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Dimension of a measure

by

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Abstract. We propose a framework to define dimensions of Borel measures in a metric space by formulating a set of natural properties for a measure-dimension mapping, namely monotonicity, bi-Lipschitz invariance, (σ-)stability, etc. We study the behaviour of most popular definitions of measure dimensions in regard to our list, with special attention to the standard correlation dimensions and their modified versions.

1. Introduction. There is a well defined set of properties that a sensible definition of dimension of sets should satisfy, namely monotonicity, bi-Lipschitz invariance, stability, σ-stability, etc. (see [3] for a typical list). In this paper we propose a counterpart for the case of dimension of measures. In Section 2 we propose a sort of general definition of a measure dimension in a metric space X as a mapping from the Borel measures in X to the non-negative reals satisfying a list of natural properties (see items (a) to (g) in Section 2). We then check for a number of popular definitions of measure dimensions in fractal geometry whether each property is satisfied or not.

In particular, we concentrate on the study of the correlation dimension of a measure (see the definition in (2)), which is the most important dimension in chaotic time series analysis. Whereas correlation dimension mainly owes its popularity to its good computational accessibility through the fast algorithm introduced by Grassberger and Procaccia in [5], our analysis shows that it fails some important theoretical properties of our list. We thus thoroughly study the *modified* correlation dimension introduced by Pesin in [8] (see the definition in (4)), which turns out to be a much better theoretical tool according to our list (see Theorem 2.10 below).

In the case where a finite measure μ is *exact-dimensional*, i.e. there exists $\alpha \geq 0$ such that

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$$\mu\Big(X\setminus\Big\{x:\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\Big\}\Big)=0,$$

many different definitions of dimensions of μ collapse to the value α (see [9, 8]). In particular, modified correlation dimensions and Hausdorff dimension coincide. It has recently been proved in [1] that invariant measures of C^1 hyperbolic diffeomorphisms are exact-dimensional.

2. Measure-dimension mappings and correlation dimension.

Given a metric space (X,d), let $\mathcal{B}(X)$ denote the Borel σ-algebra in X, and let $\mathcal{BM}(X)$ be the class of non-null finite Borel measures defined on X. We will denote by $\|\mu\|$ the total mass of μ. Let dim be a non-negative real-valued function defined in $\mathcal{BM}(X)$. For a Borel measurable mapping $g:X\to Y$, the measure $g_\sharp\mu:=\mu\circ g^{-1}$ belongs to the class $\mathcal{BM}(Y)$. Natural properties that a measure-dimension mapping dim should satisfy are:

(a) (Monotonicity) If $\mu,\nu\in\mathcal{BM}(X)$ are such that μ is absolutely continuous with respect to ν (we write $\mu\ll\nu$), then $\dim\mu\ge\dim\nu$.

(b) (Lipschitz mappings) Let $g:X\to Y$ be a Lipschitz mapping, $\mu\in\mathcal{BM}(X)$, and assume that the mapping dim is also defined in $\mathcal{BM}(Y)$. Then $\dim(g_\sharp\mu)\le\dim\mu$.

(c) (Boundedness) If $X=\mathbb{R}^m$, then $\dim\mu\le m$ for any $\mu\in\mathcal{BM}(\mathbb{R}^m)$.

(d) (Absolutely continuous measures) If $X=\mathbb{R}^m$, and $\mu\in\mathcal{BM}(\mathbb{R}^m)$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^m$, then $\dim\mu=m$.

(e) (Discrete measures) If $\mu\in\mathcal{BM}(X)$ is a discrete measure, then $\dim\mu=0$.

(f) (Stability) $\dim(\mu+\nu)=\min\{\dim\mu,\dim\nu\}$ for any $\mu,\nu\in\mathcal{BM}(X)$.

(g) (σ-stability) $\dim(\sum_{i=1}^\infty\mu_i)=\inf_{i\in\mathbb{N}}\{\dim\mu_i\}$ for any collection $\{\mu_i\}_{i\in\mathbb{N}}$ such that $\sum_{i=1}^\infty\mu_i\in\mathcal{BM}(X)$.

There are further useful properties that follow from the above ones.

Since $\nu=\mu\circ g^{-1}$ implies that $\mu=\nu\circ g$ provided that g is injective, it is readily seen that property (b) above implies

(b′) If $g:X\to Y$ is bi-Lipschitz (i.e. both g and g^{-1} are Lipschitz), then $\dim\mu=\dim\mu\circ g^{-1}$.

Usually, an interesting definition of a mapping dim depends only on the metric structure of the space X. Property (b′) implies that such a definition remains unchanged if the space X is endowed with a metric d' which is equivalent to d, i.e. there exist positive constants c_1,c_2 such that $c_1d(x,y)\le d'(x,y)\le c_2d(x,y)$ for $x,y\in X$. Indeed, the identity mapping $\mathrm{id}:(X,d)\to(X,d')$ is obviously bi-Lipschitz in this case and therefore $\dim\mu=\dim'\mu$, where $\dim'\mu$ denotes the dimension mapping defined in the metric space (X,d').

Also, notice that property (a) implies that $\dim(C\mu)=\dim\mu$ for any constant $C>0$ so that the dimension of any measure remains unchanged if the measure is probabilized.

For $\mu,\nu\in\mathcal{BM}(X)$, write $\mu\le C\nu$ if there is a constant $C>0$ such that $\mu(A)\le C\nu(A)$ for $A\in\mathcal{B}(X)$. The weakest form of monotonicity for a dimension mapping dim may be formulated as

$$\text{(1)}\qquad \dim\mu\ge\dim\nu\quad\text{whenever } \mu\le C\nu.$$

Of course, $\mu\le C\nu$ if and only if μ has an L_∞-density with respect to ν. Also, notice that (a) implies (1) whereas the opposite implication does not hold in general. However, it does provided that dim also satisfies (g).

LEMMA 2.1. *Assume that* dim *satisfies* (1) *and* (g). *Then* dim *also satisfies* (a).

Proof. From the Radon–Nikodym theorem there exists $f\in L_1(\nu)$ such that $\mu(A)=\int_A f(x)\,d\nu$ for any $A\in\mathcal{B}(X)$. For every $n\in\mathbb{N}\cup\{0\}$ define $E_n=\{x:n\le f(x)<n+1\}$ and $\mu_n=\mu|_{E_n}$, where $\mu|_{E_n}$ denotes the measure μ restricted to the set E_n. Since $\mu=\sum_{n\in\mathbb{N}}\mu_n$ and $\mu_n\le(n+1)\nu$ for every n, from (g) and (1) we get $\dim\mu=\inf_{n\in\mathbb{N}}\dim\mu_n\ge\dim\nu$. ■

Notice that property (e) holds provided that property (1) does and that the dimension of any Dirac measure is zero.

