Contents of Volume 142, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (2000): Vols. 138–143 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department
Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman

PRINTED IN POLAND

ISSN 0039-3223

On α-times integrated C-semigroups and the abstract Cauchy problem

by

CHUNG-CHENG KUO (Taipei) and SEN-YEN SHAW (Chung-Li)

Abstract. This paper is concerned with α-times integrated C-semigroups for $\alpha > 0$ and the associated abstract Cauchy problem: $u'(t) = Au(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}x$, $t > 0$; $u(0) = 0$. We first investigate basic properties of an α-times integrated C-semigroup which may not be exponentially bounded. We then characterize the generator A of an exponentially bounded α-times integrated C-semigroup, either in terms of its Laplace transforms or in terms of existence of a unique solution of the above abstract Cauchy problem for every x in $(\lambda - A)^{-1}C(X)$.

0. Introduction. Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ in a Banach space X. The *abstract Cauchy problem* associated with A is the initial value problem

$$\mathrm{ACP}(f,x) \qquad \begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = x, \end{cases}$$

where $f \in C([0,\infty);X)$. Let $[D(A)]$ denote the Banach space $D(A)$ with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$. A function u is a (*strong*) *solution* of $\mathrm{ACP}(f,x)$ if $u \in C^1((0,\infty);X) \cap C([0,\infty);[D(A)])$ and satisfies $\mathrm{ACP}(f,x)$. It is well known that the ACP is closely related to the theory of semigroups (see e.g. [5]).

Recently Li and Shaw ([8]–[10]) introduced exponentially bounded n-times integrated C-semigroups and studied their connection with the ACP. It was proved [8] that if A is the generator of an exponentially bounded n-times integrated C-semigroup $S(\cdot)$, then $\mathrm{ACP}(0,x)$ has a unique solution for every initial value $x \in C(D(A^{n+1}))$. In particular, for the simplest case $n = 0$, i.e. A is the generator of an exponentially bounded C-semigroup,

2000 *Mathematics Subject Classification*: 47D60, 47D62.

Key words and phrases: α-times integrated C-semigroup, generator, abstract Cauchy problem.

Research supported in part by the National Science Council of Taiwan.

the function $C^{-1}S(t)x$ is the unique solution of $\mathrm{ACP}(0,x)$ for every $x \in C(D(A))$.

However, as is shown by an example in Section 1, a general n-times integrated C-semigroup may not be exponentially bounded. In this paper we attempt to investigate a more general class of operator families, namely α-times integrated C-semigroups, where α may be any nonnegative number.

In Section 1, some basic properties of nondegenerate α-times integrated C-semigroups are proved. If A is the generator of such a semigroup with $\alpha > 0$ (resp. $\alpha = 0$), then $u(t;x) = C^{-1}S(t)x$ is the unique strong solution of $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0,x)$) for every value from $C(D(A))$ (Proposition 1.5). Here $j_{\alpha-1}(t)$ denotes $t^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$ and 0 for $\alpha = 0$. Section 2 is devoted to nondegenerate α-times ($\alpha > 0$) integrated C-semigroups which are exponentially bounded. In this case, $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ is solvable for every x from the set $(\lambda - A)^{-1}C(X)$ (Theorem 2.4), which is larger than $C(D(A))$ in general, and is equal to the latter when $\lambda \in \varrho(A)$, the resolvent set of A (see [16, Proposition 1.4]). We also prove a characterization of an exponentially bounded α-times integrated C-semigroup in terms of its Laplace transform (Theorem 2.3).

Conversely, if $C \in B(X)$ is an injection and A is a closed linear operator satisfying: (a) A commutes with C; (b) $\lambda - A$ is injective and $R(C) \subset D((\lambda - A)^{-1})$ for some real λ; (c) $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0,x)$ for the case $\alpha = 0$) has a unique strong solution for every initial value $x \in (\lambda - A)^{-1}C(X)$, then $C^{-1}AC$ is the generator of an α-times integrated C-semigroup (Theorem 3.2). This extends Theorem 3.1 of [16] from the case $\alpha = 0$ to cases $\alpha > 0$. Since $C^{-1}AC = A$ when $\varrho(A) \neq \emptyset$ [16, Proposition 1.4], it follows that a closed linear operator A with $\varrho(A) \neq \emptyset$ is the generator of an α-times integrated C-semigroup if and only if A commutes with C and $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0,x)$ for the case $\alpha = 0$) has a unique strong solution for every $x \in C(D(A))$ (Corollary 3.4). This extends Corollary 2.2 of [16] from the case $\alpha = 0$ to cases $\alpha > 0$. We also characterize the generator of an exponentially bounded C-semigroup in terms of the ACP (see Theorem 3.5).

Further characterizations of generators in terms of the unique existence of strong and weak solutions of $\mathrm{ACP}(f,x)$ will be established in [7]. An extension of Theorem 3.2 to the case that $\lambda - A$ is not injective is also proved there.

1. Some basic properties of α-times integrated C-semigroups. Let X be a Banach space and let $B(X)$ be the set of all bounded linear operators from X into itself. Let $C \in B(X)$ and let $\alpha > 0$. A family $\{S(t) : t \geq 0\}$ in $B(X)$ is called an *α-times integrated C-semigroup* (see [8]–[10] for the case $\alpha = n \in \mathbb{N}$) if

(1.1) $S(\cdot)x : [0,\infty) \to X$ is continuous for each $x \in X$;

(1.2) $S(t)S(s)x = (1/\Gamma(\alpha))[\int_0^{s+t} - \int_0^s - \int_0^t](t+s-r)^{\alpha-1}S(r)Cx\,dr$ for $x \in X$ and $t, s \geq 0$, $S(0) = 0$ and $CS(\cdot) = S(\cdot)C$.

It is called a (0-times integrated) *C-semigroup* (see [3], [4], [11], [15], [16]) if

(1.2′) $S(0) = C$ and $S(t)S(s) = S(t+s)C$ for all $t, s \geq 0$.

$S(\cdot)$ is said to be *nondegenerate* if

(1.3) $S(t)x = 0$ for all $t > 0$ implies $x = 0$.

$S(\cdot)$ is said to be *exponentially bounded* if

(1.4) there are constants $M, w > 0$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$.

