|$$

so we have

LEMMA 4.7. *For all $i, j \in \{1, \ldots, N\}$,*

$$|\partial B_i \wedge \partial B_j| \le |\partial B| \cdot |\partial B \wedge \partial\varrho|.$$

Proof. We can assume $\partial B_i \wedge \partial B_j \ne 0$. Set $u_l := \partial B_l / |\partial B_l|$, $l = i, j$, and let $\gamma$ be a $(1,0)$-form such that $(\partial\varrho, \gamma)$ is a normalized basis. For $l = i, j$ we have

$$u_l = a_l \partial\varrho + b_l \gamma, \quad u_l \wedge \partial\varrho = b_l \gamma \wedge \partial\varrho$$

and by an obvious computation

$$|u_i \wedge u_j| \le (|b_i| + |b_j|)|\partial\varrho \wedge \gamma| \le |u_i \wedge \partial\varrho| + |u_j \wedge \partial\varrho|.$$

Therefore

$$|\partial B_i \wedge \partial B_j| \le |\partial B|(|\partial B_i \wedge \partial\varrho| + |\partial B_j \wedge \partial\varrho|) \le |\partial B| \cdot |\partial B \wedge \partial\varrho|. \square$$

LEMMA 4.8. *For all* $a_i \in S$, $j = 1, 2$, *and* $k \in \{1, \ldots, N\}$, *on* $\{|\Phi_{a_i}| < \delta\} \cap \{\partial B_1 \wedge \partial\varrho \ne 0\} \cap \{\partial B_2 \wedge \partial\varrho \ne 0\}$ *we have*:

(i) $|\partial\Phi^j_{a_i}| \lesssim |\partial B|$,
(ii) $|\partial\Phi^j_{a_i} \wedge \partial B_k| \lesssim |\partial B| \cdot |\partial B \wedge \partial\varrho|$,
(iii) $|\partial\Phi^j_{a_i} \wedge \partial\varrho| \lesssim |\partial B \wedge \partial\varrho|$.

Proof. It is proved in [3] that $\chi_i|\partial\Phi_{a_i}| \le C\chi_i(|\partial B_1| + |\partial B_2|)$ for all $a_i \in S$ and hence (i) follows.

Let $z$ be such that $|\partial B_1(z)| + |\partial B_2(z)| \approx |\partial B_1(z)|$. For $u_1 := \partial B_1/|\partial B_1|$, on $\{|\Phi_{a_i}| < \delta\} \cap \{\partial B_1 \wedge \partial\varrho \ne 0\}$ we get

$$\partial\Phi^j_{a_i}(z) = \alpha^j_i u_1(z) + \beta^j_i \partial\varrho(z),$$
$$|\partial\Phi^j_{a_i}(z)| = |\alpha^j_i| + |\beta^j_i| \lesssim |\partial B_1(z)|.$$

Thus

$$\begin{aligned}|\partial\Phi^j_{a_i} \wedge \partial B_k(z)| &\lesssim |\partial B_1(z)|(|u_1 \wedge \partial B_k(z)| + |\partial\varrho \wedge \partial B_k(z)|)\\ &\lesssim (|\partial B_1 \wedge \partial B_k(z)| + |\partial B_1(z)| \cdot |\partial\varrho \wedge \partial B_k(z)|)\\ &\lesssim |\partial B(z)| \cdot |\partial B \wedge \partial\varrho(z)|\end{aligned}$$

by Lemma 4.7. With a similar computation we get (iii). If $z$ is such that $|\partial B_1(z)| + |\partial B_2(z)| \approx |\partial B_2(z)|$, we do the same just interchanging 1 and 2. ■

REMARK 4.9. All the majorizations below will be done on $\{\partial B_1 \wedge \partial\varrho \ne 0\} \cap \{\partial B_2 \wedge \partial\varrho \ne 0\}$, which is a set of full measure in $\mathbb{B}$.

**4.2.3.** *Majorization and estimations of the forms.* In the following we shall identify a form with its coefficients in the basis $(e_j \wedge e_k)$ or $(e_j \wedge e_k \wedge e_l)$.