LEMMA 2.2. *Suppose that* dim *satisfies* (1) *and also* $\dim(\delta_x)=0$ *for any* $x\in X$, *where* $\delta_x(A)=1$ *if* $x\in A$ *and* $\delta_x(A)=0$ *otherwise. Then* $\dim\mu=0$ *for any discrete measure* $\mu\in\mathcal{BM}(X)$.

Proof. Let $\mu=\sum_{i\in\mathbb{N}}p_i\delta_{x_i}$. Since $\mu_i:=p_i\delta_{x_i}\le\mu$ for any i, (1) gives that $0=\dim\mu_i\ge\dim\mu$. ■

Observe that it is easy to give a careless definition of measure dimension (e.g. based on the box-counting set dimension [3]) that does not satisfy (e).

The most relevant set-dimension concepts in geometric measure theory are the Hausdorff and packing dimensions, which we denote by $\dim_H$ and $\dim_P$, respectively. Definitions and properties of these dimensions can be found in [3, 7]. Hausdorff and packing measure-dimension mappings associated with these set-dimension definitions are naturally defined as follows. For $\mu\in\mathcal{BM}(X)$,

$$\dim_H\mu=\inf\{\dim_H A:\mu(A)>0,\ A\in\mathcal{B}(X)\},$$
$$\dim_P\mu=\inf\{\dim_P A:\mu(A)>0,\ A\in\mathcal{B}(X)\}.$$

Using standard properties of Hausdorff and packing dimensions of sets, it can be checked that the definitions above have properties (a) to (g).

LEMMA 2.3. *The Hausdorff and packing measure-dimension mappings defined above have properties* (a) *to* (g).

Definitions of measure-dimension mappings based on Hausdorff and packing set-dimensions might also be defined by

$$\dim_{\mathrm H}^* \mu = \inf\{\dim_{\mathrm H} A : \mu(A) = \|\mu\|,\ A \in \mathcal B(X)\},$$
$$\dim_{\mathrm P}^* \mu = \inf\{\dim_{\mathrm P} A : \mu(A) = \|\mu\|,\ A \in \mathcal B(X)\}.$$

However, these definitions do not have the monotonicity property (a) in general. A measure $\mu \in \mathcal{BM}(\mathbb R^m)$ is called of *lower exact dimension* $\underline\alpha$ if

$$\mu\Big(\mathbb R^m \setminus \Big\{x : \liminf_{r\to0} \frac{\log \mu(B(x,r))}{\log r} = \underline\alpha\Big\}\Big) = 0,$$

and μ is said to have *upper exact dimension* $\overline\alpha$ if

$$\mu\Big(\mathbb R^m \setminus \Big\{x : \limsup_{r\to0} \frac{\log \mu(B(x,r))}{\log r} = \overline\alpha\Big\}\Big) = 0.$$

It is well known that μ is of lower exact dimension $\underline\alpha$ if and only if $\dim_{\mathrm H}\mu = \dim_{\mathrm H}^*\mu = \underline\alpha$, and it is of upper exact dimension $\overline\alpha$ if and only if $\dim_{\mathrm P}\mu = \dim_{\mathrm P}^*\mu = \overline\alpha$ (see e.g. [4] for a proof of these facts). As a consequence we have the following property.

LEMMA 2.4. *Suppose that* $\nu \in \mathcal{BM}(\mathbb R^m)$ *is of lower exact dimension* $\underline\alpha$. *Then any measure* $\mu \in \mathcal{BM}(\mathbb R^m)$ *such that* $\mu \ll \nu$ *is of lower exact dimension* $\underline\alpha$. *If* ν *is of upper exact dimension* $\overline\alpha$, *then any* $\mu \ll \nu$ *is of upper exact dimension* $\overline\alpha$.

Proof. From absolute continuity we have, within the class $\mathcal B(\mathbb R^m)$,

$$\{A : \nu(A) = \|\nu\|\} \subseteq \{A : \mu(A) = \|\mu\|\} \subseteq \{A : \mu(A) > 0\} \subseteq \{A : \nu(A) > 0\}.$$

It follows that $\dim_{\mathrm H}\nu \le \dim_{\mathrm H}\mu \le \dim_{\mathrm H}^*\mu \le \dim_{\mathrm H}^*\nu$, which in turn implies that μ is of lower exact dimension $\underline\alpha$ for ν is. The proof in the case of upper exactness follows in the same way. ■

In view of Lemma 2.3, Hausdorff and packing measure-dimension mappings are good theoretical candidates to work with. There are, however, severe limitations in practice to obtain numerical estimates of such quantities. The most widely used dimension concept in chaotic time series analysis has been the correlation dimension, which we will denote by β, introduced by Grassberger and Procaccia in [5]. We adopt the theoretical approach to correlation dimension considered by Cutler [2], i.e. the upper and lower correlation dimensions of a measure $\mu \in \mathcal{BM}(X)$ are defined by

$$\underline\beta(\mu) = \liminf_{r\to0} \frac{\log\int \mu(B(x,r))\,d\mu}{\log r},\qquad \overline\beta(\mu) = \limsup_{r\to0} \frac{\log\int \mu(B(x,r))\,d\mu}{\log r}. \tag{2}$$

We study below the behaviour of the correlation dimension with regard to properties (a) to (g). The next example shows that the correlation dimension does not have properties (a), (d) and (g).

EXAMPLE 2.5. Consider the unit interval I endowed with the usual metric. Let $0 < a < 1$, and consider the sequence of subintervals of I given by $I_n = [a^{n^2} - \varepsilon_n, a^{n^2} + \varepsilon_n]$, $n \in \mathbb N$, where $0 < \varepsilon_n < \frac12(a^{n^2} - a^{(n+1)^2})$ for each n. For $n \in \mathbb N$, let $c_n = a^n/(2\varepsilon_n)$ and define a measure $\mu_n = c_n\mathcal L^1|_{I_n}$ so that the total mass of μ_n is given by $\mu_n(I) = a^n$. Since $c_n r \le \mu_n(B(x,r)) \le 2c_n r$ for $x \in I_n$ and for all $r > 0$ small enough, it follows that $\underline\beta(\mu_n) = \overline\beta(\mu_n) = 1$ for all n. Now consider the measure $\mu = \sum_{n\in\mathbb N}\mu_n$, and notice that $\mu \in \mathcal{BM}(I)$. Let $r > 0$ be small, $x \in (0,r)$, and $n = n(r) = \min\{k : a^{k^2} + \varepsilon_k < r\}$. Notice that $\mu(B(x,r)) \ge \sum_{i>n} a^i = (1-a)^{-1}a^n$, so that

$$\int \mu(B(x,r))\,d\mu(x) \ge \frac{a^n}{1-a}\mu([0,r]) \ge \frac{a^n}{1-a}\sum_{i>n} a^i = \frac{a^{2n}}{(1-a)^2}.$$

This gives

$$\underline\beta(\mu) \le \overline\beta(\mu) \le \limsup_{n\to\infty} \frac{2n\log a - 2\log(1-a)}{(n-1)^2\log a + \log 2} = 0, \tag{3}$$

which proves that property (g) does not hold in general for the correlation dimension. Since μ is absolutely continuous with respect to $\lambda = \mathcal L^1|_I$ and $\underline\beta(\lambda) = \overline\beta(\lambda) = 1$, (3) also implies that the correlation dimension does not have properties (a) and (d).

