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Received August 10, 1999
Revised version March 23, 2000 (4374)

STUDIA MATHEMATICA 142 (3) (2000)

On ideals consisting of topological zero divisors

by

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Abstract. The class $\omega(A)$ of ideals consisting of topological zero divisors of a commutative Banach algebra A is studied. We prove that the maximal ideals of the class $\omega(A)$ are of codimension one.

1. Introduction. Let A be a commutative Banach algebra over the complex field $\mathbb{C}$ with unit e. An element $a \in A$ is a *topological zero divisor* (TZD) if there exists a sequence $b_n \in A$ such that $\|b_n\| = 1$ and $\lim_{n\to\infty} ab_n = 0$. One says that an ideal $I \subset A$ consists of *joint topological zero divisors* (joint TZD) if for every finite collection $a_1, \ldots, a_k \in I$ there exists a sequence (b_n) of normalized elements of A such that

$$\lim_{n\to\infty} \sum_{j=1}^{k} \|a_j b_n\| = 0.$$

The set of all ideals of A consisting of joint TZD is denoted by $ł(A)$ while $Ł(A)$ denotes the set of those elements of $ł(A)$ which are maximal ideals of A. The class $ł(A)$ was intensively studied in the 70's. The most important results are the following theorems:

THEOREM 1.1 (Żelazko [5]). *The maximal ideals of A which belong to the Shilov boundary $S(A)$ are elements of $Ł(A)$. If A is a function algebra then $Ł(A) = S(A)$.*

THEOREM 1.2 (Słodkowski [3]). *If $J \in ł(A)$ then there exists $I \in Ł(A)$ such that $J \subset I$.*

V. Müller has proved a result conjectured by W. Żelazko which provides a complete characterization of $ł(A)$:

THEOREM 1.3 (Müller [2]). *An ideal I of A belongs to $ł(A)$ if and only if I is not removable.*

2000 *Mathematics Subject Classification*: Primary 46J20.
Research partially supported by SNI.

Recall that $I \subset A$ is *removable* if there exists a topological isomorphism of A into another Banach algebra B such that I generates B.

There exist however examples of ideals I such that every element of I is a TZD although $I \notin \L(A)$.

EXAMPLE. Let $\mathcal{A}(B)$ be the Banach algebra of continuous functions in the unit ball $B \subset \mathbb{C}^2$ which are analytic in the interior of the ball. Every $f \in \mathcal{A}(B)$ which vanishes at $0 \in B$ takes the value 0 also at some point of the boundary, that is, on the unit sphere S. The points of the sphere form the Shilov boundary of $\mathcal{A}(B)$. By Żelazko's theorem a maximal ideal of $\mathcal{A}(B)$ belongs to the Shilov boundary if and only if it consists of joint TZD. It follows that the ideal J formed by the elements of $\mathcal{A}(B)$ vanishing at $0 \in B$ consists of TZD, but obviously not of joint TZD.

In this paper we study the ideals of A which consist of TZD and we prove that a theorem analogous to Theorem 1.2 is valid.

THEOREM 1.4. *Let A be a unital commutative Banach algebra. If J is an ideal of A consisting of TZD then there exists in A a maximal ideal I also consisting of TZD such that $J \subset I$.*

In Section 3 we define a new joint spectrum which is greater than the approximate point spectrum but is contained in the rationally convex hull of the latter. Theorem 1.4 is equivalent to the fact that this spectrum obeys the spectral mapping formula.

The author is very indebted to Vladimír Müller, Andrzej Sołtysiak and Wiesław Żelazko for stimulating discussions and encouragement received.

2. Proof of Theorem 1.4. Denote by $M(A)$ the maximal ideal space of A provided with the Gelfand topology. This is a compact space. As usual, we identify $M(A)$ with the set of nonzero multiplicative functionals on A by associating with a functional φ the ideal $\ker\varphi$. The Gelfand transform of $a \in A$ is denoted by $\widehat{a}$:

$$\widehat{a}(\ker\varphi) = \varphi(a).$$

The set $\L(A)$ is closed in $M(A)$. If $x \in A$ is a TZD then the ideal xA consists of joint TZD and by Theorem 1.2 it is contained in some $I \in \L(A)$, hence $\widehat{x}(I) = 0$. This means that $a \in A$ is a TZD if and only if $\widehat{a}$ has a zero on $\L(A)$.

