On $\alpha$-times integrated $C$-semigroups
and the abstract Cauchy problem

by

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Abstract. This paper is concerned with $\alpha$-times integrated $C$-semigroups for $\alpha > 0$
and the associated abstract Cauchy problem: $u'(t) = Au(t) + \frac{t^\alpha}{\Gamma(\alpha + 1)} x$, $t > 0$; $u(0) = 0$.
We first investigate basic properties of an $\alpha$-times integrated $C$-semigroup which may
not be exponentially bounded. We then characterise the generator $A$ of an exponentially
bounded $\alpha$-times integrated $C$-semigroup, either in terms of its Laplace transforms or in
terms of existence of a unique solution of the above abstract Cauchy problem for every $x$
in $(\lambda - A)^{-1}C(X)$.

0. Introduction. Let $A$ be a closed linear operator with domain $D(A)$
and range $R(A)$ in a Banach space $X$. The abstract Cauchy problem associated
with $A$ is the initial value problem

$$
\begin{cases}
u'(t) = Au(t) + f(t), & t > 0, \\
u(0) = x,
\end{cases}
$$

where $f \in C([0, \infty); X)$. Let $[D(A)]$ denote the Banach space $D(A)$ with
the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for $x \in D(A)$. A function $u$ is a (strong)
solution of $ACP(f, x)$ if $u \in C^1([0, \infty); X) \cap C([0, \infty); [D(A)])$ and satisfies
$ACP(f, x)$. It is well known that the ACP is closely related to the theory of
semigroups (see e.g. [5]).

Recently Li and Shaw ([8]–[10]) introduced exponentially bounded $n$-times integrated $C$-semigroups and studied their connection with the ACP.
It was proved [8] that if $A$ is the generator of an exponentially bounded
$n$-times integrated $C$-semigroup $S(t)$, then ACP$(0, x)$ has a unique solution
for every initial value $x \in C(D(A^{n+1}))$. In particular, for the simplest case
$n = 0$, i.e. $A$ is the generator of an exponentially bounded $C$-semigroup,

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the function \( C^{-1}S(t)x \) is the unique solution of \( ACP(0, x) \) for every \( x \in C(D(A)) \).

However, as shown by an example in Section 1, a general \( \alpha \)-times integrated \( C \)-semigroup may not be exponentially bounded. In this paper we attempt to investigate a more general class of operator families, namely \( \alpha \)-times integrated \( C \)-semigroups, where \( \alpha \) may be any nonnegative number.

In Section 1, some basic properties of nondegenerate \( \alpha \)-times integrated \( C \)-semigroups are proved. If \( A \) is the generator of such a semigroup with \( \alpha > 0 \) (resp. \( \alpha = 0 \)), then \( u(t; x) = C^{-1}S(t)x \) is the unique strong solution of \( ACP(\alpha; x, 0) \) (resp. \( ACP(0, x) \)) for every value from \( C(D(A)) \) (Proposition 1.5). Here \( j_{\alpha-1}(\cdot,x,0) \) denotes \( t^{\alpha-1}/\Gamma(\alpha) \) for \( \alpha > 0 \) and 0 for \( \alpha = 0 \). Section 2 is devoted to nondegenerate \( \alpha \)-times \( \alpha > 0 \) integrated \( C \)-semigroups which are exponentially bounded. In this case, \( ACP(\alpha; x, 0) \) is solvable for every \( x \) from the set \( (\lambda - A)^{-1}C(X) \) (Theorem 2.4), which is larger than \( C(D(A)) \) in general, and is equal to the latter when \( \lambda \in \phi(A) \), the resolvent set of \( A \) (see [16, Proposition 1.4]). We also prove a characterization of an exponentially bounded \( \alpha \)-times integrated \( C \)-semigroup in terms of its Laplace transform (Theorem 2.3).

Conversely, if \( C \in B(X) \) is an injection and \( A \) is a closed linear operator satisfying: (a) \( A \) commutes with \( C \); (b) \( \lambda - A \) is injective and \( R(A) \subset D((\lambda - A)^{-1}) \) for some \( \lambda \); (c) \( ACP(\alpha; x, 0) \) (resp. \( ACP(0, x) \) for the case \( \alpha = 0 \)) has a unique strong solution for every initial value \( x \in (\lambda - A)^{-1}C(X) \), then \( C^{-1}AC \) is the generator of an \( \alpha \)-times integrated \( C \)-semigroup (Theorem 3.2). This extends Theorem 3.1 of [16] from the case \( \alpha = 0 \) to cases \( \alpha > 0 \). Since \( C^{-1}AC = A \) when \( \phi(A) \neq 0 \) [16, Proposition 1.4], it follows that a closed linear operator \( A \) with \( \phi(A) \neq 0 \) is the generator of an \( \alpha \)-times integrated \( C \)-semigroup if and only if \( A \) commutes with \( C \) and \( ACP(\alpha; x, 0) \) (resp. \( ACP(0, x) \) for the case \( \alpha = 0 \)) has a unique strong solution for every \( x \in C(D(A)) \) (Corollary 3.4). This extends Corollary 2.2 of [16] from the case \( \alpha = 0 \) to cases \( \alpha > 0 \). We also characterize the generator of an exponentially bounded \( C \)-semigroup in terms of the \( ACP \) (see Theorem 3.5).

Further characterizations of generators in terms of the unique existence of strong and weak solutions of \( ACP(f, x) \) will be established in [7]. An extension of Theorem 3.2 to the case that \( \lambda - A \) is not injective is also proved there.

