[19] P. Wojtaszczyk, *Some remarks on the Daugavet equation*, Proc. Amer. Math. Soc. 115 (1992), 1047–1052.

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: mmartins@ugr.es
rpaya@ugr.es

Received October 18, 1999
Revised version June 19, 2000 (4416)

STUDIA MATHEMATICA 142 (3) (2000)

Polydisc slicing in $\mathbb{C}^n$

by

KRZYSZTOF OLESZKIEWICZ and
ALEKSANDER PEŁCZYŃSKI (Warszawa)

Abstract. Let D be the unit disc in the complex plane $\mathbb{C}$. Then for every complex linear subspace H in $\mathbb{C}^n$ of codimension 1,

$$\mathrm{vol}_{2n-2}(D^{n-1}) \le \mathrm{vol}_{2n-2}(H \cap D^n) \le 2\mathrm{vol}_{2n-2}(D^{n-1}).$$

The lower bound is attained if and only if H is orthogonal to the versor e_j of the jth coordinate axis for some $j = 1, \ldots, n$; the upper bound is attained if and only if H is orthogonal to a vector $e_j + \sigma e_k$ for some $1 \le j < k \le n$ and some $\sigma \in \mathbb{C}$ with $|\sigma| = 1$.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$; by $\mathrm{vol}_k(\cdot)$ we denote the usual k-dimensional volume in $\mathbb{R}^{2n}$.

The result is a complex counterpart of Ball's [B1] result for cube slicing.

1. Introduction. In 1986 Ball [B1] discovered

THEOREM $B_{\mathbb{R}}$. *Let $I = [-1, 1]$. Let H be a linear subspace of $\mathbb{R}^n$ $(n = 2, 3, \ldots)$ of codimension 1. Then*

$$\mathrm{vol}_{n-1}(I^{n-1}) \le \mathrm{vol}_{n-1}(H \cap I^n) \le \sqrt{2}\,\mathrm{vol}_{n-1}(I^{n-1}).$$

The lower bound is attained if and only if H is orthogonal to the versor $e_j = (\delta^j_{j'})^n_{j'=1}$ of the jth coordinate axis for some $j = 1, \ldots, n$; the upper bound is attained if and only if H is orthogonal to a vector $e_j \pm e_k$ for some $1 \le j < k \le n$.

The lower estimate goes back to Hensley [H] who also used a "probabilistic approach" to establish some upper bound. Following closely Ball's approach we establish the complex counterpart of Theorem $B_{\mathbb{R}}$; we prove

THEOREM $B_{\mathbb{C}}$. *Let $D = \{z \in \mathbb{C} : |z| \le 1\}$. Let H be a complex linear subspace in $\mathbb{C}^n$ $(n = 2, 3, \ldots)$ of codimension 1. Then*

$$\mathrm{vol}_{2n-2}(D^{n-1}) \le \mathrm{vol}_{2n-2}(H \cap D^n) \le 2\,\mathrm{vol}_{2n-2}(D^{n-1}).$$

2000 *Mathematics Subject Classification*: 52A20, 33C10.
Key words and phrases: polydisc, volume of section, Bessel functions.
Research of K. Oleszkiewicz supported by the Polish KBN grant 2P03A 043 15.
Research of A. Pełczyński supported by the Polish KBN grant 2P03A 036 14.

The lower bound is attained if and only if H is orthogonal to the versor e_j of the jth coordinate axis for some $j = 1, \ldots, n$; the upper bound is attained if and only if H is orthogonal to a vector $e_j + \sigma e_k$ for $1 \le j < k \le n$ and for some $\sigma \in \mathbb{C}$ with $|\sigma| = 1$.

Throughout the paper we identify $\mathbb{C}^n$ with the Euclidean space $\mathbb{R}^{2n}$ via the map

$$(z_j)_{j=1}^n \mapsto (\Re z_1, \Im z_1, \ldots, \Re z_n, \Im z_n).$$

By $\mathrm{vol}_k(\cdot)$ we denote the usual k-dimensional volume in a Euclidean space. In particular $\mathrm{vol}_{2n-2}(D^{n-1}) = \pi^{n-1}$. A k-dimensional subspace is a k-dimensional hyperplane passing through the origin. By $\langle \cdot, \cdot \rangle$ we denote the usual complex scalar product in $\mathbb{C}^n$.

The sharp lower bound for the k-dimensional volume of sections of the n-cube by k-dimensional subspaces has been obtained by Vaaler [V]. In [MP] a lower bound of the volumes of central sections of the unit ball of a real l_p^n space $(2 \le p \le \infty)$ was considered. Similar problems (again for real spaces only) have been treated by other methods in [S] and [NP].

Our proof of the upper bound in Theorem $B_{\mathbb{C}}$ bases upon the following analytic inequality which seems to be interesting in itself.

PROPOSITION 1.1. *One has*

$$\int_0^\infty \left[\frac{2|J_1(t)|}{t}\right]^p t\,dt \le \frac{4}{p} \quad \text{for } p \ge 2, \tag{1}$$

and there is equality if and only if $p = 2$.

Here

$$J_1(t) = \sum_{k=0}^\infty (-1)^k \left(\frac{t}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}$$

is the Bessel function of order 1.

Assuming the validity of Proposition 1.1 we prove Theorem $B_{\mathbb{C}}$ in Section 2. The proof of Proposition 1.1 is given in Section 3. Section 4 contains some remarks.