When $C = I$, an α-times integrated C-semigroup reduces to an α-times integrated semigroup (see [1], [14] for the case $\alpha = n \in \mathbb{N}$ and [6], [12] and [13] for the case $\alpha \in \mathbb{R}_+$), and a C-semigroup is a classical C_0-semigroup. In general, an α-times integrated C-semigroup may not be exponentially bounded. For example, let $\{S(t) : t \geq 0\}$ be the family of linear operators on $X = L^2(\mathbb{R})$ defined by

$$(S(t)f)(s) = \frac{1}{\Gamma(\alpha)}\Big[\int_0^t (t-r)^{\alpha-1}e^{rs}\,dr\Big]e^{-s^2}f(s).$$

It is clear that $S(\cdot)$ is an α-times integrated C-semigroup with $(Cf)(s) = e^{-s^2}f(s)$ for all $s \in \mathbb{R}$, $f \in X$ and

$$\begin{aligned}
\|S(t)\| &= \sup_{s\in\mathbb{R}}\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{rs}e^{-s^2}\,dr\Big] \\
&= \sup_{s\in\mathbb{R}}\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{r^2/4}e^{-(s-r/2)^2}\,dr\Big] \\
&\geq \sup_{s\geq 0}\Big[\frac{1}{\Gamma(\alpha)}\int_{t/2}^t (t-r)^{\alpha-1}e^{r^2/4}e^{-(s-r/2)^2}\,dr\Big] \\
&\geq \sup_{s\geq 0}\Big[\frac{1}{\Gamma(\alpha)}e^{t^2/4^2}\int_{t/2}^t (t-r)^{\alpha-1}e^{-(s-r/2)^2}\,dr\Big] \\
&\geq \sup_{s\geq 3t/8}\Big[\frac{1}{\Gamma(\alpha)}e^{t^2/4^2}\int_{t/2}^t (t-r)^{\alpha-1}e^{-(s-t/4)^2}\,dr\Big] \\
&\geq \sup_{s\geq 3t/8}\Big[\frac{1}{\Gamma(\alpha)}e^{t^2/4^2}\frac{(t/2)^\alpha}{\alpha}e^{-(s-t/4)^2}\Big]
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha+1)} e^{t^2/4^2} \left(\frac{t}{2}\right)^{\alpha} e^{-(t/8)^2}$$
$$\geq \frac{1}{\Gamma(\alpha+1)} e^{3t^2/8^2} \quad \text{for all } t \geq 2.$$

Under the assumption that $S(\cdot)$ is nondegenerate, one can define its generator.

DEFINITION 1.1. Let $\alpha \geq 0$. The *(integral) generator* A of a nondegenerate α-times integrated C-semigroup $S(\cdot)$ is defined as

(1.5) $x \in D(A)$ and
$$Ax = y \Leftrightarrow S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)} Cx = \int_0^t S(s)y\,ds \quad \text{for all } t \geq 0.$$

The assumption that $S(\cdot)$ is nondegenerate implies that C is injective and the operator A is well defined. It is also clear that A is linear and closed.

In what follows, we use the notations
$$j_{-1}(t) \equiv 0; \quad j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)} \quad \text{for } \beta > -1.$$

We shall need the following lemma which is originally proved in [13, Proposition 2.6] for the case $C = I$. Since C is injective, the same proof works for the general case.

LEMMA 1.2. *Let A be the generator of an α-times integrated C-semigroup $S(\cdot)$ and let $T > 0$. If $u \in C([0,T];X)$ satisfies*

$$u(t) = A\int_0^t u(s)\,ds \quad \text{for all } 0 \leq t \leq T, \tag{1.6}$$

then $u \equiv 0$ on $[0,T]$.

PROPOSITION 1.3. *Let $\alpha \geq 0$. The generator A of an α-times integrated C-semigroup $S(\cdot)$ satisfies*

(1.7) *$S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;*

(1.8) *$\int_0^t S(r)x\,dr \in D(A)$ and $A\int_0^t S(r)x\,dr = S(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $t \geq 0$;*

(1.9) *$S(\cdot)$ is uniquely determined by A;*

(1.10) *$C^{-1}AC = A$.*

Proof. (1.7) follows from the commutativity of the family $S(\cdot)$ and the definition of A. To prove (1.8), it suffices to show that

$$\frac{d}{ds} S(s)\int_0^t S(r)x\,dr = S(s)(S(t)x - j_\alpha(t)Cx) + j_{\alpha-1}(s)C\int_0^t S(r)x\,dr. \tag{1.11}$$

For this, we define $\widetilde{S}(t) := \int_0^t S(r)\,dr$. Then $\widetilde{S}(\cdot)$ is an $\alpha+1$-times integrated C-semigroup. It follows that $\widetilde{S}(\cdot)\widetilde{S}(t)x \in C^1([0,\infty);X)$ for each $t \geq 0$ and

$$S(t)\widetilde{S}(s)x + j_\alpha(s)\widetilde{S}(t)Cx = \frac{d}{dt}\widetilde{S}(t)\widetilde{S}(s)x + j_\alpha(s)\widetilde{S}(t)Cx$$
$$= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{t+s} - \int_0^t - \int_0^s\Big)(t+s-r)^{\alpha-1}\widetilde{S}(r)Cx\,dr.$$

Since this function is symmetric in s and t, it follows that
$$S(s)\widetilde{S}(t)x + j_\alpha(t)\widetilde{S}(s)Cx = S(t)\widetilde{S}(s)x + j_\alpha(s)\widetilde{S}(t)Cx.$$

Hence
$$\frac{d}{ds}S(s)\widetilde{S}(t)x = S(t)S(s)x + j_{\alpha-1}(s)\widetilde{S}(t)Cx - j_\alpha(t)S(s)Cx.$$

(1.9) is a direct consequence of Lemma 1.2. To prove (1.10), we first show $A \subset C^{-1}AC$. Let $x \in D(A)$. Then by (1.2) and the definition of A, we have $S(t)Cx - j_\alpha(t)C^2x = C(S(t)x - j_\alpha(t)Cx) = C\int_0^t S(r)Ax\,dr = \int_0^t S(r)CAx\,dr$, or equivalently $Cx \in D(A)$ and $ACx = CAx \in R(C)$. That is, $A \subset C^{-1}AC$. To show the inclusion $C^{-1}AC \subset A$, let $x \in D(C^{-1}AC)$. We have $Cx \in D(A)$ and $ACx \in R(C)$. By the commutativity of C and $S(\cdot)$ and by the definition of generator, we have

$$C[S(t)x - j_\alpha(t)Cx] = S(t)Cx - j_\alpha(t)C^2x = \int_0^t S(r)ACx\,dr$$
$$= \int_0^t S(r)CC^{-1}ACx\,dr = C\int_0^t S(r)C^{-1}ACx\,dr.$$

Since C is injective, this implies $x \in D(A)$ and $Ax = C^{-1}ACx$. Consequently, $A = C^{-1}AC$.