LEMMA 4.10. *We have* $|\omega_{2,2}| \lesssim \widetilde{f}|\partial B| \cdot |\partial B \wedge \partial\varrho|$ *and* $|\omega_{2,3}| \lesssim \widetilde{f}|\partial B| \cdot |\partial B \wedge \partial\varrho|$, *where* $\widetilde{f}$ *is the Poisson–Szegő integral of* $|f|$.

Proof. The components of $\omega_{2,3}$ are those of $\omega_{2,2}$ times $\overline{B}_l/|B|^2$ which is bounded on $\operatorname{supp}\omega_{2,3}$, hence the majorizations for $\omega_{2,3}$ and $\omega_{2,2}$ are the same; we shall prove them for $\omega_{2,3}$.

Because $\operatorname{supp}\omega_{2,3} \subset \{|B| > \eta\}$, the proposition is a consequence of the formula (4.4); the modulus of the sum is majorized by the sum of moduli, $|f|$ by $\widetilde{f}$ and then we apply Lemmas 4.7, 4.8(ii) and 4.6(i). ■



Now we decompose $\omega_{1,1} = ft'_1 + t'_2$ with

$$t'_1 := \Big\{\Big(1 - \sum_i \chi_i\Big)\overline{\partial}\Big(\frac{\overline{B}_j}{|B|^2}\Big) - \Big(\sum_i \chi'_i\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle\Big)\frac{\overline{B}_j}{|B|^2}\Big\}$$
$$t'_2 := \Big(\sum_i \chi'_i\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle G^j_i\Big),$$

and $\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle := \Phi^1_{a_i}\overline{\partial\Phi}^1_{a_i} + \Phi^2_{a_i}\overline{\partial\Phi}^2_{a_i}$.

In the same way we decompose $\omega_{1,2} = ft_1 + t_2$ with $t_i := t'_i\overline{B}_k/|B|^2$, $i = 1, 2$.

REMARK 4.11. We notice that estimates for $t_i, \partial t_i$ imply analogous estimates for $t'_i$, $\partial t'_i$ hence it is enough to get estimates for $\omega_{1,2}$ in order to have them for $\omega_{1,1}$.

LEMMA 4.12. *On* $\mathbb{B}$ *we have*

(i) $-\varrho(|t_1|^2 + |\partial t_1|) \lesssim -\varrho|\partial B|^2$,
(ii) $\sqrt{-\varrho}\,|\partial t_1 \wedge \overline{\partial}\varrho| \lesssim \sqrt{-\varrho}\,|\partial B| \cdot |\partial B \wedge \partial\varrho|$,
(iii) $|t_1 \wedge \overline{\partial}\varrho|^2 \lesssim |\partial B \wedge \partial\varrho|^2$,

*where the implied constants are independent of* $z$.

Proof. $\partial t_1$ is the sum of terms like:

$$\sum_i \chi'_i\langle\overline{\Phi}_{a_i}, \partial\Phi_{a_i}\rangle\overline{\partial}\Big(\frac{\overline{B}_j}{|B|^2}\Big)\frac{\overline{B}_k}{|B|^2}, \quad \Big(1 - \sum_i \chi_i\Big)\frac{\overline{B}_k\overline{\partial B}_j \wedge \partial B_l}{|B|^6},$$
$$\sum_i \chi''_i\langle\overline{\Phi}_{a_i}, \partial\Phi_{a_i}\rangle\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle\frac{\overline{B}_j\overline{B}_k}{|B|^4},$$
$$\sum_i \chi_i\langle\partial\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle\frac{\overline{B}_j\overline{B}_k}{|B|^4}, \quad \sum_i \chi'_i\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle\frac{\overline{B}_j\overline{B}_k\overline{B}_l\partial B_l}{|B|^6}.$$

The assertions are direct consequences of:
(i) : Lemma 4.8(i),
(ii) : Lemma 4.8(i) and (iii),
(iii) : Lemma 4.8(iii). ■

LEMMA 4.13. *We have*

(i) $$|t_2| \lesssim \widetilde{f}\sum_i \chi'_i|\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle|,$$

(ii) $$\Big|\frac{t_2 \wedge \overline{\partial}\varrho}{\sqrt{-\varrho}}\Big| \lesssim \widetilde{f}\sum_i \chi'_i\frac{|\langle\Phi_{a_i}, \overline{\partial\Phi}_{a_i}\rangle \wedge \overline{\partial}\varrho|}{\sqrt{-\varrho}},$$

*where the implied constants are independent of* $z \in \mathbb{B}$.