2. Proof of Theorem $B_{\mathbb{C}}$

2.1. *Introductory reduction.* The general form of an $(n-1)$-dimensional hyperplane in $\mathbb{C}^n$ is $H_a^c = \{\mathfrak{z} \in \mathbb{C}^n : \langle \mathfrak{z}, a \rangle = c\}$ for some $a = (a_j)_{j=1}^n \in \mathbb{C}^n$ with $\|a\|_2 = 1$ and some $c \in \mathbb{C}$; if $c = 0$ we write H_a instead of H_a^0. If a is a coordinate versor then $\mathrm{vol}_{2n-2}(H_a^c \cap D^n) = \pi^{n-1}$ for $|c| \le 1$ and $\mathrm{vol}_{2n-2}(H_a^c \cap D^n) = 0$ for $|c| > 1$. Note that, in general, there is a unitary transformation of $\mathbb{C}^n$, say U, which is a composition of rotations in each coordinate plane and a permutation, say $p(\cdot)$, of coordinates such that $U(a) = (|a_{p(1)}|, \ldots, |a_{p(n)}|)$ and $|a_{p(1)}| \ge \ldots \ge |a_{p(n)}|$. Clearly U preserves D^n and $U(H_a^c \cap D^n) = H_{U(a)}^c \cap D^n$. Therefore

$$\mathrm{vol}_{2n-2}(H_a^c \cap D^n) = \mathrm{vol}_{2n-2}(H_{U(a)}^c \cap D^n).$$

Thus in what follows without loss generality we can assume that $a = (a_j)_{j=1}^n$ is a vector with real coordinates which satisfies

$$\sum_{j=1}^n a_j^2 = 1; \quad a_1 \ge \ldots \ge a_n \ge 0; \quad a_2 > 0. \tag{i}$$

(The last condition $a_2 > 0$ just means that a is not the coordinate versor e_1.)

2.2. *Probabilistic intermezzo.* Let $X : \Omega \to \mathbb{R}^2$ be a 2-dimensional r.v. (= random variable) defined on a probability space (P, Ω) with density $g : \mathbb{R}^2 \to \mathbb{R}$. Denote by $\mathcal{E}_X$ the characteristic function of X defined by

$$\mathcal{E}_X(\xi, \eta) := \iint_{\mathbb{R}^2} g(x, y) \exp i(x\xi + y\eta)\,dx\,dy.$$

Let Z denote the r.v. uniformly distributed on D, i.e. the density of Z is $\pi^{-1} 1_D$ where 1_D denotes the indicator function of D. Since D is centrally symmetric,

$$\mathcal{E}_Z(\xi, \eta) = \pi^{-1} \iint_{\mathbb{R}^2} 1_D(x, y) \exp i(x\xi + y\eta)\,dx\,dy = \pi^{-1} \iint_D \cos(x\xi + y\eta)\,dx\,dy.$$

Since 1_D is rotation invariant, so is $\mathcal{E}_Z$. Thus

$$\mathcal{E}_Z(\xi, \eta) = \mathcal{E}_Z(\sqrt{\xi^2 + \eta^2}, 0) = \pi^{-1} \iint_D \cos(\sqrt{\xi^2 + \eta^2})\,dx\,dy.$$

Passing to polar coordinates, expanding into a power series the "outer" cosinus of the integrand $\varrho \cos(\sqrt{\xi^2 + \eta^2}\,\varrho \cos\phi)$ and integrating term by term we get (cf. [KK] for details)

$$\mathcal{E}_Z(\xi, \eta) = j_1(\sqrt{\xi^2 + \eta^2}), \tag{2}$$

where

$$j_1(t) = 2J_1(t)t^{-1} \quad \text{for } t > 0 \quad \text{and} \quad j_1(0) = 1. \tag{3}$$

Let $Z^{(1)}, \ldots, Z^{(n)}$ be independent r.v.'s, each distributed as Z. Fix $a = (a_j)_{j=1}^n$ satisfying (i) and consider the r.v.

$$Z_a := \sum_{j=1}^n a_j Z^{(j)}.$$

The independence of the $Z^{(j)}$'s and the equality $\mathcal{E}_{Z^{(j)}} = \mathcal{E}_Z$ (because $Z^{(j)}$ has the same distribution as Z for $j = 1, \ldots, n$) yield

$$\mathcal{E}_{Z_a}(\xi, \eta) = \prod_{j=1}^{n} \mathcal{E}_{Z^{(j)}}(a_j\xi, a_j\eta) = \prod_{j=1}^{n} \mathcal{E}_Z(a_j\xi, a_j\eta).$$

Note that, by the Parseval identity, $\mathcal{E}_Z \in L^2(\mathbb{R}^2)$ because $1_D \in L^2(\mathbb{R}^2)$. Thus if a satisfies (i) then $\mathcal{E}_{Z_a} \in L^1(\mathbb{R}^2)$, being a product of at least two functions belonging to $L^2(\mathbb{R}^2)$ and bounded. Hence, by the Fourier inversion formula (cf. [F], Chapt. XV, §7),

(∗) *if a satisfies* (i) *then Z_a has a density, say g_a, which is a continuous function given by*

$$g_a(x,y) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \mathcal{E}_{Z_a}(\xi,\eta) \exp(-i(x\xi + y\eta))\, d\xi\, d\eta \quad \textit{for } (x,y) \in \mathbb{R}^2.$$

Next consider the function $(x,y) \mapsto \mathrm{vol}_{2n-2}(H_a^z \cap D^n)$, where $z = x + iy$. Note that $H_a^{z+w} = H_a^z + wa$ for $w \in \mathbb{C}$ (here wa denotes the scalar w-multiple of the vector $a \in \mathbb{C}^n$). Thus, by the Cavalieri Principle, for $z \in \mathbb{C}$ we have

$$\mathrm{vol}_{2n}\Big(\bigcup_{|w|<\varepsilon} \{H_a^z + wa\} \cap D^n\Big) = \mathrm{vol}_{2n}\Big(\bigcup_{|w|<\varepsilon} H_a^{z+w} \cap D^n\Big) = \iint_{|w|<\varepsilon} \mathrm{vol}_{2n-2}(H_a^{z+w} \cap D^n)\, d\Re w\, d\Im w.$$