REMARK. (1.8) shows that a nondegenerate α-times integrated C-semigroup is an example of a $K(t)$-evolution operator with $K(t) = j_{\alpha-1}(t)C$ (see [2] for results on this class).

PROPOSITION 1.4. *Let A be the generator of an α-times integrated C-semigroup $S(\cdot)$ with $\alpha > 0$ (resp. $\alpha = 0$), and let*
$$C^1 = \{x \in X : S(\cdot)x \text{ is continuously differentiable on } [0,\infty)\}.$$

Then

(1) *$S(t)C^1 \subset D(A)$ for all $t \geq 0$;*

(2) *for $x \in C^1$, $u(\cdot) = S(\cdot)x$ is the unique solution of* ACP$(j_{\alpha-1}(\cdot)Cx, 0)$ *(resp.* ACP$(0, Cx)$*).*

In particular, (2) *holds for all $x \in D(A)$.*

Proof. Let $x \in C^1$ and $t > 0$. Differentiating the equation in (1.8), we have $S(t)x \in D(A)$ and

$$AS(t)x = \frac{d}{dt}S(t)x - j_{\alpha-1}(t)Cx.$$

The uniqueness of solution of ACP$(j_{\alpha-1}(\cdot)Cx, 0)$ (resp. ACP$(0, Cx)$ for the case $\alpha = 0$) is obvious from Lemma 1.2. Finally, the definition of A implies that $D(A) \subset C^1$ so that (2) holds for all $x \in D(A)$.

PROPOSITION 1.5. *Let $S(\cdot)$ be an α-times integrated C-semigroup with $\alpha > 0$ (resp. $\alpha = 0$). If $S(\cdot)x \in R(C)$ and $C^{-1}S(\cdot)x$ is continuously differentiable on $[0,\infty)$, then $C^{-1}S(\cdot)x \in D(A)$, and $u(\cdot) = C^{-1}S(\cdot)x$ is the unique solution of ACP$(j_{\alpha-1}(\cdot)x, 0)$ (resp.* ACP$(0, x)$*). This holds in particular for $x \in C(D(A))$.*

Proof. The uniqueness is obvious and hence we only need to show that $u(\cdot) = C^{-1}S(\cdot)x$ is a solution of ACP$(j_{\alpha-1}(\cdot)x, 0)$. Clearly, $S(\cdot)x = CC^{-1}S(\cdot)x$ is continuously differentiable on $[0,\infty)$. By Proposition 1.4 we have

$$\begin{aligned} C\frac{d}{dt}C^{-1}S(t)x &= \frac{d}{dt}S(t)x = AS(t)x + j_{\alpha-1}(t)Cx \\ &= ACC^{-1}S(t)x + j_{\alpha-1}(t)Cx. \end{aligned}$$

Hence $C^{-1}S(t)x \in D(C^{-1}AC) = D(A)$ and

$$\begin{aligned} \frac{d}{dt}C^{-1}S(t)x &= (C^{-1}AC)C^{-1}S(t)x + j_{\alpha-1}(t)x \\ &= AC^{-1}S(t)x + j_{\alpha-1}(t)x. \end{aligned}$$

2. Exponentially bounded α-times integrated C-semigroups

LEMMA 2.1. *Let A be the generator of an α-times integrated C-semigroup $S(\cdot)$. For $\lambda > 0$, let D_λ denote the set of all those $x \in X$ for which $L_\lambda x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$ exists and $\int_0^\infty e^{-\lambda t}\|\int_0^t S(s)x\,ds\|\,dt < \infty$. Then*

(i) *$L_\lambda D_\lambda \subset D(A)$ and $(\lambda - A)\lambda^\alpha L_\lambda x = Cx$ for $x \in D_\lambda$;*

(ii) *$S(t)L_\lambda D_\lambda \subset R(C)$, and $C^{-1}S(\cdot)L_\lambda x$ is continuously differentiable on $[0,\infty)$ for $x \in D_\lambda$.*

Proof. (i) Integration by parts yields

$$e^{-\lambda\tau}\int_0^\tau S(s)x\,ds = \int_0^\tau e^{-\lambda t}S(t)x\,dt - \lambda\int_0^\tau e^{-\lambda t}\int_0^t S(s)x\,ds\,dt,$$

which converges as $\tau \to \infty$ if $x \in D_\lambda$. This and $\int_0^\infty e^{-\lambda t}\|\int_0^t S(s)x\,ds\|\,dt < \infty$

imply that $e^{-\lambda\tau}\int_0^\tau S(s)x\,ds \to 0$ as $\tau \to \infty$. Therefore we have

$$L_\lambda x = \lambda\int_0^\infty e^{-\lambda t}\int_0^t S(r)x\,dr\,dt.$$

From (1.8) and the closedness of A we deduce that $\lambda^{\alpha+1}\int_0^\tau e^{-\lambda t}\int_0^t S(r)x\,dr\,dt \in D(A)$ and

$$\begin{aligned} \lambda^{\alpha+1}A\int_0^\tau e^{-\lambda t}\int_0^t S(r)x\,dr\,dt &= \lambda^{\alpha+1}\int_0^\tau e^{-\lambda t}(S(t)x - j_\alpha(t)Cx)\,dt \\ &\to \lambda^{\alpha+1}L_\lambda x - Cx \end{aligned}$$

as $\tau \to \infty$. Hence the closedness of A implies that $L_\lambda x \in D(A)$ and $\lambda^\alpha AL_\lambda x = \lambda^{\alpha+1}L_\lambda x - Cx$ for $x \in D_\lambda$. Consequently, $(\lambda - A)\lambda^\alpha L_\lambda x = Cx$ for $x \in D_\lambda$.