Recall that the essential supremum of a ν-measurable function $f : X \to \mathbb R$ is defined by $\operatorname{ess\,sup}_\nu f = \inf\{K : \nu(\{x : f(x) \ge K\}) = 0\}$. Notice that the measure μ in Example 2.5 has an $\mathrm L_1$-density with respect to the Lebesgue measure given by $h(x) = \sum_{n\in\mathbb N} c_n \mathbf 1_{I_n}(x)$, where $\mathbf 1_A$ denotes the characteristic function of A. Notice also that $\operatorname{ess\,supp}_\mu h = \infty$. In view of Example 2.5, we consider weaker versions of properties (a) and (d) above.

(a*) If $\mu, \nu \in \mathcal{BM}(X)$ are such that μ has a density $h \in \mathrm L_\infty(\nu)$ with respect to ν, then $\dim\mu \ge \dim\nu$.

(d*) If $X = \mathbb R^m$ and $\mu \in \mathcal{BM}(X)$ has a density $h \in \mathrm L_\infty(\mathcal L^m)$, then $\dim\mu = m$.

Notice that (a*) is a restatement of the monotonicity property defined in (1) and it requires a form of absolute continuity which is stronger than that demanded by (a).

THEOREM 2.6. *The upper and lower correlation dimensions have properties* (a*), (b), (c), (d*) *and* (e), *and the lower correlation dimension also has property* (f) *for measures in* $\mathcal{BM}(X)$.

Proof. (a*) Let $\mu,\nu\in\mathcal{BM}(X)$ be such that μ has a density $h\in \mathrm{L}_\infty(\nu)$ with respect to ν and ess $\sup_\nu h<M$. Therefore, for any $r>0$ and $x\in X$, $\mu(B(x,r))\le M\nu(B(x,r))$. This gives, for $r>0$,

$$\int\mu(B(x,r))\,d\mu(x)=\int\mu(B(x,r))h(x)\,d\nu(x)\le M^2\int\nu(B(x,r))\,d\nu(x).$$

Taking lim sup (respectively lim inf), after some algebra, we get $\overline{\beta}(\mu)\ge\overline{\beta}(\nu)$ (resp. $\underline{\beta}(\mu)\ge\underline{\beta}(\nu)$).

(e) follows from Lemma 2.2 since trivially $\overline{\beta}(\delta_x)=0$.

(b) Assume that $\nu=\mu\circ g^{-1}$, where $\mu\in\mathcal{BM}(X)$ and $g:X\to Y$ is a Lipschitz mapping with Lipschitz constant $K>0$. It is easy to check that $g(B(x,r/K))\subseteq B(g(x),r)$ for any $x\in X$ and any $r>0$. This fact along with the change of variable gives

$$\begin{aligned}\int\nu(B(x,r))\,d\nu(x)&=\int\nu(B(g(x),r))\,d\mu(x)\\&=\int\mu(g^{-1}(B(g(x),r)))\,d\mu(x)\ge\int\mu(B(x,r/K))\,d\mu(x),\end{aligned}$$

which in turn gives

$$\frac{\log\int\nu(B(x,r))\,d\nu(x)}{\log r}\le\frac{\log(r/K)}{\log r}\cdot\frac{\log\int\mu(B(x,r/K))\,d\mu(x)}{\log(r/K)},$$

so that, letting $r\to0$, we get $\overline{\beta}(\nu)\le\overline{\beta}(\mu)$ and $\underline{\beta}(\nu)\le\underline{\beta}(\mu)$.

(c) Let $\mu\in\mathcal{BM}(\mathbb{R}^m)$. The Lebesgue decomposition theorem gives $\mu=\mu^{\mathrm{a}}+\mu^s$, where $\mu^{\mathrm{a}}\ll\mathcal{L}^m$ and μ^s is singular with respect to $\mathcal{L}^m$. By standard differentiation theorems of measures (see e.g. [7, Theorem 2.12]), $\liminf_{r\to0}\mu^{\mathrm{a}}(B(x,r))/r^m=cf(x)>0$ for μ^{a}-almost all $x\in\mathbb{R}^m$, where f is the Radon–Nikodym derivative of μ^{a}, and $\liminf_{r\to0}\mu^s(B(x,r))/r^m=\infty$ for μ^s-almost all $x\in\mathbb{R}^m$. Hence the set

$$E=\Big\{x:0<\liminf_{r\to0}\frac{\mu(B(x,r))}{r^m}\Big\}$$

has positive μ-measure. Fix a decreasing sequence $r_n\to0$ as $n\to\infty$, and define, for each $n\in\mathbb{N}$,

$$E_n=\{x\in E:n^{-1}r^m<\mu(B(x,r))\text{ for all }r<r_n\}.$$

Choose n large enough so that $\mu(E_n)>0$ (notice that $\bigcup_n E_n=E$ with the union increasing). We have

$$n^{-1}\mu(E_n)r^m<\int_{E_n}\mu(B(x,r))\,d\mu(x)\le\int\mu(B(x,r))\,d\mu(x)$$

so that, for all $r<r_n$,

$$\frac{\log\int\mu(B(x,r))\,d\mu(x)}{\log r}\le\frac{\log(n^{-1}\mu(E_n))}{\log r}+m.$$

Letting $r\to0$, we get $\overline{\beta}(\mu)\le m$.

(d*) Let $\mu\in\mathcal{BM}(\mathbb{R}^m)$ be absolutely continuous with respect to $\mathcal{L}^m$, with density $h\in\mathrm{L}_\infty(\mathcal{L}^m)$. As in the proof of (a), we see that $\mu(B(x,r))\le M\mathcal{L}^m(B(x,r))=Mk_mr^m$ for all x and $r>0$, where k_m is a constant independent of x and r. This gives

$$\frac{\int\log\mu(B(x,r))\,d\mu(x)}{\log r}\ge\frac{\log Mk_m}{\log r}+m,$$

so that letting $r\to0$ we obtain $\underline{\beta}(\mu)\ge m$. The equality $\overline{\beta}(\mu)=\underline{\beta}(\mu)=m$ thus follows from (c).