On the other hand the definition of density yields

$$[\mathrm{vol}_{2n}(D^n)]^{-1} \mathrm{vol}_{2n}\Big(\bigcup_{|w|<\varepsilon} H_a^{z+w} \cap D^n\Big) = P\Big(\bigcup_{|w|<\varepsilon} \Big\{\sum_{j=1}^{n} a_j Z^{(j)} = z+w\Big\}\Big) = \iint_{|w|<\varepsilon} g_a(z+w)\, d\Re w\, d\Im w.$$

Thus at each point z of continuity of the function $z \mapsto \mathrm{vol}_{2n-2}(H_a^z \cap D^n)$, remembering that $\mathrm{vol}_{2n}(D^n) = \pi^n$, one has

$$\begin{aligned} g_a(z) &= \lim_{\varepsilon\to 0} (\varepsilon^2\pi)^{-1} \iint_{|w|<\varepsilon} g_a(z+w)\, d\Re w\, d\Im w \\ &= \lim_{\varepsilon\to 0} (\varepsilon^2\pi)^{-1} \pi^{-n} \mathrm{vol}_{2n}\Big(\bigcup_{|w|<\varepsilon} H_a^{z+w} \cap D^n\Big) \\ &= \pi^{-n} \mathrm{vol}_{2n-2}(H_a^z \cap D^n). \end{aligned}$$

It follows from geometric considerations that if a satisfies (i) then $z \mapsto \mathrm{vol}_{2n-2}(H_a^z \cap D^n)$ is a continuous function at each point $z \in \mathbb{C}$. This is obvious if either $H_a^z \cap D^n$ contains an interior point of D^n, or the hyperplane H_a^z is disjoint from D^n. Otherwise the assumption that a satisfies (i) implies that $H_a^z \cap D^n$ is contained in a hyperplane of dimension less than $2n-2$, hence $\mathrm{vol}_{2n-2}(H_a^z \cap D^n) = 0$. In the latter case also $g_a(z) = 0$ because g_a is continuous and replacing z by tz with $t > 1$ we easily conclude that $H_a^{tz} \cap D^n = \emptyset$, hence $g_a(tz) = 0$. Thus $\mathrm{vol}_{2n-2}(H_a^z \cap D^n) = \pi^n g_a(z)$ for all $z \in \mathbb{C}$.

Concluding, for all a satisfying (i), and therefore satisfying (∗), we obtain

$$\mathrm{vol}_{2n-2}(H_a^z \cap D^n) = \frac{\pi^{n-2}}{4} \iint_{\mathbb{R}^2} \mathcal{E}_{Z_a}(\xi,\eta) \exp(-i(\Re z\xi + \Im z\eta))\, d\xi\, d\eta.$$

In particular

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) = \frac{\pi^{n-2}}{4} \iint_{\mathbb{R}^2} \mathcal{E}_{Z_a}(\xi,\eta)\, d\xi\, d\eta = \frac{\pi^{n-2}}{4} \iint_{\mathbb{R}^2} \prod_{j=1}^{n} \mathcal{E}_Z(a_j\xi, a_j\eta)\, d\xi\, d\eta. \tag{4}$$

Combining (4) with (2) and (3), and passing to polar coordinates we finally get

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) = \frac{\pi^{n-1}}{2} \int_0^\infty \prod_{j=1}^{n} j_1(a_j t) t\, dt \quad \text{for } a \text{ satisfying (i)}. \tag{5}$$

2.3. *The upper bound.* We assume that a satisfies (i) and consider separately two cases.

Case (I): $a_1 \le 1/\sqrt{2}$. Let $n(a)$ denote the last index j such that $a_j > 0$. Clearly $2 \le n(a) \le n$. It follows from (5) and (3) that

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) = \frac{\pi^{n-1}}{2} \int_0^\infty \prod_{j=1}^{n(a)} j_1(a_j t) t\, dt. \tag{6}$$

Put $p_j = a_j^{-2}$ for $j = 1, \ldots, n(a)$. Invoking (i) and remembering that we consider Case (I), we have

$$\sum_{j=1}^{n(a)} 1/p_j = 1; \quad p_j \ge 2 \quad (j = 1, \ldots, n(a)).$$

Applying the Hölder Inequality with respect to the measure $t dt$ and making appropriate rescalings we get

$$\int_0^\infty \Big|\prod_{j=1}^{n(a)} j_1(a_jt)\Big| t\,dt \le \prod_{j=1}^{n(a)} \Big(\int_0^\infty |j_1(a_jt)|^{p_j} t\,dt\Big)^{1/p_j}$$
$$= \prod_{j=1}^{n(a)} a_j^{-2/p_j} \Big(\int_0^\infty |j_1(t)|^{p_j} t\,dt\Big)^{1/p_j}.$$

Now we apply Proposition 1.1 to get further domination,

$$\le \prod_{j=1}^{n(a)} a_j^{-2/p_j} \Big(\frac{4}{p_j}\Big)^{1/p_j} = 4.$$

In the latter step, equality holds if and only if $p_j = 2$ for $j = 1, \ldots, n(a)$, which forces $n(a) = 2$ and $a_1 = a_2 = \sqrt{1/2}$. Invoking (6) we finally get

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) \le 2\pi^{n-1},$$

and we have equality if and only if $a = (\sqrt{1/2}, \sqrt{1/2}, 0, \ldots, 0)$.