1. Some basic properties of \( \alpha \)-times integrated \( C \)-semigroups.

Let \( X \) be a Banach space and let \( B(X) \) be the set of all bounded linear operators from \( X \) into itself. Let \( C \in B(X) \) and let \( \alpha > 0 \). A family \( \{ S(t) : t \geq 0 \} \) in \( B(X) \) is called an \( \alpha \)-times integrated \( C \)-semigroup (see [8]–[10] for the case \( \alpha = n \in \mathbb{N} \) if

\[
\begin{align*}
(1.1) & \quad \text{\( S(\cdot)x : [0, \infty) \to X \) is continuous for each \( x \in X \);} \\
(1.2) & \quad \text{\( S(t)S(s)x = (I/\Gamma(\alpha)[t^{\alpha-1} - s^{\alpha-1}](t + s - r)^{\alpha-1}S(r)Cdr \) for \( x \in X \) and \( t, s > 0, S(0) = 0 \) and \( C\cdot S(\cdot) = S(\cdot)C \).}
\end{align*}
\]

It is called a \((0\text{-times integrated}) C\text{-semigroup} \) (see [3], [4], [11], [15], [16]) if

\[
\begin{align*}
(1.2') & \quad \text{\( S(0) = C \) and \( S(t)S(s) = S(t + s)C \) for all \( t, s \geq 0 \).}
\end{align*}
\]

\( S(\cdot) \) is said to be nondegenerate if

\[
\begin{align*}
(1.3) & \quad \text{\( S(t)x = 0 \) for all \( t > 0 \) implies \( x = 0 \).}
\end{align*}
\]

\( S(\cdot) \) is said to be exponentially bounded if

\[
\begin{align*}
(1.4) & \quad \text{there are constants } M, w > 0 \text{ such that } \|S(t)\| \leq Me^{-w t} \text{ for all } t \geq 0.
\end{align*}
\]

When \( C = I \), an \( \alpha \)-times integrated \( C \)-semigroup reduces to an \( \alpha \)-times integrated semigroup (see [1], [14] for the case \( \alpha = n \in \mathbb{N} \) and [8], [12] and [13] for the case \( \alpha \in \mathbb{R}_+ \), and a \( C \)-semigroup is a classical \( C_0 \)-semigroup. In general, an \( \alpha \)-times integrated \( C \)-semigroup may not be exponentially bounded. For example, let \( \{S(t) : t \geq 0\} \) be the family of linear operators on \( X = L^2(\mathbb{R}) \) defined by

\[
(S(t)f)(s) = \frac{1}{r(t)} \int_0^t (t-r)^{\alpha-1}e^{rs}e^{-r^2}dr
\]

It is clear that \( S(\cdot) \) is an \( \alpha \)-times integrated \( C \)-semigroup with \( (Cf)(s) = e^{-s^2}f(s) \) for all \( s \in \mathbb{R}, f \in X \) and

\[
\|S(t)\| = \sup_{s \in \mathbb{R}} \left[ \frac{1}{r(t)} \int_0^t (t-r)^{\alpha-1}e^{rs}e^{-r^2}dr \right] = \sup_{s \in \mathbb{R}} \left[ \frac{1}{r(t)} \int_0^t (t-r)^{\alpha-1}e^{-r^2/4}e^{-(s-r/2)^2}dr \right] \geq \sup_{t \geq 0} \left[ \frac{1}{r(t)} \int_0^{t/2} (t-r)^{\alpha-1}e^{-r^2/4}e^{-(s-r/2)^2}dr \right] \geq \sup_{t \geq 0} \left[ \frac{1}{r(t)} e^{t^2/4} \int_0^{t/2} (t-r)^{\alpha-1}e^{-(s-r/2)^2}dr \right] \geq \sup_{t \geq 3t/8} \left[ \frac{1}{r(t)} e^{t^2/4} \frac{\alpha}{\alpha} e^{-(s-t/4)^2} \right].
\]
For this, we define $\tilde{S}(t) := \int_0^t S(r) dr$. Then $\tilde{S}(\cdot)$ is an $\alpha + 1$-times integrated $C$-semigroup. It follows that $\tilde{S}(\cdot) \tilde{S}(t)x = \mathcal{C}(\cdot) \tilde{S}(t)x \in C^1(0, \infty; X)$ for each $t \geq 0$ and

$$S(t) \tilde{S}(s)x + j_\alpha(s) \tilde{S}(t)Cx = \frac{d}{dt} \tilde{S}(t) \tilde{S}(s)x + j_\alpha(s) \tilde{S}(t)Cx$$

$$= \frac{1}{\Gamma(\alpha)} \left( \int_0^t \int_0^s (t+\theta)^{\alpha-1} \tilde{S}(\cdot) dr d\theta \right)(t-s)^{\alpha-1} \tilde{S}(s)x dr.$$

Since this function is symmetric in $s$ and $t$, it follows that

$$S(s) \tilde{S}(t)x + j_\alpha(t) \tilde{S}(s)Cx = \mathcal{S}(t) \mathcal{S}(s)x + j_\alpha(t) \tilde{S}(t)Cx.$$

Hence

$$\frac{d}{ds} S(s) \tilde{S}(t)x = S(s) \tilde{S}(t)x + j_\alpha(t) \tilde{S}(s)Cx - j_\alpha(t) S(s)Cx.$$