The ideal generated in the algebra A by the elements $a_1, \ldots, a_k$ is denoted by $I_A(a_1, \ldots, a_k)$. If K is a Hausdorff compact space, we denote by $C(K)$ the Banach algebra of continuous functions on K equipped with the supremum norm $\|\cdot\|$.

The following property of subalgebras of $C(K)$ is a decisive step in the proof of Theorem 1.4.

PROPOSITION 2.1. *Let K be a compact Hausdorff space and let $\mathcal{A}$ be a unital subalgebra of $C(K)$. Suppose that for some $f_1, \ldots, f_k \in \mathcal{A}$ every function in the ideal $I_{\mathcal{A}}(f_1, \ldots, f_k)$ has a zero in K. Then for every $g \in \mathcal{A}$ there exists $\mu \in \mathbb{C}$ such that every function in the ideal $I_{\mathcal{A}}(f_1, \ldots, f_k, g-\mu)$ has a zero in K.*

Proof. Assume the contrary: there exists $g \in \mathcal{A}$ such that for every $\mu \in \mathbb{C}$ we can find $\phi_j^\mu \in \mathcal{A}$, $1 \le j \le k+1$, such that

$$u^\mu = \sum_{j=1}^{k} \phi_j^\mu f_j + \phi_{k+1}^\mu (g-\mu)$$

nowhere vanishes on K. Obviously, the functions ϕ_j^μ can be chosen in such a way that $|u^\mu| > 1$. Denote by Δ a closed disc in the complex plane centered at zero and containing $g(K)$.

LEMMA 2.2. *There exist a collection of functions $\psi_j^\mu \in \mathcal{A}$ and $D > 0$ such that $\|\psi_{k+1}^\mu\| \le D$ and $|\sum_{j=1}^{k} \psi_j^\mu f_j + \psi_{k+1}^\mu (g-\mu)| > 1$ for every $\mu \in \Delta$.*

Proof. For every fixed $\mu \in \Delta$ there exists $r(\mu) > 0$ such that for λ obeying $|\lambda - \mu| < r(\mu)$ we still have

$$\Big|\sum_{j=1}^{k} \phi_j^\mu f_j + \phi_{k+1}^\mu (g-\lambda)\Big| = |u^\mu + \phi_{k+1}^\mu (\mu-\lambda)| > 1.$$

By the compactness of Δ there exist a finite set $\{\mu_i\}_{i=1}^m \subset \Delta$ and the corresponding finite collection of functions $\{\phi_j^{\mu_i}\}$ such that for every $\mu \in \Delta$ and for i such that $|\mu - \mu_i| < r(\mu_i)$ we have

$$\Big|\sum_{j=1}^{k} \phi_j^{\mu_i} f_j + \phi_{k+1}^{\mu_i} (g-\mu)\Big| > 1.$$

For given $\mu \in \Delta$ let $i_0 = \min\{i \mid |\mu - \mu_i| < r(\mu_i)\}$. We define $\psi_j^\mu = \phi_j^{\mu_{i_0}}$. Taking $D = \max_{1\le i\le m} \|\phi_{k+1}^{\mu_i}\|$ we end the proof of Lemma 2.2.

Set $r = 1/D$ and cover Δ with the discs $\{D(\nu_i, r)\}_{i=1}^l$, where $\nu_i \in \Delta$. We define

$$h_i = u^{\nu_i} = \sum_{j=1}^{k} \psi_j^{\nu_i} f_j + \psi_{k+1}^{\nu_i} (g-\nu_i).$$

Since $|h_i| > 1$ for $1 \le i \le l$, the inverse functions $t_i = h_i^{-1}$ satisfy $\|t_i\| < 1$. Denote by B the smallest closed unital subalgebra of $C(K)$ which contains $\mathcal{A}$ and the functions t_i, $1 \le i \le l$.

LEMMA 2.3. *The closed ideal M generated in B by the functions f_j, $1 \le j \le k$, is proper.*

Proof. The set of functions of the form

$$p = \sum_{j=1}^{k} f_j \sum_{N_j=(n_{j1},\ldots,n_{jl})\in F(p)} b_{N_j} t_1^{n_{j1}} \ldots t_l^{n_{jl}},$$

where $F(p)$ is a finite subset of $\mathbb{N}^l$ and $b_{N_j} \in \mathcal{A}$, is dense in the ideal M. Let $L = \max_{ji} n_{ji}$. We obtain

$$q = h_i^L \ldots h_l^L p = \sum_{j=1}^{k} f_j \sum_{N_j=(n_{j1},\ldots,n_{jl})\in F(p)} b_{N_j} h_1^{L-n_{j1}} \ldots h_l^{L-n_{jl}}.$$

This function belongs to the ideal $I_{\mathcal{A}}(f_1,\ldots,f_k)$, so $q(x) = 0$ for some $x \in K$. Obviously q is not invertible in B and $p = t_i^N \ldots t_l^N q$ is not invertible either. The ideal M contains a dense subset of noninvertible elements. Thus it is proper. This proves the lemma.