CASE (II): $a_1 > 1/\sqrt{2}$. Consider the "cylinder"

$$C_n := \{\mathfrak{z} \in \mathbb{C}^n : |\mathfrak{z}_j| \le 1 \text{ for } j = 2, \ldots, n\}.$$

Clearly $D^n \subseteq C_n$. Thus $H_a \cap D^n \subseteq H_a \cap C_n$. Hence

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) \le \mathrm{vol}_{2n-2}(H_a \cap C_n). \tag{7}$$

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the nonsingular linear map defined by $T(za + h) = ze_1 + Q(h)$ for $z \in \mathbb{C}$ and $h \in H_a$, where Q is the orthogonal projection onto the hyperplane $H_{e_1} = \{\mathfrak{z} \in \mathbb{C}^n : \mathfrak{z}_1 = 0\}$ and $e_1 = (1, 0, \ldots, 0)$. It is easy to check that in the unit vector basis T is given by the matrix

$$M_T = \begin{pmatrix} a_1 & -a_1a_2 & -a_1a_3 & \ldots & -a_1a_n \\ a_2 & 1-a_2^2 & -a_2a_3 & \ldots & -a_2a_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & -a_na_2 & -a_na_3 & \ldots & 1-a_n^2 \end{pmatrix}.$$

A straightforward calculation gives $\det M_T = a_1$. Let $\widetilde{T} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the map induced by T under our identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and let $M_{\widetilde{T}}$ be the matrix corresponding to $\widetilde{T}$. Then $M_{\widetilde{T}} = M_T \otimes I_{\mathbb{R}^2}$ is the Kronecker product of M by the matrix corresponding to the identity operator $I_{\mathbb{R}^2}$ on $\mathbb{R}^2$. Thus $\det M_{\widetilde{T}} = a_1^2 > 1/2$ because we are considering Case (II). Hence, by the Jacobian Formula,

$$\mathrm{vol}_{2n}(A) = a_1^{-2}\, \mathrm{vol}_{2n}(T(A)) \quad \text{for Borel } A \subset \mathbb{C}^n.$$

Fix $B \subset H_a$ with $\mathrm{vol}_{2n-2}(B) > 0$. Let $B_\varepsilon = \bigcup_{|z|<\varepsilon}\{za+B\}$. Then $\mathrm{vol}_{2n-2}(B) = \lim_{\varepsilon\to 0}(\pi\varepsilon^2)^{-1}\mathrm{vol}_{2n}(B_\varepsilon)$. Since $T(B)$ is orthogonal to $e_1 = T(a)$, we have $T(B_\varepsilon) = \bigcup_{|z|<\varepsilon}\{ze_1 + T(B)\}$ and

$$\mathrm{vol}_{2n-2}(T(B)) = \lim_{\varepsilon\to 0}(\pi\varepsilon^2)^{-1}\mathrm{vol}_{2n}(T(B_\varepsilon)).$$

Hence

$$\mathrm{vol}_{2n-2}(B) = a_1^{-2}\,\mathrm{vol}_{2n-2}(T(B)) < 2\,\mathrm{vol}_{2n-2}(T(B)).$$

Since $T(H_a \cap C_n) = \{\mathfrak{z} \in \mathbb{C}^n : \mathfrak{z}_1 = 0 \text{ and } |\mathfrak{z}_j| \le 1 \text{ for } j = 2, \ldots, n\}$, we have $\mathrm{vol}_{2n-2}(T(H_a \cap C_n)) = \pi^{n-1}$. Thus invoking (7) we see that in Case (II),

$$\mathrm{vol}_{2n-2}(H_a \cap D^n) \le \mathrm{vol}_{2n-2}(H_a \cap C_n) < 2\pi^{n-1}. \ \blacksquare$$

2.4. *The lower bound.* Let λ denote the usual Lebesgue measure in $\mathbb{R}^2$. We need

LEMMA 2.1. *Let*

$$K = \Big\{g : \mathbb{R}^2 \to \mathbb{R} : g \ge 0 \ \&\ \int_{\mathbb{R}^2} g\,d\lambda = 1\Big\}.$$

Then, for $g \in K$,

$$\|g\|_\infty \int_{\mathbb{R}^2} \|x\|_2^2 g(x)\,\lambda(dx) \ge (2\pi)^{-1},$$

and there is equality if and only if g *is the indicator function of a disc centered at the origin divided by the area of the disc.*

Proof. Put $\lambda_g(dx) = g(x)\lambda(dx)$. Then

$$\int_{\mathbb{R}^2} \|x\|_2^2 g(x)\,\lambda(dx) = \int_0^\infty \lambda_g(\mathbb{R}^2 \setminus B_2(0,\sqrt{t}))\,dt$$
$$\ge \int_0^\infty (1 - \|g\|_\infty \lambda(B_2(0,\sqrt{t})))_+\,dt$$
$$= \int_0^{(\pi\|g\|_\infty)^{-1}} (1 - \pi\|g\|_\infty t)\,dt = (2\pi)^{-1}\|g\|_\infty^{-1}.$$

Clearly there is equality if and only if $g = \|g\|_\infty 1_{B_2(0,(\pi\|g\|_\infty)^{-1/2})}$ λ-a.e. $\blacksquare$