(1.9) is a direct consequence of Lemma 1.2. To prove (1.10), we first show $A \subset C^{1-\alpha}AC$. Let $x \in D(A)$. Then by (1.2) and the definition of $A$, we have

$$S(t)Cx - j_\alpha(t) C^2x = C(S(t)x - j_\alpha(t)Cx) = C \left( \int_0^t S(r) Ax dr \right) CAx = \int_0^t S(r) C^2 Ax dr,$$

or equivalently $Cx \in D(A)$ and $ACx = C Ax \in R(C)$. That is, $A \subset C^{1-\alpha}AC$. To show the inclusion $C^{1-\alpha}AC \subset A$, let $x \in D(C^{1-\alpha}AC)$. We have $Cx \in D(A)$ and $ACx \in R(C)$. By the commutativity of $C$ and $S(\cdot)$, and the definition of generator, we have

$$C[S(t)x - j_\alpha(t)Cx] = S(t)Cx - j_\alpha(t) C^2x = \int_0^t S(r) ACx dr$$

$$= \int_0^t S(r) C^{1-\alpha}ACx dr = C \int_0^t S(r) C^{1-\alpha}ACx dr.$$

Since $C$ is injective, this implies $x \in D(A)$ and $Ax = C^{1-\alpha}ACx$. Consequently, $A = C^{1-\alpha}AC$.

**Remark.** (1.8) shows that a nondegenerate $\alpha$-times integrated $C$-semigroup is an example of a $K(t)$-evolution operator with $K(t) = j_{\alpha-1}(t)C$ (see [2] for results on this class).

**Proposition 1.4.** Let $A$ be the generator of an $\alpha$-times integrated $C$-semigroup $S(\cdot)$ with $\alpha > 0$ (resp. $\alpha = 0$), and let

$$C^1 = \{ x \in X : S(\cdot)x \text{ is continuously differentiable on } [0, \infty) \}.$$

Then

1. $S(t)C^1 \subset D(A)$ for all $t \geq 0$;
2. for $x \in C^1$, $u(\cdot) = S(\cdot)x$ is the unique solution of $ACP(j_{\alpha-1}(\cdot)Cx, 0)$ (resp. $ACP(0, Cx)$).

In particular, (2) holds for all $x \in D(A)$. 

**Proof.** (1.7) follows from the commutativity of the family $S(\cdot)$ and the definition of $A$. To prove (1.8), it suffices to show that

$$\frac{d}{ds} S(s) \int_0^t S(r)x dr = S(s)(S(t)x - j_\alpha(t)Cx)$$

$$+ j_{\alpha-1}(s) C \int_0^t S(r)x dr.$$

**Lemma 1.2.** Let $A$ be the generator of an $\alpha$-times integrated $C$-semigroup $S(\cdot)$ and let $T > 0$. If $u \in C([0, T]; X)$ satisfies

$$u(t) = A \int_0^t u(s) ds \quad \text{for all } 0 \leq t \leq T,$$

then $u \equiv 0$ on $[0, T]$.

**Proposition 1.3.** Let $\alpha \geq 0$. The generator $A$ of an $\alpha$-times integrated $C$-semigroup $S(\cdot)$ satisfies

1. $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
2. $\int_0^t S(r)x dr \in D(A)$ and $A \int_0^t S(r)x dr = S(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $t \geq 0$;
3. $S(\cdot)$ is uniquely determined by $A$;
4. $C^{1-\alpha}AC = A$.

**Proof.** (1.7) follows from the commutativity of the family $S(\cdot)$ and the definition of $A$. To prove (1.8), it suffices to show that

$$\frac{d}{ds} S(s) \int_0^t S(r)x dr = S(s)(S(t)x - j_\alpha(t)Cx)$$

$$+ j_{\alpha-1}(s) C \int_0^t S(r)x dr.$$
Proof. Let $x \in C^1$ and $t > 0$. Differentiating the equation in (1.8), we have $S(t)x \in D(A)$ and

$$AS(t)x = \frac{d}{dt}S(t)x + j\alpha(t)Cx.$$  

The uniqueness of solution of $ACP(j\alpha(t)Cx, 0)$ (resp. $ACP(0, Cx)$ for the case $\alpha = 0$) is obvious from Lemma 1.2. Finally, the definition of $A$ implies that $D(A) \subset C^1$ so that (2) holds for all $x \in D(A)$.

PROPOSITION 1.5. Let $S(\cdot)$ be an $\alpha$-times integrated C-semigroup with $\alpha > 0$ (resp. $\alpha = 0$). If $S(\cdot)x \in R(C)$ and $C^{-1}S(\cdot)x$ is continuously differentiable on $[0, \infty)$, then $C^{-1}S(\cdot)x \in D(A)$, and $u(\cdot) = C^{-1}S(\cdot)x$ is the unique solution of $ACP(j\alpha(\cdot)x, 0)$ (resp. $ACP(0, x)$). This holds in particular for $x \in C(D(A))$.

Proof. The uniqueness is obvious and hence we only need to show that $u(\cdot) = C^{-1}S(\cdot)x$ is a solution of $ACP(j\alpha(\cdot)x, 0)$. Clearly, $S(\cdot)x = CC^{-1}S(\cdot)x$ is continuously differentiable on $[0, \infty)$. By Proposition 1.4 we have

$$C^{-1}S(\cdot)x = AS(\cdot)x + j\alpha(t)Cx = C^{-1}S(\cdot)x + j\alpha(t)Cx.$$  

Hence $C^{-1}S(\cdot)x \in D(C^{-1}AC) = D(A)$ and

$$\frac{d}{dt}C^{-1}S(\cdot)x = (C^{-1}AC)C^{-1}S(\cdot)x + j\alpha(t)Cx = AC^{-1}S(\cdot)x + j\alpha(t)Cx.$$  

\[2.\] **Exponentially bounded $\alpha$-times integrated C-semigroups**

**LEMMA 2.1.** Let $A$ be the generator of an $\alpha$-times integrated C-semigroup $S(\cdot)$. For $\lambda > 0$, let $D(\alpha)$ denote the set of all those $x \in X$ for which $L_x = \int_0^\infty e^{-\lambda t}S(t)x \ dt$ exists and $\|C^{-1}S(t)\| \leq M e^{\lambda t}$ for some $M > 0$, $w \geq 0$ and all $t \geq 0$. For $\lambda > w$ define $R_x = \int_0^\infty e^{-\lambda t}S(t)x \ dt$ for $x \in D(\alpha)$.