(f) Let $\mu_1,\mu_2\in\mathcal{BM}(X)$. To prove $\underline{\beta}(\mu_1+\mu_2)\le\min\{\underline{\beta}(\mu_1),\underline{\beta}(\mu_2)\}$ observe that, for all $r>0$,

$$\int(\mu_1+\mu_2)(B(x,r))\,d(\mu_1+\mu_2)(x)\ge\max\Big\{\int\mu_i(B(x,r))\,d\mu_i(x):i=1,2\Big\}.$$

Notice that this also applies to countably many measures.

We now prove the reverse inequality. For $r>0$ let $A=\{x:\mu_1(B(x,2r))\le\mu_2(B(x,2r))\}$ and $B=X\setminus A$. If $A\cap B(x,r)\ne\emptyset$, then there is $y\in A\cap B(x,r)$. Then $B(x,r)\subset B(y,2r)\subset B(x,3r)$, whence, as $y\in A$,

$$\mu_1(A\cap B(x,r))\le\mu_1(B(y,2r))\le\mu_2(B(y,2r))\le\mu_2(B(x,3r)).$$

This is trivial if $A\cap B(x,r)=\emptyset$, so by Fubini's theorem and the definition of B,

$$\begin{aligned}\int\mu_2(B(x,r))\,d\mu_1(x)&=\int_A\mu_2(B(x,r))\,d\mu_1(x)+\int_B\mu_2(B(x,r))\,d\mu_1(x)\\&\le\int\mu_1(A\cap B(x,r))\,d\mu_2(x)+\int\mu_1(B(x,2r))\,d\mu_1(x)\\&\le\int\mu_2(B(x,3r))\,d\mu_2(x)+\int\mu_1(B(x,2r))\,d\mu_1(x).\end{aligned}$$

Assume that $\beta<\underline{\beta}(\mu_1)\le\underline{\beta}(\mu_2)$. By Fubini's theorem again, it follows from above that, for arbitrarily small values of r,

$$\begin{aligned}\int(\mu_1+\mu_2)(B(x,r))\,d(\mu_1+\mu_2)(x)&=\int\mu_1(B(x,r))\,d\mu_1(x)\\&\quad+\int\mu_2(B(x,r))\,d\mu_2(x)\\&\quad+2\int\mu_2(B(x,r))\,d\mu_1(x)\\&<2\cdot r^\beta+4\cdot3^\beta r^\beta\end{aligned}$$

so that we get $\underline{\beta}(\mu_1+\mu_2)\ge\beta$. This proves (f) for the lower correlation dimension. ■

Below we shall give an example which shows that (f) fails for the upper correlation dimension.

Monotonicity is a key property for a measure-dimension mapping in order to compare the geometric sizes of two different measures. Example 2.5 shows that the correlation dimension is far from satisfying the general monotonicity property (a). The bad behaviour of the correlation dimension in Example 2.5 is a consequence of the poor regularity of the density h, which does not belong to L_p for any $p > 1$. The lemma below shows that the monotonicity property of the correlation dimension is related to the L_p-space which contains the relative density of the measures. In particular, the monotonicity stated by property (a*) is obtained as the limit case $p \to \infty$ of Lemma 2.7.

LEMMA 2.7. *Let $\mu, \nu \in \mathcal{BM}(X)$ be such that μ has a density $f \in L_p(\nu)$, $1 < p < \infty$, with respect to ν. Then*

$$\underline{\beta}(\mu) \ge \frac{p-1}{p}\underline{\beta}(\nu) \quad \text{and} \quad \overline{\beta}(\mu) \ge \frac{p-1}{p}\overline{\beta}(\nu).$$

Proof. Let q be such that $1/p + 1/q = 1$. Applying Hölder's inequality twice, we obtain

$$\int \mu(B(x,r))\,d\mu(x) \le \|f\|_p^2 \Big(\int \nu(B(x,r))\,d\nu(x)\Big)^{1/q}$$

so that, after some algebra and letting $r \to 0$, the result follows. ∎

Pesin defined in [8] the *modified correlation dimensions* as follows: for $\mu \in \mathcal{BM}(X)$ with total mass $\|\mu\|$, let

$$(4)\qquad \begin{aligned} \underline{\beta}_{\rm M}(\mu) &= \lim_{\delta\to 0} \sup_{\{Z\in\mathcal{B}(X):\mu(Z)\ge\|\mu\|-\delta\}} \liminf_{r\to 0} \frac{\log\int_Z \mu(B(x,r))\,d\mu(x)}{\log r},\\ \overline{\beta}_{\rm M}(\mu) &= \lim_{\delta\to 0} \sup_{\{Z\in\mathcal{B}(X):\mu(Z)\ge\|\mu\|-\delta\}} \limsup_{r\to 0} \frac{\log\int_Z \mu(B(x,r))\,d\mu(x)}{\log r}.\end{aligned}$$

It turns out that in $\mathbb{R}^m$ these modified versions of correlation dimensions have all the properties listed in Section 1, except that $\overline{\beta}_{\rm M}$ does not satisfy (f) (see Example 2.14 below). This is the content of Theorem 2.10 below. We first give an alternative useful definition of modified correlation dimensions, by expressing modified dimensions in terms of unmodified ones.

LEMMA 2.8. *For $\mu \in \mathcal{BM}(\mathbb{R}^m)$,*

$$(5)\qquad \begin{aligned}\underline{\beta}_{\rm M}(\mu) &= \lim_{\delta\to 0} \sup\{\underline{\beta}(\mu|_Z) : Z \in \mathcal{B}(\mathbb{R}^m),\ \mu(Z) \ge \|\mu\| - \delta\},\\ \overline{\beta}_{\rm M}(\mu) &= \lim_{\delta\to 0} \sup\{\overline{\beta}(\mu|_Z) : Z \in \mathcal{B}(\mathbb{R}^m),\ \mu(Z) \ge \|\mu\| - \delta\},\end{aligned}$$

where $\mu|_Z$ denotes the measure defined by $\mu|_Z(A) = \mu(A\cap Z)$ for $A \in \mathcal{B}(\mathbb{R}^m)$.