(ii) By (1.2) we have, for $x \in D_\lambda$,

$$\begin{aligned} S(t)L_\lambda x &= \int_0^\infty e^{-\lambda s}S(t)S(s)x\,ds \\ &= C\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda s}\Big[\int_0^{s+t} - \int_0^s - \int_0^t\Big](t+s-r)^{\alpha-1}S(r)Cx\,dr\,ds, \end{aligned}$$

from which it is clear that $S(t)L_\lambda x \in R(C)$ and $C^{-1}S(\cdot)L_\lambda x$ is continuously differentiable.

The next proposition generalizes Theorem 3.1 of [1], where the case $\alpha \in \mathbb{N}$, $C = I$ was proven.

PROPOSITION 2.2. *Let $\alpha > 0$, and let $\{S(t) : t \ge 0\} \subset B(X)$ be a strongly continuous function such that $\|S(t)\| \le Me^{wt}$ for some $M > 0$, $w \ge 0$ and all $t \ge 0$. For $\lambda > w$ define*

$$R_\lambda x = \lambda^\alpha L_\lambda x = \lambda^\alpha\int_0^\infty e^{-\lambda t}S(t)x\,dt \quad \text{for } x \in X.$$

Then $\{R_\lambda : \lambda > w\}$ is a C-pseudo-resolvent, i.e.

$$(\lambda - \mu)R_\mu R_\lambda = R_\mu C - R_\lambda C \quad \text{for } \lambda, \mu > w,$$

if and only if $S(\cdot)$ satisfies

$$S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}S(r)Cx\,dr$$

for $x \in X$ and $t, s \ge 0$.

Proof. Interchanging the order of integrations we obtain the following equalities:

$$\int_0^\infty e^{-\lambda s}\int_0^s (t+s-r)^{\alpha-1}S(r)Cx\,dr\,ds$$

$$= \int_0^\infty\int_r^\infty e^{-\lambda s}(t+s-r)^{\alpha-1}ds\,S(r)Cx\,dr$$

$$= \int_0^\infty e^{-\lambda r}\int_0^\infty e^{-\lambda s'}(t+s')^{\alpha-1}\,ds'\,S(r)Cx\,dr$$

$$= e^{\lambda t}\int_t^\infty e^{-\lambda s}s^{\alpha-1}\,ds\,\lambda^{-\alpha}R_\lambda Cx,$$

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_0^s (t+s-r)^{\alpha-1}S(r)Cx\,dr\,ds\,dt$$

$$= \lambda^{-\alpha}\int_0^\infty e^{-(\mu-\lambda)t}\int_t^\infty e^{-\lambda s}s^{\alpha-1}\,ds\,dt\,R_\lambda Cx$$

$$= \lambda^{-\alpha}\int_0^\infty \Big(\int_0^s e^{-(\mu-\lambda)t}\,dt\Big)e^{-\lambda s}s^{\alpha-1}\,ds\,R_\lambda Cx$$

$$= \frac{\Gamma(\alpha)\lambda^{-\alpha}}{\lambda-\mu}(\mu^{-\alpha}-\lambda^{-\alpha})R_\lambda Cx,$$

$$\int_0^\infty e^{-\lambda s}\int_t^{t+s}(t+s-r)^{\alpha-1}S(r)Cx\,dr\,ds$$

$$= \int_t^\infty\int_{r-t}^\infty e^{-\lambda s}(t+s-r)^{\alpha-1}\,ds\,S(r)Cx\,dr$$

$$= \int_t^\infty e^{\lambda(t-r)}\Big(\int_0^\infty e^{-\lambda s'}s'^{\alpha-1}\,ds'\Big)S(r)Cx\,dr$$

$$= \Gamma(\alpha)\lambda^{-\alpha}\int_t^\infty e^{-\lambda(r-t)}S(r)Cx\,dr,$$

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\int_t^{t+s}(t+s-r)^{\alpha-1}S(r)Cx\,dr\,ds\,dt$$

$$= \Gamma(\alpha)\lambda^{-\alpha}\int_0^\infty e^{-\mu t}\int_t^\infty e^{-\lambda(r-t)}S(r)Cx\,dr\,dt$$

$$= \Gamma(\alpha)\lambda^{-\alpha}\int_0^\infty e^{-\lambda r}\Big(\int_0^r e^{(\lambda-\mu)t}\,dt\Big)S(r)Cx\,dr$$

$$= \frac{\Gamma(\alpha)\lambda^{-\alpha}}{\lambda-\mu}(\mu^{-\alpha}R_\mu Cx-\lambda^{-\alpha}R_\lambda Cx).$$

Combining these equalities we obtain

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\Big[S(t)S(s)x - \frac{1}{\Gamma(\alpha)}\Big(\int_t^{t+s}-\int_0^s\Big)(t+s-r)^{\alpha-1}S(r)Cx\,dr\Big]ds\,dt$$

$$= \lambda^{-\alpha}\mu^{-\alpha}[R_\mu R_\lambda x-(\lambda-\mu)^{-1}(R_\mu Cx-R_\lambda Cx)]$$

for all $x\in X$ and $s,t\ge 0$. The conclusion now follows from the uniqueness theorem for Laplace transforms.

Proposition 2.2 can be applied to prove the following characterization of an exponentially bounded α-times integrated C-semigroup, in terms of its Laplace transform. The case of n-times integrated C-semigroup was proved in [8, 9].

THEOREM 2.3. *Let $C\in B(X)$ be an injection. A strongly continuous function $S(\cdot)$ satisfying* (1.4) *is an α-times integrated C-semigroup with generator A if and only if $CS(\cdot)=S(\cdot)C$, $C^{-1}AC=A$, and $\lambda-A$ is injective, $R(C)\subset R(\lambda-A)$, and*

$$\lambda^\alpha L_\lambda(\lambda-A)\subset\lambda^\alpha(\lambda-A)L_\lambda=C \tag{2.1}$$

for each $\lambda>w$.