(i) $L_xD(\alpha) \subset D(A)$ and $(\lambda - A)L_x = Cx$ for $x \in D(\alpha)$.

(ii) $(S(\cdot)L_x)_{x \in D(\alpha)} \subset R(C)$, and $C^{-1}S(\cdot)L_x$ is continuously differentiable on $[0, \infty)$ for $x \in D(\alpha)$.

Proof. (i) Integration by parts yields

$$e^{-\lambda t} \int_0^t S(t)x \ dt = e^{-\lambda t}S(t)x \ dt - \int_0^t e^{-\lambda s}S(s)x \ ds \ dt,$$

which converges as $t \to \infty$ if $x \in D\alpha$. This and $\int_0^\infty e^{-\lambda t} \|S(t)\| \ dt < \infty$ imply that $e^{-\lambda t} \int_0^t S(t)x \ dt \to 0$ as $t \to \infty$. Therefore we have

$$L_x = \int_0^\infty e^{-\lambda t}S(t)x \ dt.$$  

From (1.8) and the closedness of $A$ we deduce that $\lambda^{-1} \int_0^\infty e^{-\lambda t}S(t)x \ dt \in D(A)$ and

$$\lambda^{-1}A \int_0^\infty e^{-\lambda t}S(t)x \ dt = \lambda^{-1} \int_0^\infty e^{-\lambda t}(S(t)x - j\alpha(t)Cx) \ dt$$  

$$\to \lambda^{-1}L_x - Cx$$

as $t \to \infty$. Hence the closedness of $A$ implies that $L_x \in D(A)$ and

$$\lambda^{-1}AL_x = \lambda^{-1}L_x - Cx$$

for $x \in D(\alpha)$. Consequently, $(\lambda - A)\lambda^{-1}L_x = Cx$ for $x \in D(\alpha)$.

(ii) By (1.2) we have, for $x \in D(\alpha)$,

$$S(t)L_x = \int_0^\infty e^{-\lambda t}S(t)x \ dt$$

$$= \int_0^\infty e^{-\lambda t}S(t)x \ dt.$$  

from which it is clear that $S(t)L_x \in R(C)$ and $C^{-1}S(\cdot)L_x$ is continuously differentiable.

The next proposition generalizes Theorem 3.1 of [1], where the case $\alpha = 2$ was proven.

**PROPOSITION 2.2.** Let $\alpha > 0$, and let $\{S(t) : t \geq 0\} \subset B(X)$ be a strongly continuous function such that $\|S(t)\| \leq Me^{\alpha w}$ for some $M > 0$, $w \geq 0$ and all $t \geq 0$. For $\lambda > w$ define

$$ R_x = \lambda^{-1}L_x = \int_0^\infty e^{-\lambda t}S(t)x \ dt.$$  

Then $\{R_x : x \geq 0\}$ is a C-pseudo-resolvent, i.e.

$$(\lambda - \mu)R_x - R_y = \mu^{-1}R_x \quad \text{for } \lambda, \mu > w,$$

if and only if $S(\cdot)$ satisfies

$$S(t)= \int_0^\infty \left( \left( \int_0^\infty \int_0^\infty \right) \right) S(t)S(x) \ dt \ ds \ dx$$

for $x \in X$ and $t, s \geq 0$. 

\[\]
Proof. Interchanging the order of integrations we obtain the following equalities:

\[
\begin{align*}
\int_0^\infty e^{-\lambda s} \int_0^s (t + s - r)^{\alpha - 1} S(r) Cx \, dr \, ds & = \int_0^\infty \int_0^r e^{-\lambda s} (t + s - r)^{\alpha - 1} ds \, S(r) Cx \, dr \\
& = \int_0^\infty e^{-\lambda s} \int_0^s (t + s')^{\alpha - 1} ds' \, S(r) Cx \, dr \\
& = e^{\lambda t} \int_0^\infty e^{-\lambda s} s^{\alpha - 1} ds \, \lambda^{-\alpha} R_\lambda Cx,
\end{align*}
\]

\[
\begin{align*}
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^s (t + s - r)^{\alpha - 1} S(r) Cx \, dr \, ds \, dt & = \lambda^{-\alpha} \int_0^\infty e^{-(\mu - \lambda)t} \int_0^t e^{-\lambda s} s^{\alpha - 1} ds \, dt \, R_\lambda Cx \\
& = \lambda^{-\alpha} \int_0^\infty \left( \int_0^t e^{-(\mu - \lambda)t} dt \right) e^{-\lambda s} s^{\alpha - 1} ds \, R_\lambda Cx \\
& = \Gamma(\alpha) \lambda^{-\alpha} \frac{\mu^{-\alpha} - \lambda^{-\alpha}}{\lambda - \mu} R_\lambda Cx,
\end{align*}
\]

\[
\begin{align*}
\int_0^\infty e^{-\lambda s} \int_0^s (t + s - r)^{\alpha - 1} S(r) Cx \, dr \, ds & = \int_0^\infty \int_0^r e^{-\lambda s} (t + s - r)^{\alpha - 1} ds \, S(r) Cx \, dr \\
& = \int_0^\infty \int_0^{t+s} e^{-\lambda(t-r)} \left( \int_0^{r-t} e^{-\lambda s'} s'^{\alpha - 1} ds' \right) S(r) Cx \, dr \\
& = \Gamma(\alpha) \lambda^{-\alpha} \int_0^\infty e^{-\lambda(r-t)} S(r) Cx \, dr,
\end{align*}
\]