Now we complete the proof of the lower bound part of Theorem $B_{\mathbb{C}}$. Fix a satisfying (i). Let g_a be the density of $Z_a = \sum_{j=1}^n a_j Z^{(j)}$. Then $g_a \in K$. Let as usual $E|X|^2$ denote the second moment of the 2-dimensional r.v. X. Since the $Z^{(j)}$'s are independent and they have the same distribution as Z, (remembering that $\sum_{j=1}^n a_j^2 = 1$) we have

$$\int_{\mathbb{R}^2} \|x\|_2^2 g_a(x)\,\lambda(dx) = E|Z_a|^2 = \sum_{j=1}^n a_j^2 E|Z^{(j)}|^2 = E|Z|^2 = \frac{1}{2}.$$

Thus, by Lemma 2.1, $\|g_a\|_\infty \ge \pi^{-1}$. As shown in 2.2, g_a is continuous because a satisfies (i). Thus, by the second part of Lemma 2.1, $\|g_a\|_\infty > \pi^{-1}$. We have also shown in 2.2 that (i) implies that $g_a(\Re z, \Im z) = \pi^{-n}\operatorname{vol}_{2n-2}(H_a^z \cap D^{n-1})$. Therefore, by a corollary to the Brunn–Minkowski Theorem (cf., e.g., [B1], p. 466), g_a attains its maximum at the origin. Thus, if a satisfies (i) then

$$\operatorname{vol}_{2n-2}(H_a \cap D^n) = \pi^n g_a(0,0) > \pi^{n-1}.$$

Obviously $\operatorname{vol}_{2n-2}(H_{e_1} \cap D^n) = \pi^{n-1}$. ■

3. Proof of Proposition 1.1. Note that $0 \le |j_1(t)| \le 1$ for $t \ge 0$ because $(\xi,\eta) \mapsto j_1(\sqrt{\xi^2+\eta^2})$ is the characteristic function of a density.

We consider two cases: (I) $p \ge 8/3$ and (II) $2 \le p \le 8/3$.

We need two lemmas.

Lemma 3.1. *One has*

$$|j_1(t)| \le \exp\left(-\frac{t^2}{8} - \frac{t^4}{3\cdot 2^7}\right) \quad \text{for } 0 \le t \le 4.$$

Lemma 3.2. *Let*

$$I_p := \int_0^\infty t \exp\left(-\frac{pt^2}{8} - \frac{pt^4}{3\cdot 2^7}\right) dt.$$

Then

$$I_p \le \frac{4}{p} - \frac{4}{3p^2} + \frac{4}{3p^3}.$$

First assuming the validity of the lemmas we give

Proof of Proposition 1.1 in case (I). By Lemma 3.1 and the definition of I_p it is sufficient to show that if $p \ge 8/3$ then

$$I_p + \Delta_p < 4/p, \quad \text{where} \quad \Delta_p := \int_4^\infty |j_1(t)|^p t\, dt.$$

Since $|J_1(t)| \le \sqrt{2/\pi}\,(t^2-1)^{-1/4}$ for $t > 1$ (cf. [W], p. 447), we get

$$|j_1(t)| \le \sqrt{\frac{2}{\pi}}\,(t^2-1)^{-1/4} 2t^{-1} < \frac{8t^{-3/2}}{\sqrt{2\pi}\sqrt[4]{15}} \quad \text{for } t > 4.$$

Thus

$$\Delta_p < \left(\frac{8}{\sqrt{2\pi}\sqrt[4]{15}}\right)^p \int_4^\infty t^{1-3p/2}\, dt = \frac{32}{3p-4}(\sqrt{2\pi}\sqrt[4]{15})^{-p}.$$

The desired conclusion now follows from Lemma 3.2 and the elementary inequality

$$(\sqrt{2\pi}\sqrt[4]{15})^p > 26p > \frac{24p^3}{(3p-4)(p-1)}$$

for $p \ge 8/3$. ■

We return to the lemmas. Lemma 3.1 is a special case for $n = 4$ of Proposition 11 in the unpublished preprint [K]; see also [KK], Remark after Proposition 12.

Proof of Lemma 3.2. Let

$$I_{u,v} = \int_0^\infty t \exp(-ut^2 - vt^4)\, dt \quad \text{for } u, v > 0.$$

Substituting $s = ut^2$ we get

$$I_{u,v} = \frac{1}{2u}\int_0^\infty \exp\left(-s - \frac{v}{u^2}s^2\right) ds = \frac{1}{2u}\int_0^\infty \exp\left(-\frac{v}{u^2}s^2\right)\exp(-s)\, ds.$$

Since $\exp(-x) \le 1 - x + x^2/2$ for $x \ge 0$, (remembering that $\int_0^\infty s^k \exp(-s)\, ds = k!$ for $k = 0, 1, \ldots$) we get with $x = \frac{v}{u^2}s^2$,

$$I_{u,v} \le \frac{1}{2u}\int_0^\infty \left(1 - \frac{v}{u^2}s^2 + \frac{v^2}{2u^4}s^4\right)\exp(-s)\, ds$$
$$= \frac{1}{2u}\left(1 - 2\cdot\frac{v}{u^2} + 24\cdot\frac{v^2}{2u^4}\right) = \frac{1}{2u} - \frac{v}{u^3} + 6\frac{v^2}{u^5}.$$

Specifying $u = p/8$, $v = p/(3\cdot 2^7)$ we get

$$I_p = I_{p/8,\,p/(3\cdot 2^7)} \le \frac{4}{p} - \frac{4}{3p^2} + \frac{4}{3p^3}. \ ■$$

The argument presented above is due to Professor Hermann König and is published here with his permission. The authors' original proof was more complicated and based on the asymptotic expansion of the Erf function (cf. [RG]; 6.234).