Proof. Write $\underline{\beta}^*_{\rm M}(\mu)$ for the right-hand side of (5). It can be easily proved that $\underline{\beta}_{\rm M}(\mu) \le \underline{\beta}^*_{\rm M}(\mu)$. To prove the opposite inequality, take $\beta < \underline{\beta}^*_{\rm M}(\mu)$ and let $\delta > 0$. It follows from the above that there exists $Z_0 = Z_0(\delta)$ such that $\mu(Z_0) \ge \|\mu\| - \delta/2$ and $\int_{Z_0} \mu(B(x,r)\cap Z_0)\,d\mu < r^\beta$ for all r small enough. Borel's density theorem plus Egorov's theorem [6] imply that there is a Borel set $Z \subset Z_0$ and a number $r_0 > 0$ such that $\mu(Z) \ge \mu(Z_0) - \delta/2 \ge \|\mu\| - \delta$ and $\mu(B(x,r)) \le 2\mu(B(x,r)\cap Z_0)$ for all $x \in Z$ and $0 < r < r_0$. We thus get, for any $0 < r < r_0$,

$$\begin{aligned}\int_Z \mu(B(x,r))\,d\mu(x) &\le 2\int_Z \mu(B(x,r)\cap Z_0)\,d\mu(x)\\ &\le 2\int_{Z_0} \mu(B(x,r)\cap Z_0)\,d\mu(x) < 2r^\beta.\end{aligned}$$

Since $\delta > 0$ is arbitrary this gives $\underline{\beta}_{\rm M}(\mu) > \beta$. This completes the proof since $\beta < \underline{\beta}^*_{\rm M}(\mu)$ was arbitrary. The proof for the upper dimension follows in a similar way. ∎

We next show that the modification of the correlation dimension introduced above gives something new only once. We give a proof in a more general context.

LEMMA 2.9. *Suppose that* dim *is a measure-dimension mapping satisfying property* (a*) *and let*

$$\dim_{\rm M}\mu = \lim_{\delta\to 0} \sup\{\dim \mu|_Z : \mu(Z) \ge \|\mu\| - \delta\}.$$

Then $(\dim_{\rm M})_{\rm M}\,\mu = \dim_{\rm M}\mu$. *In particular,* $(\underline{\beta}_{\rm M})_{\rm M} = \underline{\beta}_{\rm M}$ *and* $(\overline{\beta}_{\rm M})_{\rm M} = \overline{\beta}_{\rm M}$.

Proof. Since $\mu|_Z \le \mu$ for any set Z of positive μ-measure, property (a*) implies that $\dim_{\rm M}\mu \ge \dim\mu$. Notice that $\dim_{\rm M}$ also satisfies (a*) and thus we get $(\dim_{\rm M})_{\rm M}\,\mu \ge \dim_{\rm M}\mu$.

To obtain the reverse inequality, consider $\beta < (\dim_{\rm M})_{\rm M}\,\mu$ and let $\delta > 0$. It follows that there exist sets Z_1 and Z_2 such that $\mu(Z_1) \ge \|\mu\| - \delta/2$, $\dim_{\rm M}(\mu|_{Z_1}) > \beta$, $\mu|_{Z_1}(Z_2) \ge \|\mu|_{Z_1}\| - \delta/2 \ge \|\mu\| - \delta$ and $\dim(\mu|_{Z_1\cap Z_2}) > \beta$. Now taking $Z = Z_1 \cap Z_2$ we have $\mu(Z) \ge \|\mu\| - \delta$ and $\dim(\mu|_Z) > \beta$. This means that $\dim_{\rm M}\mu > \beta$. ∎

THEOREM 2.10. *The lower and upper modified correlation dimensions $\underline{\beta}_{\rm M}$ and $\overline{\beta}_{\rm M}$ have* (a)–(e), *and $\underline{\beta}_{\rm M}$ also satisfies* (f) *and* (g) *for measures in $\mathcal{BM}(\mathbb{R}^m)$.*

Proof. We prove the statements of the theorem for the lower modified correlation dimension. The proofs of (a)–(e) for the upper dimension follow in the same manner.

(a) Let $\xi > 0$ be given. It follows from the ε-δ characterization of absolute continuity that there is a $\delta > 0$ such that, for any $A \in \mathcal{B}(\mathbb{R}^m)$, $\mu(A) \geq \|\mu\| - \xi/2$ provided that $\nu(A) \geq \|\nu\| - \delta$.

Let $\underline{\beta}_M(\nu) > \beta$. The definition of $\underline{\beta}_M(\nu)$ implies that there exists $Z_0 = Z_0(\delta)$ such that $\nu(Z_0) \geq \|\nu\| - \delta$ and $\int_{Z_0} \nu(B(x,r))\,d\nu(x) < r^\beta$ for all r small enough. Notice that we also have $\mu(Z_0) \geq \|\mu\| - \xi/2$.

From Lemma 2.8 it is enough to show that there exists a Borel set Z such that $\mu(Z) \geq \|\mu\| - \xi$ and $\underline{\beta}(\mu|_Z) > \beta$. Let $f \in L_1(\nu)$ be the density of μ with respect to ν. There exist $M = M(\delta) < \infty$ and $\Lambda = \{x : f(x) < M\}$ such that $\nu(\Lambda) \geq \|\nu\| - \delta$, and thus $\mu(\Lambda) \geq \|\mu\| - \xi/2$. Define $Z = Z(\delta) = Z_0 \cap \Lambda$ and notice that $\mu(Z) \geq \|\mu\| - \xi$. We see from the above that

$$\int_Z \mu(B(x,r) \cap Z)\,d\mu(x) \leq M \int_Z \mu(B(x,r) \cap Z)\,d\nu(x) \leq M^2 \int_Z \nu(B(x,r))\,d\nu(x) < M^2 r^\beta$$

for all r small enough. This gives $\underline{\beta}(\mu|_Z) > \beta$ and, since $\mu(Z) \geq \|\mu\| - \xi$, property (a) follows.

The proofs of statements (b), (c) and (e) for the modified correlation dimensions $\underline{\beta}_M$ and $\overline{\beta}_M$ follow from Lemma 2.8 along with the equivalent statements for plain correlation dimensions, proved in Theorem 2.6.

(d) Suppose that $\mu \ll \mathcal{L}^m$. Since $\mu|_Z$ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^m$ whenever Z has positive μ-measure, property (d) for $\underline{\beta}_M$ follows from property (a) above and Lemma 2.8.

(f) The inequality $\underline{\beta}_M(\sum_{i\in I} \mu_i) \leq \inf_{i\in I} \underline{\beta}_M(\mu_i)$ is easily proved (even for I countable) as in Theorem 2.6, (f).

We next prove the reverse inequality for I finite. To this end, let $\min\{\underline{\beta}_M(\mu_1), \underline{\beta}_M(\mu_2)\} > \beta$. Decompose $\mu_2 = \mu_2^a + \mu_2^s$ where $\mu_2^a \ll \mu_1$ and $\mu_2^s \perp \mu_1$, and write $\mu = \mu_1 + \mu_2 = \widehat{\mu}_1 + \widehat{\mu}_2$ where $\widehat{\mu}_1 = \mu_1 + \mu_2^a$ and $\widehat{\mu}_2 = \mu_2^s$. We have $\widehat{\mu}_1 \perp \widehat{\mu}_2$.