Combining these equalities we obtain

\[
\begin{align*}
\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \left[ S(t) S(s) x \right] ds \, dt & = \frac{1}{\Gamma(\alpha)} \left( \int_0^t \left( \int_0^{t+s} (t + s - r)^{\alpha - 1} S(r) Cx \, dr \right) ds \, dt \\
& = \lambda^{-\alpha} \mu^{-\alpha} [R_\mu R_\lambda x - (\lambda - \mu)^{-1} (R_\mu Cx - R_\lambda Cx)]
\end{align*}
\]

for all \( x \in X \) and \( s, t \geq 0 \). The conclusion now follows from the uniqueness theorem for Laplace transforms.

Proposition 2.2 can be applied to prove the following characterization of an exponentially bounded \( \alpha \)-times integrated \( C \)-semigroup, in terms of its Laplace transform. The case of \( n \)-times integrated \( C \)-semigroup was proved in [8, 9].

**Theorem 2.3.** Let \( C \in B(X) \) be an injection. A strongly continuous function \( S(\cdot) \) satisfying (1.4) is an \( \alpha \)-times integrated \( C \)-semigroup with generator \( A \) if and only if \( CS(\cdot) = S(\cdot)C, C^{-1}AC = A, \) and \( \lambda - A \) is injective, \( R(C) \subset R(\lambda - A), \) and

\[
(2.1) \quad \lambda^\alpha L_\lambda (\lambda - A) \subset \lambda^\alpha (\lambda - A) L_\lambda = C
\]

for each \( \lambda > \omega \).

**Proof.** When \( S(\cdot) \) is an exponentially bounded \( \alpha \)-times integrated \( C \)-semigroup, for large \( \lambda \) the set \( D_\lambda \) as defined in Lemma 2.1 is clearly equal to \( X \). Then Lemma 2.1 together with (1.7) yields that for each \( \lambda > \omega, \lambda - A \) is injective, \( R(C) \subset R(\lambda - A), L_\lambda \in B(X), R(L_\lambda) \subset D(A), \) and (2.1) holds.

Conversely, (2.1) implies

\[
R_\mu C - R_\lambda C = R_\mu C - CR_\lambda = R_\mu (\lambda - A) R_\lambda - R_\mu (\mu - A) R_\lambda
\]

so that, by Proposition 2.2, \( S(\cdot) \) is an \( \alpha \)-times integrated \( C \)-semigroup. Since \( \lambda - A \) and \( C \) are injective, it is seen from (2.1) that \( S(\cdot) \) is nondegenerate. Let \( B \) be its generator. Then the "only if" part of the theorem asserts that \( C^{-1} BC = B \), and for each \( \lambda > \omega, \lambda - B \) is injective, \( R(C) \subset R(\lambda - B) \), and \( R_\lambda (\lambda - B) \subset (\lambda - B) R_\lambda = C \). If \( x \in D(A), \) then \( Cx = R_\lambda (\lambda - A) x \in D(B) \) and \( (\lambda - B) Cx = (\lambda - B) R_\lambda (\lambda - A) x = C(\lambda - A) x \), so that \( x \in D(C^{-1} BC) \).
$D(B)$ and $Ax = C^{-1}BCx = Bx$. Hence $A \subset B$. By symmetry we also have $B \subset A$. This completes the proof.

We obtain the following theorem by applying Proposition 1.5, Lemma 2.1(ii), and Theorem 2.3.

**Theorem 2.4.** Let $\alpha > 0$ (resp. $\alpha = 0$) and $A$ be the generator of an exponentially bounded $\alpha$-times integrated $C$-semigroup $S(\cdot)$ with $\|S(t)\| \leq Me^{\omega t}$ for some $M, \omega > 0$ and all $t \geq 0$, and let $\lambda > w$. Then for each $x \in (\lambda - A)^{-1}CX = L^\alpha_x L_x X$, $u(\cdot) = C^{-1}S(\cdot)x$ is the unique solution of $ACP(j_{\alpha-1}(\cdot)x, 0)$ (resp. $ACP(0, x)$ for the case $\alpha = 0$). Moreover, $\|u(t)\| = O(e^{\omega t})$ and $\|u(\cdot')\| = O(e^{\omega t})$ as $t \to \infty$.

Note that this theorem extends the particular case $x \in C(D(A))$ of Proposition 1.5 to exponentially bounded $\alpha$-times integrated $C$-semigroups because it is shown in the following lemma of Tanaka and Miyadera [16, Proposition 1.4] that $C(D(A)) \subset (\lambda - A)^{-1}CX$ in general. This lemma will also be needed in Section 3.

**Lemma 2.5.** Let $\lambda \in \mathbb{R}$ and let $A$ be a closed linear operator satisfying

(a) $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$;
(b) $\lambda - A$ is injective and $D((\lambda - A)^{-1}) \supseteq R(C)$.

Then

(i) $C(D(A)) \subset C(D(C^{-1}AC)) \subset (\lambda - A)^{-1}CX$;
(ii) $C(D(A)) = (\lambda - A)^{-1}CX$ if and only if $\lambda \in \sigma(A)$;
(iii) $C^{-1}AC = A$ if $\sigma(A) \neq \emptyset$.