Proof of Proposition 1.1 in case (II). First consider $p = 2$. The identity $\int_0^\infty j_1(t)^2 t\, dt = 2$ follows by passing from polar to Cartesian coordinates in the plane and applying the Parseval identity for the Fourier transform:

$$2\pi\int_0^\infty j_1(t)^2 t\, dt = \iint_{\mathbb{R}^2} j_1(\sqrt{\xi^2+\eta^2})^2\, d\xi d\eta$$
$$= (2\pi)^2 \iint_{\mathbb{R}^2} [\pi^{-1}1_D(x,y)]^2\, dx\, dy = 4\pi.$$

Now, let $2 < p \le 8/3$. It suffices to show that

$$\int_0^\infty |j_1(t)|^{8/3} t\, dt < 2e^{-1/3}. \tag{8}$$

Indeed, by the Hölder Inequality for $2 < p \le 8/3$ we have

$$\begin{aligned}\int_0^\infty |j_1(t)|^p t\,dt &= \int_0^\infty |j_1(t)|^{8-3p}|j_1(t)|^{4p-8}t\,dt\\ &\le \Big(\int_0^\infty |j_1(t)|^2 t\,dt\Big)^{(8-3p)/2}\Big(\int_0^\infty |j_1(t)|^{8/3}t\,dt\Big)^{(3p-6)/2}\\ &< 2^{(8-3p)/2}(2e^{-1/3})^{(3p-6)/2}\\ &= 2/e^{(p-2)/2} \le 2\Big/\Big(1+\frac{p-2}{2}\Big) = 4/p.\end{aligned}$$

The proof of (8) is numerical. One can argue as follows. Invoke the inequality $|J_1(t)| \le \sqrt{2/\pi}\,(t^2-1)^{-1/4}$ for $t > 1$ (cf. [W], p. 447). Hence $t \ge 5$ yields

$$|j_1(t)| \le 2\sqrt[4]{\frac{25}{24}}\sqrt{\frac{2}{\pi}}\,t^{-3/2}.$$

Thus

$$\int_5^\infty |j_1(t)|^{8/3}t\,dt \le \Big[\sqrt{\frac{8}{\pi}}\sqrt[4]{\frac{25}{24}}\Big]^{8/3}\int_5^\infty \frac{dt}{t^3} = 4\pi^{-4/3}(25/3)^{2/3}/50 < 0.08.$$

On the other hand we have

$$\int_0^5 |j_1(t)|^{8/3}t\,dt < 1.35.$$

The latter integral is estimated by its Riemann sums (the interval $[0,5]$ is divided into 25 intervals of length $1/5$ each):

$$\int_0^5 |j_1(t)|^{8/3}t\,dt \le \frac{1}{5}\sum_{k=0}^{24}\max(|j_1(k/5)|^{8/3}, |j_1((k+1)/5)|^{8/3});$$

here we use the fact that the function j_1 is nonincreasing in the interval $[0,5]$. Indeed, observe that for $t \in [0,5]$,

$$-2j_1'(t)/t = \sum_{k=0}^\infty (-t^2/4)^k\frac{1}{k!(k+2)!} \ge P(t^2/4),$$

where

$$P(a) = \sum_{k=0}^5 \frac{(-1)^k a^k}{k!(k+2)!}.$$

Now, as $P^{(5)} = -1/7! < 0$, $P^{(4)}(25/4) > 0$, $P'''(25/4) < 0$, $P''(25/4) > 0$, $P'(25/4) < 0$ and $P(25/4) > 0$ we prove that $P^{(4)}, -P''', P'', -P'$ and P are nonnegative on $[0, 25/4]$, which proves our assertion. Taking the values of J_1 from tables we compute the values $j_1(k/5)$ for $k = 0, 1, \ldots, 24$.

Thus, we get

$$\int_0^\infty |j_1(t)|^{8/3}t\,dt < 1.35 + 0.08 = 1.43 < 2e^{-1/3}. \ \blacksquare$$

4. Remarks

4.1. The upper bound estimate in Theorem $B_{\mathbb{C}}$ is valid for an arbitrary hyperplane section of a polydisc because if $0 \ne c \in \mathbb{C}$ then

$$\mathrm{vol}_{2n-2}(H_{\mathfrak{z}}^c \cap D^n) \le \mathrm{vol}_{2n-2}(H_{\mathfrak{z}} \cap D^n)$$

for arbitrary $\mathfrak{z} \in \mathbb{C}^n$. This is an easy consequence of the Brunn–Minkowski Theorem (cf., e.g., [B1], p. 466).

4.2. Lemma 2.1 generalizes as follows:

LEMMA 2.1′. *Let $0 < p < \infty$, let $n = 1, 2, \ldots$ and let $g : \mathbb{R}^n \to \mathbb{R}$ satisfy $g \ge 0$ and $\int_{\mathbb{R}^n} g\,d\lambda_n = 1$. Then*

$$\|g\|_\infty \int_{\mathbb{R}^n} |x|_2^p g(x)\ \lambda_n(dx) \le \frac{n}{n+p}(\mathrm{vol}_n(B_n))^{-p/n}$$

and there is equality if and only if $g = (\mathrm{vol}_n(cB_n))^{-1}1_{(cB_n)}$ for some $c > 0$.

Here B_n denotes the Euclidean unit ball centered at the origin and λ_n the usual Lebesgue measure on $\mathbb{R}^n$.

The case $n = 1$ is due to Ball ([B1], Lemma 1).

Lemma 2.1′ for $p = 2$ can be used to determine the sharp lower bound for the $(N-1)n$th volume of sections of the Cartesian product $(B_n)^N$ by $(N-1)n$-dimensional subspaces of $\mathbb{R}^N$.