Every proper ideal in a commutative Banach algebra is contained in the kernel of a multiplicative functional. Let φ be a multiplicative functional on B such that $\varphi(M) = 0$. The function $g - \varphi(g)$ also belongs to the kernel of φ. This implies that $\nu_0 := \varphi(g)$ belongs to Δ, because for $\mu \notin \Delta$ the element $g - \mu$ is invertible.

There exists i such that $|\nu_0 - \nu_i| < r$. Set

$$u = \sum_{j=1}^{k} \psi_j^{\nu_i} f_j + \psi_{k+1}^{\nu_i}(g - \nu_0) = h_i + (\nu_i - \nu_0)\psi_{k+1}^{\nu_i}.$$

Since $\|t_i\| < 1$, we have

$$\|ut_i - 1\| = \|(\nu_i - \nu_0)\psi_{k+1}^{\nu_i} t_i\| < 1.$$

It follows that ut_i is invertible in B, so u is also invertible. This is a contradiction, because u belongs to the ideal generated by the $k+1$-tuple $f_1,\ldots,f_k, g-\nu$ so it belongs to $\ker\varphi$. Proposition 2.1 is proved. ∎

The next result can be called the projection property of the family of ideals consisting of TZD.

THEOREM 2.4. *Let $J \subset A$ be an ideal consisting of TZD. Let $a_1,\ldots,a_k \in J$. For every $c \in A$ there exists $\lambda \in \mathbb{C}$ such that the ideal generated by $a_1,\ldots,a_k, c - \lambda e$ consists of TZD.*

Proof. Let $\chi : A \to C(Ł(A))$ be the Banach algebra homomorphism defined by $\chi(a) = \widehat{a}|Ł(A)$. Take $\mathcal{A} = \chi(A)$ and let $f_i = \chi(a_i)$, $g = \chi(c)$. The functions f_i, $1 \le i \le k$, satisfy the assumptions of Proposition 2.1. There exists a complex number μ such that the Gelfand transform of an arbitrary element of $I_A(f_1,\ldots,f_k, g-\mu)$ vanishes at some point of $Ł(A)$. This means that $I_A(a_1,\ldots,a_k, c-\mu)$ consists of TZD. ∎

Proof of Theorem 1.4. Having obtained the projection property in Theorem 2.4, we can apply the standard method which works in the case of ideals consisting of joint TZD. For a fixed k-tuple $a_1,\ldots,a_k \in A$ generating an ideal consisting of TZD and for any $c \in A$ denote by $\delta(a_1,\ldots,a_k)$ the set of all $\lambda \in \mathbb{C}$ such that the ideal generated by $a_1,\ldots,a_k, c-\lambda e$ consists of TZD. The set $\delta(a_1,\ldots,a_k)$ is nonempty by Theorem 2.1. It is closed, bounded and satisfies

$$\delta(a_1,\ldots,a_k,b_1,\ldots,b_m) \subset \delta(a_1,\ldots,a_k) \cap \delta(b_1,\ldots,b_m). \tag{1}$$

Considering the collection of all ideals consisting of TZD as a set ordered by inclusion we see by the Kuratowski–Zorn lemma that there exist maximal ideals consisting of TZD. Denote by $\Omega(A)$ the set of all maximal ideals consisting of TZD. Let $J \in \Omega(A)$. If J is not of codimension 1 then there exists $c \in A$ such that $x - \lambda e \notin J$ for all $\lambda \in \mathbb{C}$.