Notice first that $\min\{\underline{\beta}_M(\widehat{\mu}_1), \underline{\beta}_M(\widehat{\mu}_2)\} > \beta$. This is because of part (a): $\widehat{\mu}_2 \leq \mu_2$ and thus $\underline{\beta}_M(\widehat{\mu}_2) \geq \underline{\beta}_M(\mu_2)$, and also $\widehat{\mu}_1 \ll \mu_1$ so that $\underline{\beta}_M(\widehat{\mu}_1) \geq \underline{\beta}_M(\mu_1)$. Let $\xi > 0$ and let Z_1 and Z_2 be such that $\widehat{\mu}_1(Z_2) = \widehat{\mu}_2(Z_1) = 0$ and $\widehat{\mu}_1(Z_1) \geq \|\widehat{\mu}_1\| - \xi/2$, $\widehat{\mu}_2(Z_2) \geq \|\widehat{\mu}_2\| - \xi/2$, and $\min\{\underline{\beta}(\widehat{\mu}_1|_{Z_1}), \underline{\beta}(\widehat{\mu}_2|_{Z_2})\} > \beta$. For $Z = Z_1 \cup Z_2$, we have

$$\mu(Z) = \widehat{\mu}_1(Z_1) + \widehat{\mu}_2(Z_2) \geq \|\widehat{\mu}_1\| + \|\widehat{\mu}_2\| - \xi = \|\mu\| - \xi,$$

and also, using stability of the lower correlation dimension (Theorem 2.6, (f)), we have

$$\underline{\beta}(\mu|_Z) = \underline{\beta}(\widehat{\mu}_1|_{Z_1} + \widehat{\mu}_2|_{Z_2}) = \min\{\underline{\beta}(\widehat{\mu}_1|_{Z_1}), \underline{\beta}(\widehat{\mu}_2|_{Z_2})\} > \beta.$$

(g) Let $\mu = \sum_{i=1}^\infty \mu_i$. We only need to prove that $\underline{\beta}_M(\mu) \geq \inf_{i\in\mathbb{N}} \underline{\beta}_M(\mu_i)$. Define head and tail measures by $\alpha_j = \sum_{i=1}^j \mu_i$ and $\nu_j = \sum_{i>j} \mu_i$ for $j \in \mathbb{N}$. Suppose that $\nu_j \ll \alpha_j$ for some j. It follows from properties (a) and (f) proved above that

$$\underline{\beta}_M(\mu) = \min\{\underline{\beta}_M(\alpha_j), \underline{\beta}_M(\nu_j)\} = \underline{\beta}_M(\alpha_j) = \min_{1\leq i\leq j} \underline{\beta}_M(\mu_i) \geq \inf_{i\in\mathbb{N}} \underline{\beta}_M(\mu_i).$$

Otherwise, for every j there is a decomposition $\nu_j = \nu_j^a + \nu_j^s$, where $\nu_j^a \ll \alpha_j$ and ν_j^s is a non-trivial measure which is singular with respect to α_j. This implies that the sum measure μ can be split as $\mu = \eta_j + \nu_j^s$, where $\eta_j = \alpha_j + \nu_j^a$ is clearly singular with respect to ν_j^s. Let $\delta > 0$, and choose an integer N such that $\|\nu_N^s\| \leq \delta$. Since $\|\mu\| \leq \|\eta_N\| + \|\nu_N^s\| \leq \|\eta_N\| + \delta$, there exists a set Z_N such that $\eta_N(Z_N) \geq \|\mu\| - \delta$ and $\nu_N^s(Z_N) = 0$. Using properties (a) and (f) again, we have

$$\underline{\beta}_M(\mu|_{Z_N}) = \underline{\beta}_M(\eta_N|_{Z_N}) = \underline{\beta}_M(\alpha_N|_{Z_N}) \geq \min_{1\leq i\leq N} \underline{\beta}_M(\mu_i) \geq \inf_{i\in\mathbb{N}} \underline{\beta}_M(\mu_i),$$

which in turn means $\underline{\beta}_M(\mu) = (\underline{\beta}_M)_M(\mu) \geq \inf_{i\in\mathbb{N}} \underline{\beta}_M(\mu_i)$ via Lemma 2.9. ■

Notice that the proof of Theorem 2.10 can be easily adapted to the following general case.

THEOREM 2.11. *Let* dim *be a measure-dimension mapping in* $\mathcal{BM}(\mathbb{R}^m)$ *with properties* (a*), (b), (c), (d*), (e) *and* (f). *Then the modified dimension* $\dim_M$ *defined by*

$$\dim_M \mu = \lim_{\delta\to 0} \sup\{\dim \mu|_Z : Z \in \mathcal{B}(X),\ \mu(Z) \geq \|\mu\| - \delta\}$$

has properties (a)–(g).

REMARK 2.12. Below we shall show that (f) fails for $\overline{\beta}$ and $\overline{\beta}_M$. Of course, (f) and (g) hold for $\overline{\beta}_M$ for absolutely continuous measures μ_i in $\mathbb{R}^m$, since $\overline{\beta}_M(\mu_i) = m$ for them. However, we shall also show that (f) fails for $\overline{\beta}$ even for absolutely continuous measures.

REMARK 2.13. An inspection of the proof of Theorem 2.6 reveals that we always have

$$\overline{\beta}(\mu_1 + \mu_2) \geq \max\{\min\{\overline{\beta}(\mu_1), \underline{\beta}(\mu_2)\}, \min\{\underline{\beta}(\mu_1), \overline{\beta}(\mu_2)\}\}.$$

EXAMPLE 2.14. We now show that (f) fails for the upper and modified upper correlation dimensions. We shall construct Borel measures μ_1 and μ_2 on $\mathbb{R}$ with compact support such that

$$\overline{\beta}(\mu_j) = \overline{\beta}_M(\mu_j) = 1 \quad \text{for } j = 1, 2, \tag{6}$$

and

$$\overline{\beta}(\mu_1 + \mu_2) = \overline{\beta}_M(\mu_1 + \mu_2) = 0. \tag{7}$$

Let I be a closed interval in $\mathbb{R}$ of length $d < 1$ and let $0 < \varepsilon_i < 1/2$ be a non-increasing sequence. We shall construct measures μ_1 and μ_2 on I such that there are sequences $(r_{j,i})$, $(s_{j,i})$ and $(t_{j,i})$, $j = 1,2$, such that for $j = 1,2$ and for all i,