3. The abstract Cauchy problem. In this section we try to characterize the generator of an $\alpha$-times integrated $C$-semigroup in terms of the unique existence of a strong solution of $ACP(j_{\alpha-1}(\cdot)x, 0)$.

**Lemma 3.1.** For all $\alpha > 0$ and $t, s \geq 0$ we have

$$
(\int_0^t - \int_0^s)(t + s - r)^{\alpha - 1}r^{\alpha - 1} \, dr = 0.
$$

**Proof.** Since $S(t) = j_\alpha(t)$ is an $\alpha$-times integrated semigroup on $\mathbb{R}$, using (1.2) and integration by parts one has

$$
j_\alpha(t)j_\alpha(s) = \frac{1}{\Gamma(\alpha)} \left( \int_0^{s+t} - \int_0^{t+s} \right)(t + s - r)^{\alpha - 1}r^{\alpha - 1} \, dr = \frac{1}{\Gamma(\alpha + 1) \Gamma(\alpha + 1)} \left[ 3^{\alpha}s^\alpha + \alpha \left( \int_0^t - \int_0^s \right)(t + s - r)^{\alpha - 1}r^{\alpha - 1} \, dr \right],
$$

so that

$$
-t^\alpha s^\alpha = \alpha \left( \int_0^{t+s} - \int_0^s \right)(t + s - r)^{\alpha - 1}r^{\alpha - 1} \, dr.
$$

Differentiation with respect to $t$ yields

$$
-\alpha t^{\alpha - 1}s^\alpha = -\alpha s^\alpha t^{\alpha - 1} + \alpha^2 \left( \int_0^{t+s} - \int_0^s \right)(t + s - r)^{\alpha - 1}r^{\alpha - 1} \, dr
$$

and hence (3.1).

**Theorem 3.2.** Let $\alpha > 0$ (resp. $\alpha = 0$) and $\lambda \in \mathbb{R}$, and let $A$ be a closed linear operator satisfying

(a) $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$;
(b) $\lambda - A$ is injective and $D((\lambda - A)^{-1}) \supseteq R(C)$;
(c) $ACP(j_{\alpha-1}(\cdot)x, 0)$ (resp. $ACP(0, x)$ for the case $\alpha = 0$) has a unique solution for each $x \in (\lambda - A)^{-1}CX$.

Then there exists an $\alpha$-times integrated $C$-semigroup $S(\cdot)$ on $X$ with generator $C^{-1}AC$.

We shall need the next lemma.

**Lemma 3.3.** If (c) holds, then it also holds when $\alpha - 1$ is replaced by $\alpha$.

**Proof.** For any $x \in (\lambda - A)^{-1}CX$, (c) implies there is a solution $v$ of $ACP(j_{\alpha-1}(\cdot)x, 0)$ (resp. $ACP(0, x)$ for the case $\alpha = 0$). Then $v(t) = \int_0^t u(r) \, dr$ is a solution of $ACP(j_{\alpha-1}(\cdot)x, 0)$. That the solution $v$ is unique follows from the uniqueness of solution of $ACP(0, x) = ACP(j_{\alpha-1}(\cdot)0, 0)$, which is guaranteed by (c).

**Proof of Theorem 3.2.** We denote the unique solution of $ACP(j_{\alpha-1}(\cdot)x, 0)$ by $u(t; x)$ and define the operator $S(t) : X \to X$ for $t \geq 0$ by

$$
S(t)x = (\lambda - A)u(t; (\lambda - A)^{-1}Cx) \quad \text{for} \quad x \in X.
$$

Then $S(0) = 0$ and $S(\cdot)x : [0, \infty) \to X$ is continuous for $x \in X$. Now, the uniqueness of solution implies that $S(\cdot)$ is linear, $Cu(t; (\lambda - A)^{-1}Cx) = u(t; (\lambda - A)^{-1}Cx)$ and so $CS(\cdot) = S(\cdot)C$.

Let $C([0, \infty); [D(A)])$ be the Fréchet space with the quasi-norm

$$
||u|| := \sum_{k=1}^{\infty} \frac{||u||_k}{2^k(1 + ||v||_k)} \quad \text{for} \quad v \in C([0, \infty); [D(A)]),
$$

where $||v||_k = \max_{t_k \leq t < t_{k+1}} ||u(t)||_k$, $k \in \mathbb{N}$, $\{t_k\}$ being an increasing sequence in $(0, \infty)$ with $\lim_{k \to \infty} t_k = \infty$. Consider the linear map $\eta : X \to C([0, \infty); [D(A)])$ given by $\eta(x) = u(\cdot; (\lambda - A)^{-1}Cx)$. We show that $\eta$ is a closed linear operator.
In fact, let \( x_n \to x \) in \( X \) and \( \eta(x_n) \to v \) in \( C([0, \infty); [D(A)]) \). Then
\[
 u(t; (\lambda - A)^{-1}C x_n) = \int_0^t A u(r; (\lambda - A)^{-1}C x_n) \, dr + j_a(t)(\lambda - A)^{-1}C x_n.
\]

Letting \( n \to \infty \) we obtain \( v(t) = \int_0^t A u(r; (\lambda - A)^{-1}C x) \, dr \) for all \( t \geq 0 \). Thus \( v(\cdot) \) is a solution of \( \text{ACP}(j_{a-1}(())(\lambda - A)^{-1}C x, 0) \). Therefore, from the uniqueness of solution it follows that \( v(\cdot) = u(\cdot; (\lambda - A)^{-1}C x) = \eta(x) \). We have shown that \( \eta \) is closed. By the closed graph theorem, \( \eta \) is continuous from \( X \) into \( C([0, \infty); [D(A)]) \). This shows that \( S(\cdot) \) is a strongly continuous function into bounded linear operators on \( X \).