Proof. This is a direct consequence of Lemma 4.6. ■

LEMMA 4.14. *$-\varrho|\partial B|^2$, $|\partial B\wedge\partial\varrho|^2$, $\sqrt{-\varrho}\,|\partial B|\cdot|\partial B\wedge\partial\varrho|$, $\sum_i \chi_i'|\langle\Phi_{a_i},\overline{\partial\Phi}_{a_i}\rangle|$ and $\sum_i \chi_i'|\langle\Phi_{a_i},\overline{\partial\Phi}_{a_i}\rangle\wedge\overline{\partial}\varrho|/\sqrt{-\varrho}$ are Carleson measures on $\mathbb{B}$.*

Proof. Recall that $B\in(H^\infty(\mathbb{B}))^N$; it was shown in [2] that $-\varrho|\partial B|^2$ and $|\partial B\wedge\partial\varrho|^2$ are Carleson measures. This implies that $\sqrt{-\varrho}\,|\partial B|\cdot|\partial B\wedge\partial\varrho|$ is a Carleson measure by the Schwarz inequality. The last two points are already shown in [3]. ∎

PROPOSITION 4.15. *$\omega_{2,3}\in L_2^3$ and $\omega_{2,2}\in L_2^2$.*

Proof. $\sqrt{-\varrho}|\omega_{2,3}|$ and $\sqrt{-\varrho}|\omega_{2,2}|$ are majorized by $\widetilde{f}\sqrt{-\varrho}|\partial B|\cdot|\partial B\wedge\partial\varrho|$ up to a constant (Lemma 4.10) where $\widetilde{f}$ is the Poisson integral of $|f|$, $f\in H^p(\mathbb{B})$, and $\sqrt{-\varrho}\,|\partial B|\cdot|\partial B\wedge\partial\varrho|$ is a Carleson measure by Lemma 4.14. Thus $\widetilde{f}\sqrt{-\varrho}\,|\partial B|\cdot|\partial B\wedge\partial\varrho|$ is a Carleson measure of order $1-1/p$ (see [4]) and so are $\sqrt{-\varrho}\,|\omega_{2,2}|$ and $\sqrt{-\varrho}\,|\omega_{2,3}|$; moreover $\omega_{2,2},\omega_{2,3}\in\mathcal{C}^\infty_{(0,2)}(\mathbb{B})$. ∎

PROPOSITION 4.16. *$\omega_{1,1}\in L_1^1$ and $\omega_{1,2}\in L_1^2$.*

Proof. We have $\omega_{1,1}=ft_1'+t_2'$, $\omega_{1,2}=ft_1+t_2$. For any $(0,1)$-form $u$ and any smooth $(1,0)$-vector field $\mathcal{L}$ on $\overline{\mathbb{B}}$, $|\mathcal{L}u|$ and $|\mathcal{L}u\wedge\overline{\partial}\varrho|$ are bounded by $|\partial u|$ and $|\partial u\wedge\overline{\partial}\varrho|$ respectively, thus by Lemmas 4.12 and 4.14, $t_1'$ and $t_1$ belong to $CW_{(0,1)}$. By Lemmas 4.13 and 4.14, $t_2'$ and $t_2$ belong to $W^\alpha_{(0,1)}$. Hence by the definition of $L_1^i$, $\omega_{1,1}\in L_1^1$ and $\omega_{1,2}\in L_1^2$. ∎

This finishes the proof of the necessary estimates used for the main theorem, Theorem 3.1. As already said in the introduction, the case of the unit ball of $\mathbb{C}^n$, $n\geq 3$, might be handled the same way thanks to the recent results by Andersson and Carlsson [5] valid in any dimension.

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*Received December 24, 1997*
*Revised version May 15, 2000* (4023)