(8) $$\|\mu_j\| = d,$$

(9) $$0 < t_{j,i+1} < r_{j,i} < s_{j,i} < t_{j,i}, \qquad \lim_{i\to\infty} t_{j,i} = 0,$$

(10) $$s_{1,i} \geq s_{2,i} \geq r_{1,i} \geq s_{1,i+1} \geq r_{2,i} \geq s_{2,i+1} \geq r_{1,i+1},$$

(11) $$\mu_j(B(x,t_{j,i})) \leq 2t_{j,i}^{1-\varepsilon_i} \quad \text{for } x \in \mathbb{R},$$

(12) $$\mu_j(B(x,r)) \geq r^{\varepsilon_i}/4 \quad \text{for } r_{j,i} \leq r \leq s_{j,i} \text{ and } x \in \operatorname{spt}\mu_j.$$

Choosing the sequence (ε_i) such that $\lim_{i\to\infty}\varepsilon_i = 0$, from (11) we obtain $\overline{\beta}(\mu_j) = \overline{\beta}_{\mathrm{M}}(\mu_j) = 1$ for $j = 1,2$. If $0 < r < s_{1,1}$, there are, by (10), j and i such that $r_{j,i} \leq r \leq s_{j,i}$, and then (12) gives, for any Borel set Z with $\mu_j(Z) > 0$,
$$\int_Z \mu_j(B(x,r))\,d\mu_j(x) \geq r^{\varepsilon_i}\mu_j(Z)/4,$$
which implies $\underline{\beta}(\mu_1+\mu_2) = \underline{\beta}_{\mathrm{M}}(\mu_1+\mu_2) = 0$.

To construct μ_1 we first take $t_{1,1} = d$ and define $s_{1,1}$ by

(13) $$s_{1,1}^{\varepsilon_1} = d.$$

Anticipating the notation used below we put $I_{1,1}^t = I$ and let $I_{1,1}^s$ be the closed interval with the same centre as I and of length $s_{1,1}$. We perform m times the "ε_1-dimensional symmetric Cantor operation" on $I_{1,1}^s$, that is, we first delete a middle interval of $I_{1,1}^s$, so that the remaining two intervals each have length d_1 satisfying $2d_1^{\varepsilon_1} = s_{1,1}^{\varepsilon_1}$. Then we continue as in the standard Cantor constructions having after m steps the closed subintervals $I_{1,1}^r,\ldots,I_{1,2^m}^r$ of $I_{1,1}^s$, each of length $d_m = r_{1,1}$ which satisfies
$$2^m r_{1,1}^{\varepsilon_1} = s_{1,1}^{\varepsilon_1}.$$
At this first stage we can take, for example, $m = 1$, but later the constructions of μ_1 and μ_2 will depend on each other in the following way: at stage i, we first choose $t_{1,i}$, then $s_{1,i} < t_{1,i}$, then $t_{2,i}$ and $s_{2,i} < t_{2,i} < s_{1,i}$, then $r_{1,i} \leq s_{2,i}$, then $t_{1,i+1}$ and $s_{1,i+1} < t_{1,i+1} < r_{1,i}$, and finally $r_{2,i} \leq s_{1,i+1}$. How this works in practice should be clear from the first steps which we explain below.

Before continuing the construction, observe that if μ_1 is any Borel measure with $\operatorname{spt}\mu_1 \subset \bigcup_{k=1}^{2^m} I_{1,k}^r$ and $\mu_1(I_{1,k}^r) = r_{1,1}^{\varepsilon_1}$ for all k, then (12) is satisfied for $j = i = 1$ (see [7, p. 62]).

Next, let n be an integer such that
$$n^{\varepsilon_2} > r_{1,1}^{\varepsilon_1+\varepsilon_2-1}.$$

Set $t_{1,2} = r_{1,1}/n$ and divide each $I_{1,k}^r$ into n subintervals of length $t_{1,2}$. We denote all these intervals by $I_{2,l}^t$. Let $s_{1,2}$ be defined by
$$n s_{1,2}^{\varepsilon_2} = r_{1,1}^{\varepsilon_1}.$$
Then

(14) $$s_{1,2}^{\varepsilon_2} < t_{1,2}^{1-\varepsilon_2}.$$

Now continue as before. We define $I_{2,k}^s$ to be the closed interval with the same centre as $I_{2,k}^t$ and of length $s_{1,2}$. Then we perform on each $I_{2,k}^s$ several times the "ε_2-dimensional symmetric Cantor operation" to get the intervals $I_{2,l}^r$ of length $r_{1,2}$. But, as explained above, before this we proceed in the construction of μ_2 up to the level where we have defined $s_{2,2}$. Then we require that $r_{1,2} \leq s_{2,2}$.

If μ_1 is a Borel measure with $\operatorname{spt}\mu_1 \subset \bigcup_k I_{2,k}^r$ and $\mu_1(I_{2,k}^r) = r_{1,2}^{\varepsilon_2}$ for all k, we also have $\mu_1(I_{1,k}^r) = r_{1,1}^{\varepsilon_1}$ for all k, which implies (12) for $j = 1$ and $i = 1,2$. Moreover, (14) implies (11) for $j = 1$ and $i = 2$ (for $i = 1$, (11) is trivial).

Continuing this process indefinitely we obtain a Cantor measure μ_1 which satisfies (8), (11) and (12) for $j = 1$. Also (9) holds true.

We construct μ_2 by a similar process denoting the corresponding intervals by $J_{i,k}^t$, $J_{i,k}^s$ and $J_{i,k}^r$. We start by taking $t_{2,1} = t_{1,1}$, $s_{2,1} = s_{1,1}$, $J_{1,1}^t = I_{1,1}^t$ and $J_{1,1}^s = I_{1,1}^s$. Then we take $r_{2,1} \leq s_{1,2}$, as we may. This will also give us $t_{2,2}$ and $s_{2,2}$, and we now require that, in the construction of μ_1, $r_{1,2} \leq s_{2,2}$. We may continue this indefinitely choosing always $r_{j,i}$ sufficiently small so that (10) will hold for all j and i. This gives us the desired measures μ_1 and μ_2 as the natural limit (Cantor) measures on the sets $\bigcap_{i=1}^{\infty}\bigcup_k I_{i,k}^r$ and $\bigcap_{i=1}^{\infty}\bigcup_k J_{i,k}^r$.