4.3. Ball's inequality (cf. [B1], Lemma 3), which is a paradigm of Proposition 1.1, can be stated as follows:

$$\int_0^\infty \Big(\frac{|\sin t|}{t}\Big)^p dt < \frac{\pi}{2}\Big(\frac{2}{p}\Big)^{1/2} \quad \text{for } p > 2. \tag{9}$$

One can prove it in the same way as Proposition 1.1, by considering the cases $p \ge 3$ and $2 < p < 3$. The second case follows from the numerical inequality

$$b(3) < e^{-1/4}, \quad \text{where} \quad b(p) = \frac{2}{\pi}\int_0^\infty \Big|\frac{\sin t}{t}\Big|^p dt \quad \text{for } p > 2.$$

A common formulation of both inequalities (1) and (9) is the following. Let $a \ge 1/2$. Recall that the Bessel function of order a is defined by

$$J_a(t) := \sum_{m=0}^\infty (-1)^m\Big(\frac{t}{2}\Big)^{2m+a}\frac{1}{m!\Gamma(m+1+a)} \quad \text{for } t \ge 0.$$

Define $j_a : [0,\infty) \to \mathbb{R}$ by

$$j_a(t) = 2^a \Gamma(a+1)\frac{J_a(t)}{t^a} \quad \text{for } t > 0; \quad j_a(0) = 1.$$

Then we ask if it is true that

$$(10) \qquad \int_0^\infty |j_a(t)|^p t^{2a-1}\,dt < \Big(\int_0^\infty (j_a(t))^2 t^{2a-1}\,dt\Big)\frac{2^a}{p^a} \qquad \text{for } p > 2.$$

Note that for $a = 1/2$, (10) is equivalent to (9) because

$$J_{1/2}(t) = \left(\frac{2}{\pi t}\right)^{1/2}\frac{\sin t}{t} \quad \text{for } t \in \mathbb{R}$$

(cf. [W], p. 54, formula (1)). For $a = 1$, (10) is equivalent to Proposition 1.1. From an old formula due to Weber (cf. [W], §13.42, p. 405, formula (1)) we get

$$(11) \qquad \int_0^\infty [j_a(t)]^2 t^{2a-1}\,dt = 2^{2a}[\Gamma(a+1)]^2 \int_0^\infty \frac{[J_a(t)]^2}{t}\,dt = 2^{2a-1}a[\Gamma(a)]^2.$$

If $a = n/2$ for $n = 1,2,\ldots$ then the function $\Xi \mapsto j_{n/2}(|\Xi|_2)$ for $\Xi \in \mathbb{R}^n$ is the characteristic function of the n-dimensional r.v. with density $(\mathrm{vol}_n(B_n))^{-1}1_{B_n}$ (cf. [KK]). Thus, for $a = n/2$, (11) can be proved in the same way as at the beginning of Case (II) of Proposition 1.1. It is plausible that (10) is true iff $1/2 \le a \le 1$. Professor Hermann König noticed that (10) is false for $a = n/2$ for $n = 3,4,\ldots$

4.4. The upper bound part of Theorem $B_{\mathbb{R}}$ is a simple consequence of the upper bound part of Theorem $B_{\mathbb{C}}$ while the lower estimate in Theorem $B_{\mathbb{C}}$ follows from the lower estimate in Theorem $B_{\mathbb{R}}$.

This observation is due to Stanisław Szarek who kindly permitted to include his argument in the paper.

Let E and F be orthogonal subspaces of $\mathbb{R}^N$. Let $U \subset E$ and $V \subset F$ be convex sets symmetric with respect to the origin. Put

$$(12) \qquad U\oplus_2 V := \{x \in \mathbb{R}^N : x = su+tv;\ u \in U,\ v \in V,\ s^2+t^2 = 1,\ s,t \in \mathbb{R}\}.$$

It follows easily from Fubini's Theorem that

$$\mathrm{vol}_{k+l}(U \oplus_2 V) = c\,\mathrm{vol}_k(U)\cdot\mathrm{vol}_l(V),$$

where the constant $c = c_{k,l}$ depends only on k and l but not on U, V, E, and F $(k,l = 1,2,\ldots)$. Let $N = 2n$. Let H_a be a complex $(n-1)$-dimensional subspace of $\mathbb{C}^n$ orthogonal to a versor $a \in \mathbb{C}^n$ with all coordinates real. Under our identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ we identify H_a with a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$. We set E to be the subspace of H_a with even coordinates zero, F the subspace of H_a with odd coordinates zero, $U = I^{2n} \cap E$, $V = I^{2n} \cap F$. Then the sets U and V are isometric to the intersection of I^n with the subspace, say $H_a^{\mathbb{R}}$, of $\mathbb{R}^n$ orthogonal to the versor a regarded as element of $\mathbb{R}^n$; moreover $U \oplus_2 V \subset H_a \cap D^n$. Thus using (12) for $k = l = n-1$ we get

$$(13) \qquad c_{n-1,n-1}[\mathrm{vol}_{n-1}(H_a^{\mathbb{R}} \cap I^n)]^2 = (\pi/4)^{n-1}[\mathrm{vol}_{n-1}(H_a^{\mathbb{R}} \cap I^n)]^2 \le \mathrm{vol}_{2n-2}(H_a \cap D^{n-1}),$$

because specifying $a = e_1$ we get $c_{n-1,n-1} = (\pi/4)^{n-1}$. Clearly (13) yields the desired conclusion.