Proof. When $S(\cdot)$ is an exponentially bounded α-times integrated C-semigroup, for large λ the set D_λ as defined in Lemma 2.1 is clearly equal to X. Then Lemma 2.1 together with (1.7) yields that for each $\lambda>w$, $\lambda-A$ is injective, $R(C)\subset R(\lambda-A)$, $L_\lambda\in B(X)$, $R(L_\lambda)\subset D(A)$, and (2.1) holds.

Conversely, (2.1) implies

$$R_\mu C-R_\lambda C = R_\mu C-CR_\lambda = R_\mu(\lambda-A)R_\lambda - R_\mu(\mu-A)R_\lambda = (\lambda-\mu)R_\mu R_\lambda,$$

so that, by Proposition 2.2, $S(\cdot)$ is an α-times integrated C-semigroup. Since $\lambda-A$ and C are injective, it is seen from (2.1) that $S(\cdot)$ is nondegenerate. Let B be its generator. Then the "only if" part of the theorem asserts that $C^{-1}BC=B$, and for each $\lambda>w$, $\lambda-B$ is injective, $R(C)\subset R(\lambda-B)$, and $R_\lambda(\lambda-B)\subset(\lambda-B)R_\lambda=C$. If $x\in D(A)$, then $Cx=R_\lambda(\lambda-A)x\in D(B)$ and $(\lambda-B)Cx=(\lambda-B)R_\lambda(\lambda-A)x=C(\lambda-A)x$, so that $x\in D(C^{-1}BC)=$

$D(B)$ and $Ax = C^{-1}BCx = Bx$. Hence $A \subset B$. By symmetry we also have $B \subset A$. This completes the proof.

We obtain the following theorem by applying Proposition 1.5, Lemma 2.1(ii), and Theorem 2.3.

THEOREM 2.4. *Let $\alpha > 0$ (resp. $\alpha = 0$) and A be the generator of an exponentially bounded α-times integrated C-semigroup $S(\cdot)$ with $\|S(t)\| \le Me^{wt}$ for some $M, w > 0$ and all $t \ge 0$, and let $\lambda > w$. Then for each $x \in (\lambda - A)^{-1}CX = \lambda^\alpha L_\lambda X$, $u(\cdot) = C^{-1}S(\cdot)x$ is the unique solution of $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0, x)$ for the case $\alpha = 0$). Moreover, $\|u(t)\| = O(e^{wt})$ and $\|u'(t)\| = O(e^{wt})$ as $t \to \infty$.*

Note that this theorem extends the particular case $x \in C(D(A))$ of Proposition 1.5 to exponentially bounded α-times integrated C-semigroups because it is shown in the following lemma of Tanaka and Miyadera [16, Proposition 1.4] that $C(D(A)) \subset (\lambda - A)^{-1}CX$ in general. This lemma will also be needed in Section 3.

LEMMA 2.5. *Let $\lambda \in \mathbb{R}$ and let A be a closed linear operator satisfying*

(a) *$Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$;*
(b) *$\lambda - A$ is injective and $D((\lambda - A)^{-1}) \supset R(C)$.*

Then

(i) *$C(D(A)) \subset C(D(C^{-1}AC)) \subset (\lambda - A)^{-1}CX$;*
(ii) *$C(D(A)) = (\lambda - A)^{-1}CX$ if and only if $\lambda \in \varrho(A)$;*
(iii) *$C^{-1}AC = A$ if $\varrho(A) \neq \emptyset$.*

3. The abstract Cauchy problem. In this section we try to characterize the generator of an α-times integrated C-semigroup in terms of the unique existence of a strong solution of $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$.

LEMMA 3.1. *For all $\alpha > 0$ and $s, t \ge 0$ we have*

$$\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1} r^{\alpha-1}\, dr = 0. \tag{3.1}$$

Proof. Since $S(t) = j_\alpha(t)$ is an α-times integrated semigroup on $\mathbb{R}$, using (1.2) and integration by parts one has

$$j_\alpha(t)j_\alpha(s) = \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1} j_\alpha(r)\, dr$$
$$= \frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha+1)}\Big[3t^\alpha s^\alpha + \alpha\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^\alpha r^{\alpha-1}\, dr\Big],$$

so that

$$-t^\alpha s^\alpha = \alpha\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^\alpha r^{\alpha-1}\, dr.$$

Differentiation with respect to t yields

$$-\alpha t^{\alpha-1}s^\alpha = -\alpha s^\alpha t^{\alpha-1} + \alpha^2\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1} r^{\alpha-1}\, dr$$

and hence (3.1).

THEOREM 3.2. *Let $\alpha > 0$ (resp. $\alpha = 0$) and $\lambda \in \mathbb{R}$, and let A be a closed linear operator satisfying*

(a) *$Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$;*
(b) *$\lambda - A$ is injective and $D((\lambda - A)^{-1}) \supset R(C)$;*
(c) *$\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0, x)$ for the case $\alpha = 0$) has a unique solution for each $x \in (\lambda - A)^{-1}C(X)$.*

Then there exists an α-times integrated C-semigroup $S(\cdot)$ on X with generator $C^{-1}AC$.

We shall need the next lemma.

LEMMA 3.3. *If* (c) *holds, then it also holds when $\alpha - 1$ is replaced by α.*

Proof. For any $x \in (\lambda - A)^{-1}C(X)$, (c) implies there is a solution u of $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ (resp. $\mathrm{ACP}(0, x)$ for the case $\alpha = 0$). Then $v(t) := \int_0^t u(r)\, dr$ is a solution of $\mathrm{ACP}(j_\alpha(\cdot)x, 0)$. That the solution v is unique follows from the uniqueness of solution of $\mathrm{ACP}(0,0) = \mathrm{ACP}(j_{\alpha-1}(\cdot)0, 0)$, which is guaranteed by (c).