EXAMPLE 2.15. The measures μ_1 and μ_2 in Example 2.14 are of course singular with respect to the Lebesgue measure. In the case of $\overline{\beta}_{\mathrm{M}}$ they also must be singular because of Remark 2.12. We now show that in the case of $\overline{\beta}$ they can be chosen to be absolutely continuous. Thus we next construct absolutely continuous measures μ_1 and μ_2 on $[0,1]$ such that $\overline{\beta}(\mu_1) = \overline{\beta}(\mu_2) = 1$ but $\underline{\beta}(\mu_1+\mu_2) = 0$.

Let $0 < \varepsilon_i < 1/2$ tend to zero. We claim that there are closed subintervals I_k of $[0,1]$, sequences of positive numbers (d_k), (p_k), (r_k), (s_k) and $(u_{j,k})$, $j = 1,2$, and absolutely continuous measures $\mu_{j,k}$, $j = 1,2$, such that setting

(15) $$\nu_k = \mu_{1,k} + \mu_{2,k}, \qquad \mu_j = \sum_k \mu_{j,k}, \qquad \mu = \mu_1 + \mu_2 = \sum_k \nu_k,$$

we have, for all $k,l = 1,2,\ldots$ and $j = 1,2$,

(16) $$2I_k \cap 2I_l = \emptyset \quad \text{for } k \neq l \quad (\text{with } 2(x-r,x+r) = (x-2r,x+2r)),$$

(17) $$u_{j,k+1} < p_k < r_k < s_{k+1} < u_{j,k} < s_k = d_k^{1/\varepsilon_k},$$

(18) $$d_{k+1} < d_k = \mathcal{L}^1(I_k) \le (u_{j,l}^{1-\varepsilon_l})^k \quad \text{for } k > l,$$

(19) $$\|\mu_{j,k}\| = d_k \le 1/4, \quad \operatorname{spt}\mu_{j,k} \subset I_k,$$

(20) $$\mu_{j,k}(B(x,r)) \le r^{1-\varepsilon_k} \quad \text{for } r = u_{j,k} \text{ and } 0 < r \le p_k \text{ and } x \in \mathbb{R},$$

(21) $$\nu_k(B(x,r)) \ge r^{\varepsilon_k} \quad \text{for } r_k \le r \le s_k \text{ and } x \in \operatorname{spt}\nu_k.$$

Let us first verify that these properties imply that $\overline{\beta}(\mu_j) = 1$ for $j = 1,2$ and $\overline{\beta}(\mu) = 0$. Let $r = u_{j,k}$ for some j and k. By (18) and (17), $d_l > d_k > r$ for $l < k$, whence by (16), $B(x,r)$ can meet at most one I_l for $l \le k$ for any given $x \in \mathbb{R}$. By (17), $r = u_{j,k} < p_l$ for $l < k$, and so by (20), for $l \le k$ and $x \in \mathbb{R}$,

$$\mu_{j,l}(B(x,r)) \le r^{1-\varepsilon_k},$$

whence

$$\sum_{l=1}^{k} \mu_{j,l}(B(x,r)) \le r^{1-\varepsilon_k} \quad \text{for } x \in \mathbb{R}$$

since at most one term can be non-zero by (16) and (19).

For $l > k$, we have $\|\mu_{j,l}\| \le (r^{1-\varepsilon_k})^l$ by (19) and (18), whence

$$\sum_{l=k+1}^{\infty} \mu_{j,l}(B(x,r)) \le \sum_{l=k+1}^{\infty} (r^{1-\varepsilon_k})^l \le 2r^{1-\varepsilon_k}.$$

Combining these inequalities we get, for all k,

$$\int \mu_j(B(x,u_{j,k}))\,d\mu_j(x) \le 2u_{j,k}^{1-\varepsilon_k}\|\mu_j\| \le 2u_{j,k}^{1-\varepsilon_k},$$

which gives $\overline{\beta}(\mu_j) = 1$.

To prove that $\overline{\beta}(\mu) = 0$, let $0 < r < s_1$. Then by (17) there is k such that $r_k \le r \le s_k$, and by (21) we get

$$\nu_k(B(x,r)) \ge r^{\varepsilon_k} \quad \text{for } x \in \operatorname{spt}\nu_k.$$

Thus by (15), (19), (17) and $r \le s_k$,

$$\int \nu_k(B(x,r))\,d\nu_k(x) \ge r^{\varepsilon_k}\|\nu_k\| \ge r^{\varepsilon_k} d_k = r^{\varepsilon_k} s_k^{\varepsilon_k} \ge r^{2\varepsilon_k}.$$

Hence by (15), given $\varepsilon > 0$, we have for all sufficiently small r,

$$\int \mu(B(x,r))\,d\mu(x) \ge r^{2\varepsilon}$$

for all $r_k \le r \le s_k$ and for all k. This gives $\overline{\beta}(\mu) = 0$ as required.

It remains to construct the intervals, positive numbers and measures such that (15)–(21) hold. We use the construction of the measures μ_1 and μ_2 in Example 2.14. Let $I = [0,1/4]$, $d_1 = 1/4$, and let s_1 and $u_{j,1}$, $j = 1,2$, be the numbers $s_{1,1}$ and $t_{j,2}$ from Example 2.14 corresponding to the constant sequence (ε_1). We choose r_1 to be some $r_{2,i}$ from Example 2.14 for a sufficiently large i; how large will depend on the beginning of the construction on I_2 and will be explained soon. Instead of going to the limit as in Example 2.14, we stop the construction at this level i defining $\mu_{1,1}$ to be the constant multiple of $\mathcal{L}^1|_{\bigcup_k I_{i,k}^r}$ such that $\mu_{1,1}(I_{i,k}^r) = r_{1,i}^{\varepsilon_1}$, and $\mu_{2,1}$ the constant multiple of $\mathcal{L}^1|_{\bigcup_k J_{i,k}^r}$ such that $\mu_{2,1}(J_{i,k}^r) = r_{2,i}^{\varepsilon_1}$. Then (19) and (21) follow from (8), (10) and (12) for $k = 1$. Also (20) holds for $r = u_{j,1}$ by (11). Finally, we choose $p_1 < r_1$ so small that (20) also holds for $0 < r \le p_1$.

Next we choose $I_2 \subset (1/2,1)$ of length $d_2 < 1/4 = d_1$ such that $d_2 < (u_{j,1}^{1-\varepsilon_1})^2$ for $j = 1,2$. Let $s_2 = d_2^{1/\varepsilon_2}$. We are still free to choose $r_1 < s_2$. Now we again apply the construction from Example 2.14 on I_2, with the constant sequence (ε_2), to obtain $p_2 < r_2 < u_{j,2} < s_2$ with $u_{j,2} < p_1$ and the measures $\mu_{j,2}$. Then (15)–(21) hold for $j = 1,2$ and $k = 1,2$ except the term involving $k+1 = 3$. Continuing this process we obtain the required sequences.

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