By (1) and Theorem 2.4 the family of all sets of the form $\delta(a_1,\ldots,a_k)$ for $a_1,\ldots,a_k \in J$ has the finite intersection property, hence the intersection of all elements of the family is nonvoid. If λ_0 belongs to the intersection then for all $a_1,\ldots,a_k \in J$ the $k+1$-tuple $a_1,\ldots,a_k, c-\lambda_0 e$ generates an ideal consisting of TZD. This means that J and $c - \lambda_0 e$ generate an ideal consisting of TZD, although J was supposed to be maximal in this class of ideals. This contradiction proves that $J \in M(A)$. All elements of $\Omega(A)$ belong to $M(A)$. ∎

3. The subspectrum defined by $\Omega(A)$.

According to the terminology introduced by W. Żelazko [7] a *subspectrum* on a commutative unital Banach algebra A is a mapping $\widetilde{\sigma}$ which assigns to every k-tuple $(a_1,\ldots,a_k) \in A^k$ a compact set $\widetilde{\sigma}(a_1,\ldots,a_k) \subset \mathbb{C}^k$ such that

(1) $\widetilde{\sigma}(a_1,\ldots,a_k) \subset \prod_{j=1}^k \sigma(a_j)$,

(2) $p(\widetilde{\sigma}(a_1,\ldots,a_k)) = \widetilde{\sigma}(p(a_1,\ldots,a_k))$ for all polynomial mappings $p : \mathbb{C}^k \to \mathbb{C}^m$.

All subspectra of commutative Banach algebras were described in the same paper.

THEOREM 3.1 (see [7]). *$\widetilde{\sigma}$ is subspectrum on A if and only if there exists a compact subset $K \subset M(A)$ such that*

$$\widetilde{\sigma}(a_1,\ldots,a_k) = \{(\widehat{a}_1(I),\ldots,\widehat{a}_k(I)) \mid I \in K\}.$$

The usual joint spectrum σ corresponds to $K = M(A)$. The approximate point spectrum τ is obtained by taking $K = Ł(A)$.

Define

$$\omega(a_1,\ldots,a_k) = \{(\widehat{a}_1(I),\ldots,\widehat{a}_k(I)) \mid I \in \Omega(A)\}.$$

Since $Ł(A) \subset \Omega(A) \subset M(A)$, we have

$$\tau(a_1, \ldots, a_k) \subset \omega(a_1, \ldots, a_k) \subset \sigma(a_1, \ldots, a_k)$$

for every k-tuple $(a_1, \ldots, a_k) \in A^k$.

The spectrum ω coincides with τ for single elements $a \in A$; however, in the example presented in the introduction we have $(0,0) \in \omega(z_1, z_2)$, while $(0,0) \notin \tau(z_1, z_2)$.

By Theorem 1.4 the subspectrum $\omega(a_1, \ldots, a_k)$ can be described as the set of all $(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that the elements $a_1 - \lambda_1 e, \ldots, a_k - \lambda_k e$ generate an ideal consisting of TZD.

Let $\widehat{\tau}(a_1, \ldots, a_k)$ be the rationally convex hull of the approximate joint spectrum. This set consists of the points $(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that for every polynomial p of k variables, $p(\lambda_1, \ldots, \lambda_k) \in p(\tau(a_1, \ldots, a_k))$.

THEOREM 3.2. $\tau(a_1, \ldots, a_k) \subset \omega(a_1, \ldots, a_k) \subset \widehat{\tau}(a_1, \ldots, a_k)$.

Proof. The first inclusion is obvious. According to the remainder theorem (see [1], p. 461),

$$p(a_1, \ldots, a_k) - p(\lambda_1, \ldots, \lambda_k) \in I_A(a_1 - \lambda_1 e, \ldots, a_k - \lambda_k e)$$

for every polynomial p of k variables.

If $(\lambda_1, \ldots, \lambda_k) \in \omega(a_1, \ldots, a_k)$ then $p(\lambda_1, \ldots, \lambda_k) \in \tau(p(a_1, \ldots, a_k)) = p(\tau(a_1, \ldots, a_k))$. Thus $(\lambda_1, \ldots, \lambda_k) \in \widehat{\tau}(a_1, \ldots, a_k)$. ∎

The example considered in the introduction can be used for showing that in some cases the second inclusion is also proper. Let us consider the set of the restrictions to the unit sphere of all elements of the ideal $J \subset \mathcal{A}(B)$. This yields a subalgebra $\mathcal{A}$ consisting of TZD in the algebra of $A = C(S)$ of all continuous functions on the sphere. The subalgebra $\mathcal{A}$ generates A hence it is not contained in any ideal from $\Omega(A)$. In particular $0 \in \widehat{\tau}(z_1, z_2)$ but $0 \notin \omega(z_1, z_2)$ in the algebra A.

The same observation shows that $\widehat{\tau}$ does not obey the spectral mapping formula, hence it is not a subspectrum.

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Received September 8, 1999
Revised version May 16, 2000
(4393)