To show that \( S(\cdot) \) is an \( \alpha \)-times integrated \( C \)-semigroup, we first consider the case \( \alpha \geq 1 \). For fixed \( x \in X \) and \( s \geq 0 \) we define
\[
 v_s(t) = \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} u(r; (\lambda - A)^{-1}C^2 x) \, dr,
\]
\[ t \geq 0. \]

Then
\[
 A v_s(t) = \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} A u(r; (\lambda - A)^{-1}C^2 x) \, dr
\]
\[ = \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} \left[ u'(r; (\lambda - A)^{-1}C^2 x) \right. \]
\[ \left. - \frac{\alpha - 1}{\Gamma(\alpha)} (\lambda - A)^{-1}C^2 x \right] \, dr
\]
\[ = \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} u'(r; (\lambda - A)^{-1}C^2 x) \, dr,
\]
by Lemma 3.1.

For \( \alpha = 1 \) we have
\[
 A v_s(t) = u(t + s; (\lambda - A)^{-1}C^2 x) - u(t; (\lambda - A)^{-1}C^2 x) - u(s; (\lambda - A)^{-1}C^2 x)
\]
and
\[
 \frac{dv_s(t)}{dt} = u(t + s; (\lambda - A)^{-1}C^2 x) - u(t; (\lambda - A)^{-1}C^2 x).
\]

For the case \( \alpha > 1 \), using integration by parts we have
\[
 A v_s(t) = - j_{a-1}(t) u(s; (\lambda - A)^{-1}C^2 x) - j_{a-1}(s) u(t; (\lambda - A)^{-1}C^2 x)
\]
\[ + \frac{1}{\Gamma(\alpha - 1)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 2} u(r; (\lambda - A)^{-1}C^2 x) \, dr
\]
and
\[
 \frac{dv_s(t)}{dt} = - j_{a-1}(t) u(t; (\lambda - A)^{-1}C^2 x)
\]
\[ + \frac{1}{\Gamma(\alpha - 1)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 2} u(r; (\lambda - A)^{-1}C^2 x) \, dr.
\]

It follows that for all \( \alpha \geq 1 \) and \( t \geq 0 \),
\[
 \frac{dv_s(t)}{dt} = A v_s(t) + j_{a-1}(t) u(s; (\lambda - A)^{-1}C^2 x)
\]
\[ = A v_s(t) + j_{a-1}(t) C u(s; (\lambda - A)^{-1}C x).
\]

The uniqueness of solution implies that \( v_s(t) = u(t; C u(s; (\lambda - A)^{-1}C^2 x)) \) and hence
\[
 \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} S(r) C x \, dr
\]
\[ = \frac{1}{\Gamma(\alpha)} \left( \int_0^t - \int_0^s \int_0^t \right) (t + s - r)^{\alpha - 1} (\lambda - A) u(r; (\lambda - A)^{-1}C^2 x) \, dr
\]
\[ = (\lambda - A) v_s(t) = (\lambda - A) u(t; C u(s; (\lambda - A)^{-1}C x))
\]
\[ = (\lambda - A) u(t; (\lambda - A)^{-1}C x) - (\lambda - A) u(t; (\lambda - A)^{-1}C^2 x)
\]
\[ = S(t) S(s)x \quad \text{for all } t, s \geq 0.
\]

Now we turn to the case \( 0 \leq \alpha < 1 \). The hypothesis (c) and Lemma 3.3 imply that \( v_s(t) := \int_0^t u(r; (\lambda - A)^{-1}C^2 x) \, dr \) is the unique solution of \( \text{ACP}(j_{a}();(\lambda - A)^{-1}C x, 0) \). Let \( S(\cdot) \) be defined by \( S(t)x = (\lambda - A) u(t; C u(x; (\lambda - A)^{-1}C x)) \). The previous argument has shown that \( S(\cdot) \) is an \( (\alpha - 1) \)-times integrated \( C \)-semigroup on \( X \). In particular, \( S(t)x \) is continuous for all \( x \in X \). Since \( S(t)x = (\lambda - A) u(t; (\lambda - A)^{-1}C^2 x) \) is continuous for all \( x \in X \), the closedness of \( A \) implies that
\[
 S(t)x = (\lambda - A) \int_0^t u(r; (\lambda - A)^{-1}C^2 x) \, dr = \int_0^t S(r)x \, dr \in C^2([0, \infty); X).
\]

Then an easy computation shows that \( S(\cdot) \) is an \( \alpha \)-times integrated \( C \)-semigroup.

In order to show that \( S(\cdot) \) is nondegenerate, suppose \( S(t)x = 0 \) for \( t > 0 \). Then by the injectivity of \( \lambda - A \) we have \( u(t; (\lambda - A)^{-1}C^2 x) = 0 \) for \( t > 0 \), so that
0 = \frac{d}{dt} u(t; (\lambda - A)^{-1}C x) \\
= A u(t; (\lambda - A)^{-1}C x) + j_{\alpha-1}(t)(\lambda - A)^{-1}C x \\
= j_{\alpha-1}(t)(\lambda - A)^{-1}C x

and hence \( x = 0 \), because \( (\lambda - A)^{-1} \) and \( C \) are injective.