Proof of Theorem 3.2. We denote the unique solution of $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ by $u(t; x)$ and define the operator $S(t) : X \to X$ for $t \ge 0$ by

$$S(t)x = (\lambda - A)u(t; (\lambda - A)^{-1}Cx) \quad \text{for } x \in X. \tag{3.2}$$

Then $S(0) = 0$ and $S(\cdot)x : [0, \infty) \to X$ is continuous for $x \in X$. Now, the uniqueness of solution implies that $S(\cdot)$ is linear, $Cu(t; (\lambda - A)^{-1}Cx) = u(t; (\lambda - A)^{-1}C^2x)$ and so $CS(\cdot) = S(\cdot)C$.

Let $C([0, \infty); [D(A)])$ be the Fréchet space with the quasi-norm

$$\|v\| := \sum_{k=1}^\infty \frac{\|v\|_k}{2^k(1 + \|v\|_k)} \quad \text{for } v \in C([0, \infty); [D(A)]),$$

where $\|v\|_k = \max_{t \in [0, t_k]} |v(t)|_A$, $k \in \mathbb{N}$, $\{t_k\}$ being an increasing sequence in $(0, \infty)$ with $\lim t_k = \infty$. Consider the linear map $\eta : X \to C([0, \infty); [D(A)])$ given by $\eta(x) = u(\cdot; (\lambda - A)^{-1}Cx)$. We show that η is a closed linear operator.

In fact, let $x_n \to x$ in X and $\eta(x_n) \to v$ in $C([0,\infty);[D(A)])$. Then

$$u(t;(\lambda-A)^{-1}Cx_n) = \int_0^t Au(r;(\lambda-A)^{-1}Cx_n)\,dr + j_\alpha(t)(\lambda-A)^{-1}Cx_n.$$

Letting $n\to\infty$ we obtain $v(t) = \int_0^t Av(r)dr + j_\alpha(t)(\lambda-A)^{-1}Cx$ for all $t \ge 0$. Thus $v(\cdot)$ is a solution of $\mathrm{ACP}(j_{\alpha-1}(\cdot)(\lambda-A)^{-1}Cx,0)$. Therefore, from the uniqueness of solution it follows that $v(\cdot) = u(\cdot;(\lambda-A)^{-1}Cx) = \eta(x)$. We have shown that η is closed. By the closed graph theorem, η is continuous from X into $C([0,\infty);[D(A)])$. This shows that $S(\cdot)$ is a strongly continuous function into bounded linear operators on X.

To show that $S(\cdot)$ is an α-times integrated C-semigroup, we first consider the case $\alpha \ge 1$. For fixed $x\in X$ and $s \ge 0$ we define

$$v_s(t) = \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}u(r;(\lambda-A)^{-1}C^2x)\,dr, \qquad t\ge 0.$$

Then

$$\begin{aligned}Av_s(t) &= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}Au(r;(\lambda-A)^{-1}C^2x)\,dr\\ &= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}\Big[u'(r;(\lambda-A)^{-1}C^2x)\\ &\qquad - \frac{r^{\alpha-1}}{\Gamma(\alpha)}(\lambda-A)^{-1}C^2x\Big]dr\\ &= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}u'(r;(\lambda-A)^{-1}C^2x)\,dr,\end{aligned}$$

by Lemma 3.1.

For $\alpha = 1$ we have

$$Av_s(t) = u(t+s;(\lambda-A)^{-1}C^2x) - u(t;(\lambda-A)^{-1}C^2x) - u(s;(\lambda-A)^{-1}C^2x)$$

and

$$\frac{dv_s(t)}{dt} = u(t+s;(\lambda-A)^{-1}C^2x) - u(t;(\lambda-A)^{-1}C^2x).$$

For the case $\alpha > 1$, using integration by parts we have

$$\begin{aligned}Av_s(t) = &-j_{\alpha-1}(t)u(s;(\lambda-A)^{-1}C^2x) - j_{\alpha-1}(s)u(t;(\lambda-A)^{-1}C^2x)\\ &+ \frac{1}{\Gamma(\alpha-1)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-2}u(r;(\lambda-A)^{-1}C^2x)\,dr\end{aligned}$$

and

$$\begin{aligned}\frac{dv_s(t)}{dt} = &-j_{\alpha-1}(s)u(t;(\lambda-A)^{-1}C^2x)\\ &+ \frac{1}{\Gamma(\alpha-1)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-2}u(r;(\lambda-A)^{-1}C^2x)\,dr.\end{aligned}$$

It follows that for all $\alpha\ge 1$ and $t \ge 0$,

$$\begin{aligned}\frac{dv_s(t)}{dt} &= Av_s(t) + j_{\alpha-1}(t)u(s;(\lambda-A)^{-1}C^2x)\\ &= Av_s(t) + j_{\alpha-1}(t)Cu(s;(\lambda-A)^{-1}Cx).\end{aligned}$$

The uniqueness of solution implies that $v_s(t) = u(t;Cu(s;(\lambda-A)^{-1}Cx))$ and hence

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}S(r)Cx\,dr\\ &\quad= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{s+t} - \int_0^s - \int_0^t\Big)(t+s-r)^{\alpha-1}(\lambda-A)u(r;(\lambda-A)^{-1}C^2x)\,dr\\ &\quad= (\lambda-A)v_s(t) = (\lambda-A)u(t;Cu(s;(\lambda-A)^{-1}Cx))\\ &\quad= (\lambda-A)u(t;(\lambda-A)^{-1}C(\lambda-A)u(s;(\lambda-A)^{-1}Cx))\\ &\quad= (\lambda-A)u(t;(\lambda-A)^{-1}CS(s)x)\\ &\quad= S(t)S(s)x \quad \text{ for all } t,s\ge 0.\end{aligned}$$

Now we turn to the case $0\le\alpha<1$. The hypothesis (c) and Lemma 3.3 imply that $v_x(t) := \int_0^t u(r;(\lambda-A)^{-1}Cx)\,dr$ is the unique solution of $\mathrm{ACP}(j_\alpha(\cdot)(\lambda-A)^{-1}Cx,0)$. Let $\widetilde{S}(\cdot)$ be defined by $\widetilde{S}(t)x = (\lambda-A)v_x(t)$ for $x\in X$ and $t\ge 0$. The previous argument has shown that $\widetilde{S}(\cdot)$ is an $(\alpha+1)$-times integrated C-semigroup on X. In particular, $\widetilde{S}(t)x$ is continuous for all $x \in X$. Since $S(t)x = (\lambda-A)u(t;(\lambda-A)^{-1}Cx)$ is continuous for all $x\in X$, the closedness of A implies that

$$\widetilde{S}(t)x = (\lambda-A)\int_0^t u(r;(\lambda-A)^{-1}Cx)\,dr = \int_0^t S(r)x\,dr \in C^1([0,\infty);X).$$

Then an easy computation shows that $S(\cdot)$ is an α-times integrated C-semigroup.