Addendum. We are indebted to Franck Barthe and Alexander Koldobsky for the following remarks:

1. Ball [B2] using his inequality (9) and the Brascamp–Lieb inequality has obtained the sharp upper bound of the volume of k-dimensional sections of the cube I^n for $n/2 \le k \le n-1$. His argument extends almost verbatim to k-dimensional (complex) sections of the polydisc. We replace (9) by the inequality (1) and the Brascamp–Lieb inequality by its complex counterpart which follows from Theorem 6 of [Bar]. One obtains:

THEOREM $B_{\mathbb{C}}^k$. *Let $D = \{z \in \mathbb{C} : |z| \le 1\}$. Let H be a complex linear subspace in $\mathbb{C}^n$ $(n = 2,3,\ldots)$ of codimension k. Then*

$$\mathrm{vol}_{2n-2k}(D^{n-k}) \le \mathrm{vol}_{2n-2k}(H \cap D^n) \le 2^k\,\mathrm{vol}_{2n-2k}(D^{n-k}).$$

The upper bound is sharp for $k \le n/2$. For the lower bound see Lemma 2.1′. Also Proposition 4 of [B2] can be transferred to the complex case yielding

$$\mathrm{vol}_{2n-2k}(H \cap D^n) \le \left(\frac{n}{n-k}\right)^{n-k}\mathrm{vol}_{2n-2k}(D^{n-k}).$$

2. The integral formula for volumes of sections goes back to Pólya ([P]). More bibliographical details can be found in [KL].

References

[B1] K. Ball, *Cube slicing in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. 97 (1986), 465–473.

[B2] —, *Volumes of sections of cubes and related problems*, in: Geometric Aspects of Functional Analysis (1987–88), Lecture Notes in Math. 1376, Springer, Berlin, 1989, 251–260.

[Bar] F. Barthe, *On a reverse form of the Brascamp–Lieb inequality*, Invent. Math. 134 (1998), 335–361.

[F] W. Feller, *Introduction to Probability Theory and its Applications*, *Vol. II*, Wiley, New York, 1996.

[H] D. Hensley, *Slicing the cube in* R^n *and probability (bounds for the measure of a central cube slice in* R^n *by probability methods)*, Proc. Amer. Math. Soc. 73 (1979), 95–100.

[KL] A. Koldobsky and M. Lifshits, *Average volume of sections of star bodies*, in: Geometric Aspects of Functional Analysis, Israel Seminar (GAFA), 1996–2000, V. Milman and G. Schechtman (eds.), Lecture Notes in Math. 1745, Springer, 2000, 119–146.

[K] H. König, *On the best constants in the Khintchine inequality for variables on the spheres*, preprint, Kiel Universität, 1998.

[KK] H. König and S. Kwapień, *Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors*, Positivity, to appear.

[MP] M. Meyer and A. Pajor, *Sections of the unit ball of* l_p^n, J. Funct. Anal. 80 (1988), 109–123.

[NP] F. L. Nazarov and A. N. Podkorytov, *Ball, Haagerup, and distribution functions*, in: Complex Analysis, Operator Theory, and Related Topics: S. A. Vinogradov—In Memoriam, V. Havin and N. Nikolski (eds.), Oper. Theory Adv. Appl. 113, Birkhäuser, Basel, 2000, 247–268.

[P] G. Pólya, *Berechnung eines bestimmten Integrals*, Math. Ann. 74 (1913), 204–212.

[RG] I. M. Ryshik and I. S. Gradstein, *Tables of Series, Products and Integrals*, Deutscher Verlag Wiss., Berlin, 1957.

[S] M. Schmuckenschläger, *Volume of intersections and sections of the unit ball of* l_p^n, Proc. Amer. Math. Soc. 126 (1998), 1527–1530.

[V] J. D. Vaaler, *A geometric inequality with applications to linear forms*, Pacific J. Math. 83 (1979), 543–553.

[W] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1995.

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: koles@mimuw.edu.pl

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: olek@impan.gov.pl

Received January 14, 2000 (4463)
Revised version May 17, 2000 and August 16, 2000

STUDIA MATHEMATICA 142 (3) (2000)

On isomorphisms of standard operator algebras

by

LAJOS MOLNÁR (Debrecen)

Abstract. We show that between standard operator algebras every bijective map with a certain multiplicativity property related to Jordan triple isomorphisms of associative rings is automatically additive.

1. Introduction. It is a surprising result of Martindale [7, Corollary] that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Therefore, one can say that the multiplicative structure of rings of that kind completely determines their ring structure. This result has been utilized by Šemrl in [11] to describe the form of the semigroup isomorphisms of standard operator algebras on Banach spaces. The aim of this paper is to generalize this result quite significantly. Other results on the additivity of multiplicative maps (in fact, *-semigroup homomorphisms) between operator algebras can be found in [3, 8].

Besides additive and multiplicative maps (that is, ring homomorphisms) between rings, sometimes one has to consider Jordan homomorphisms. The Jordan structure of associative rings has been studied by many people in ring theory. Moreover, Jordan operator algebras have serious applications in the mathematical foundations of quantum mechanics. If $\mathcal{R}, \mathcal{R}'$ are rings and $\phi : \mathcal{R} \to \mathcal{R}'$ is a transformation, then it is called a *Jordan homomorphism* if

$$\phi(A+B) = \phi(A) + \phi(B)$$

and

$$\phi(AB+BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$$

for every $A, B \in \mathcal{R}$. Clearly, every ring homomorphism is a Jordan homomorphism and the same is true for ring antihomomorphisms (a transforma-

2000 *Mathematics Subject Classification*: Primary 47B49.
This research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T030082, T031995, and by the Ministry of Education, Hungary, Reg. No. FKFP 0349/2000