In order to show that $S(\cdot)$ is nondegenerate, suppose $S(t)x = 0$ for $t > 0$. Then by the injectivity of $\lambda - A$ we have $u(t;(\lambda-A)^{-1}Cx) = 0$ for $t>0$, so that

$$
\begin{aligned}
0 &= \frac{d}{dt}u(t;(\lambda-A)^{-1}Cx)\\
&= Au(t;(\lambda-A)^{-1}Cx)+j_{\alpha-1}(t)(\lambda-A)^{-1}Cx\\
&= j_{\alpha-1}(t)(\lambda-A)^{-1}Cx
\end{aligned}
$$

and hence $x=0$, because $(\lambda-A)^{-1}$ and C are injective.

Having shown that $S(\cdot)$ is a nondegenerate α-times integrated C-semigroup on X, we next show that $C^{-1}AC$ is the generator of $S(\cdot)$. Let B be the generator of $S(\cdot)$ and let $x\in D(C^{-1}AC)$. Easy computations show that both $u(t;(\lambda-A)^{-1}Cx)-j_\alpha(t)(\lambda-A)^{-1}Cx$ and $\int_0^t u(s;(\lambda-A)^{-1}C(C^{-1}AC)x)\,ds$ are solutions of the Cauchy problem $\mathrm{ACP}(j_\alpha(\cdot)(\lambda-A)^{-1}C(C^{-1}AC)x,0)$. Then the uniqueness of solution of $\mathrm{ACP}(0,0)=\mathrm{ACP}(j_{\alpha-1}(\cdot)0,0)$ implies

$$
u(t;(\lambda-A)^{-1}Cx)-j_\alpha(t)(\lambda-A)^{-1}Cx=\int_0^t u(s;(\lambda-A)^{-1}C(C^{-1}AC)x)\,ds.
$$

Hence

$$
\begin{aligned}
S(t)x-j_\alpha(t)Cx &= (\lambda-A)[u(t;(\lambda-A)^{-1}Cx)-j_\alpha(t)(\lambda-A)^{-1}Cx]\\
&= (\lambda-A)\int_0^t u(s;(\lambda-A)^{-1}C(C^{-1}ACx))\,ds\\
&= \int_0^t (\lambda-A)u(s;(\lambda-A)^{-1}C(C^{-1}ACx))\,ds\\
&= \int_0^t S(s)C^{-1}ACx\,ds \quad \text{for all } t\ge 0.
\end{aligned}
$$

Consequently, $x\in D(B)$ and $Bx=C^{-1}ACx$, or equivalently $C^{-1}AC\subset B$.

To prove the converse, let $x\in D(B)$. We have

$$
\begin{aligned}
&(\lambda-A)[u(t;(\lambda-A)^{-1}Cx)-j_\alpha(t)(\lambda-A)^{-1}Cx]\\
&\qquad= S(t)x-j_\alpha(t)Cx=\int_0^t S(r)Bx\,dr\\
&\qquad= \int_0^t (\lambda-A)u(r;(\lambda-A)^{-1}CBx)\,dr\\
&\qquad= (\lambda-A)\int_0^t u(r;(\lambda-A)^{-1}CBx)\,dr.
\end{aligned}
$$

Since $\lambda-A$ is injective, it follows that

$$
u(t;(\lambda-A)^{-1}Cx)-j_\alpha(t)(\lambda-A)^{-1}Cx=\int_0^t u(r;(\lambda-A)^{-1}CBx)\,dr,
$$

so that $Au(t;(\lambda-A)^{-1}Cx)=u(t;(\lambda-A)^{-1}CBx)\in D(A)$. We also have

$$
\begin{aligned}
&\int_0^t Au(r;(\lambda-A)^{-1}CBx)\,dr\\
&\qquad= \int_0^t\Big[\frac{d}{dr}u(r;(\lambda-A)^{-1}CBx)-j_{\alpha-1}(r)(\lambda-A)^{-1}CBx\Big]\,dr\\
&\qquad= u(t;(\lambda-A)^{-1}CBx)-j_\alpha(t)(\lambda-A)^{-1}CBx\in D(A).
\end{aligned}
$$

From these facts, it follows that

$$
\begin{aligned}
j_\alpha(t)Cx &= S(t)x-\int_0^t S(r)Bx\,dr\\
&= (\lambda-A)u(t;(\lambda-A)^{-1}Cx)\\
&\quad -\int_0^t(\lambda-A)u(r;(\lambda-A)^{-1}CBx)dr\in D(A)
\end{aligned}
$$

and

$$
\begin{aligned}
Aj_\alpha(t)Cx &= A\Big[\lambda u(t;(\lambda-A)^{-1}Cx)-u(t;(\lambda-A)^{-1}CBx)\\
&\qquad -\lambda\int_0^t u(r;(\lambda-A)^{-1}CBx)\,dr\\
&\qquad + u(t;(\lambda-A)^{-1}CBx)-j_\alpha(t)(\lambda-A)^{-1}CBx\Big]\\
&= \lambda Au(t;(\lambda-A)^{-1}Cx)-\lambda\int_0^t Au(r;(\lambda-A)^{-1}CBx)\,dr\\
&\qquad -j_\alpha(t)A(\lambda-A)^{-1}CBx\\
&= \lambda Au(t;(\lambda-A)^{-1}Cx)-\lambda[u(t;(\lambda-A)^{-1}CBx)\\
&\qquad -j_\alpha(t)(\lambda-A)^{-1}CBx]-j_\alpha(t)A(\lambda-A)^{-1}CBx\\
&= j_\alpha(t)CBx \quad \text{for all } t\ge 0.
\end{aligned}
$$

That shows $B\subset C^{-1}AC$.

COROLLARY 3.4. *Let A be a closed linear operator with nonempty resolvent set. Then the following are equivalent:*

(i) *A is the generator of an α-times integrated C-semigroup $\{S(t) : t \geq 0\}$ with $\alpha > 0$ (resp. $\alpha = 0$).*

(ii) *A satisfies the conditions:*

(a) *$Cx \in D(A)$ and $ACx = CAx$ for all $x \in D(A)$;*

(b) *The $\mathrm{ACP}(j_{\alpha-1}(\cdot)Cx, 0)$ (resp. $\mathrm{ACP}(0, Cx)$ for the case $\alpha = 0$) has a unique solution $u(t; Cx)$ for every $x \in D(A)$.*

In this case, $u(t; Cx) = S(t)x$ for $t \geq 0$.

Proof. The implication (i)$\Rightarrow$(ii) is a direct consequence of Propositions 1.3 and 1.4. We show (ii)$\Rightarrow$(i). Since $C(D(A)) = (\lambda - A)^{-1}C(X)$ for $\lambda \in \varrho(A)$ and $C^{-1}AC = A$ (Lemma 2.5(ii), (iii)), it follows from Theorem 3.2 that A is the generator of an α-times integrated C-semigroup.

A characterization of exponentially bounded α-times integrated C-semigroup in terms of the ACP is given by

THEOREM 3.5. *Let A be a closed linear operator in X. Then the following are equivalent:*

(i) *A is the generator of an exponentially bounded α-times integrated C-semigroup $S(\cdot)$ with $\|S(t)\| \leq Me^{wt}$ for $t \geq 0$, where M,$w > 0$ are fixed.*

(ii) *A satisfies the conditions:*

(a) *$C^{-1}AC = A$;*

(b) *for some $\lambda \in \mathbb{R}$, $\lambda - A$ is injective and $D((\lambda - A)^{-1}) \supset R(C)$;*

(c) *for every $x \in (\lambda - A)^{-1}C(X)$, $\mathrm{ACP}(j_{\alpha-1}(\cdot)x, 0)$ has a unique solution $u(t; x)$ such that $\|u(t; x)\|$ and $\|u'(t; x)\|$ are $O(e^{wt})$ as $t \to \infty$.*

Proof. The implication (i)$\Rightarrow$(ii) follows from Theorem 2.4 and Proposition 1.3, eq. (1.10). In order to show the converse, by Theorem 3.2, it suffices to show that $\{S(t) : t \geq 0\}$ defined by (3.2) is exponentially bounded. From condition (c) we deduce that

$$\|e^{-wt}Au(t; (\lambda - A)^{-1}Cx)\| = \|e^{-wt}[u'(t; (\lambda - A)^{-1}Cx) - j_{\alpha-1}(t)(\lambda - A)^{-1}Cx]\|$$

and $\|e^{-wt}u(t; (\lambda - A)^{-1}Cx)\|$ are bounded for $x \in X$ as $t \to \infty$. Thus, by the uniform boundedness principle, we have

$$\sup_{t\geq 0} \|e^{-wt}S(t)\| = \sup_{t\geq 0} \sup_{\|x\|=1} \|\lambda e^{-wt}u(t; (\lambda - A)^{-1}Cx) - e^{-wt}Au(t; (\lambda - A)^{-1}Cx)\| < \infty.$$

Acknowledgements. The authors would like to thank the referees for their constructive suggestions. Following a referee's suggestion and using a result in [7, Theorem 2.3], we have proved in [7, Corollary 2.6] that the assumption in Theorem 3.2 that $\lambda - A$ be injective is not necessary, and when $\lambda - A$ is not injective, condition (b) in Theorem 3.2 can be replaced by $R(C) \subset R(\lambda - A)$.

References

[1] W. Arendt, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math. 59 (1987), 327–352.

[2] I. Cioranescu and G. Lumer, *On $K(t)$-convoluted semigroups*, in: Pitman Res. Notes Math. Ser. 324, Glasgow, 1994, 86–93.

[3] E. B. Davies and M. M. Pang, *The Cauchy problem and a generalization of the Hille–Yosida theorem*, Proc. London Math. Soc. 55 (1987), 181–208.

[4] R. deLaubenfels, *C-semigroups and the Cauchy problem*, J. Funct. Anal. 111 (1993), 44–61.

[5] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, 1985.

[6] M. Hieber, *Laplace transforms and α-times integrated semigroups*, Forum Math. 3 (1991), 595–612.

[7] C.-C. Kuo and S.-Y. Shaw, *On strong and weak solutions of abstract Cauchy problems*, preprint.

[8] Y.-C. Li, *Integrated C-semigroups and C-cosine functions of operators on locally convex spaces*, Ph.D. dissertation, National Central Univ., 1991.

[9] Y.-C. Li and S.-Y. Shaw, *N-times integrated C-semigroups and the abstract Cauchy problem*, Taiwanese J. Math. 1 (1997), 75–102.

[10] —, —, *On generators of integrated C-semigroups and C-cosine functions*, Semigroup Forum 47 (1993), 29–35.

[11] I. Miyadera, *On the generators of exponentially bounded C-semigroups*, Proc. Japan Acad. Ser. A Math. Sci. 62 (1986), 239–242.

[12] I. Miyadera, M. Okubo and N. Tanaka, *On integrated semigroups which are not exponentially bounded*, ibid. 69 (1993), 199–204.

[13] —, —, —, *α-integrated semigroups and abstract Cauchy problems*, Mem. School Sci. Engrg. Waseda Univ. 57 (1993), 267–289.

[14] F. Neubrander, *Integrated semigroups and their applications to the abstract Cauchy problem*, Pacific J. Math. 135 (1988), 111–155.

[15] N. Tanaka and I. Miyadera, *Exponentially bounded C-semigroups and integrated semigroups*, Tokyo J. Math. 12 (1989), 99–115.

[16] —, —, *C-semigroups and the abstract Cauchy problem*, J. Math. Anal. Appl. 170 (1992), 196–206.

Department of Mathematics
Fu-Jen University
Hsin-Chuang, Taipei, Taiwan

Department of Mathematics
National Central University
Chung-Li, Taiwan
E-mail: shaw@math.ncu.edu.tw

Received February 17, 1998
Revised version April 3, 2000
(4051)