Having shown that \( S(\cdot) \) is a nondegenerate \( \alpha \)-times integrated \( C \)-semigroup on \( X \), we next show that \( C^{-1} A C \) is the generator of \( S(\cdot) \). Let \( B \) be the generator of \( S(\cdot) \) and let \( x \in D(C^{-1} A C) \). Easy computations show that both \( u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}C x \) and \( \int_0^t u(s; (\lambda - A)^{-1}C(C^{-1} A C)x) \, ds \) are solutions of the Cauchy problem \( ACP(\lambda(\cdot)C^{-1} A Cx, 0) \). Then the uniqueness of solution of \( ACP(0, 0) = ACP(\lambda(\cdot)C^{-1} A Cx, 0) \) implies

\[
    u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}C x = \int_0^t u(s; (\lambda - A)^{-1}C(C^{-1} A C)x) \, ds.
\]

Hence

\[
    S(t)x - j_{\alpha}(t)C x = (\lambda - A)[u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}C x] \\
    = (\lambda - A)\int_0^t u(s; (\lambda - A)^{-1}C(C^{-1} A C)x) \, ds \\
    = \int_0^t (\lambda - A)u(s; (\lambda - A)^{-1}C(C^{-1} A C)x) \, ds \\
    = \int_0^t S(s)C^{-1}ACx \, ds \quad \text{for all } t \geq 0.
\]

Consequently, \( x \in D(B) \) and \( Bx = C^{-1} A C x \), or equivalently \( C^{-1} A C \subset B \).

To prove the converse, let \( x \in D(B) \). We have

\[
    (\lambda - A)[u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}C x] \\
    = S(t)x - j_{\alpha}(t)C x = \int_0^t S(r)B x \, dr \\
    = \int_0^t (\lambda - A)u(r; (\lambda - A)^{-1}CBx) \, dr \\
    = (\lambda - A)\int_0^t u(r; (\lambda - A)^{-1}CBx) \, dr.
\]

Since \( \lambda - A \) is injective, it follows that

\[
    u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}C x = \int_0^t u(r; (\lambda - A)^{-1}CBx) \, dr,
\]

so that \( A u(t; (\lambda - A)^{-1}C x) = u(t; (\lambda - A)^{-1}CBx) \in D(A) \). We also have

\[
    \int_0^t A u(r; (\lambda - A)^{-1}CBx) \, dr \\
    = \int_0^t \int_0^r \frac{d}{dr} u(r; (\lambda - A)^{-1}CBx) - j_{\alpha-1}(r)(\lambda - A)^{-1}CBx \, dr \\
    = u(t; (\lambda - A)^{-1}CBx) - j_{\alpha}(t)(\lambda - A)^{-1}CBx \in D(A).
\]

From these facts, it follows that

\[
    j_{\alpha}(t)C x = S(t)x - \int_0^t S(r)B x \, dr \\
    = (\lambda - A)u(t; (\lambda - A)^{-1}C x) \\
    - \int_0^t (\lambda - A)u(r; (\lambda - A)^{-1}CBx) \, dr \in D(A)
\]

and

\[
    A j_{\alpha}(t)C x = A\left[\lambda u(t; (\lambda - A)^{-1}C x) - u(t; (\lambda - A)^{-1}CBx) \\
    - \lambda \int_0^t u(r; (\lambda - A)^{-1}CBx) \, dr \\
    + u(t; (\lambda - A)^{-1}CBx) - j_{\alpha}(t)(\lambda - A)^{-1}CBx\right] \\
    = \lambda A u(t; (\lambda - A)^{-1}C x) - \lambda \int_0^t A u(r; (\lambda - A)^{-1}CBx) \, dr \\
    - j_{\alpha}(t)(\lambda - A)^{-1}CBx \\
    = A u(t; (\lambda - A)^{-1}C x) - j_{\alpha}(t)(\lambda - A)^{-1}CBx \\
    = j_{\alpha}(t)C x \quad \text{for all } t \geq 0.
\]

That shows \( B \subset C^{-1} A C \).

**Corollary 3.4.** Let \( A \) be a closed linear operator with nonempty resolvent set. Then the following are equivalent:
(i) $A$ is the generator of an $\alpha$-times integrated $C$-semigroup $\{S(t) : t \geq 0\}$ with $\alpha > 0$ (resp. $\alpha = 0$).

(ii) $A$ satisfies the conditions:

(a) $Cz \in D(A)$ and $ACx = CAz$ for all $x \in D(A)$;

(b) The ACP $\lambda_{\alpha-1}(\cdot)Cx, 0$ (resp. ACP $0, Cx$ for the case $\alpha = 0$) has a unique solution $u(t; Cx)$ for every $x \in D(A)$.

In this case, $u(t; Cx) = S(t)x$ for $t \geq 0$.

Proof. The implication (i)$\Rightarrow$(ii) is a direct consequence of Propositions 1.3 and 1.4. We show (ii)$\Rightarrow$(i). Since $C(D(A)) = (\lambda - A)^{-1}C(X)$ for $\lambda \in \rho(A)$ and $C^{-1}AC = A$ (Lemma 2.5(ii), (iii)), it follows from Theorem 3.2 that $A$ is the generator of an $\alpha$-times integrated $C$-semigroup.

A characterization of exponentially bounded $\alpha$-times integrated $C$-semigroup in terms of the ACP is given by

**Theorem 3.5.** Let $A$ be a closed linear operator in $X$. Then the following are equivalent:

(i) $A$ is the generator of an exponentially bounded $\alpha$-times integrated $C$-semigroup $S(\cdot)$ with $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$, where $M, \omega > 0$ are fixed.

(ii) $A$ satisfies the conditions:

(a) $C^{-1}AC = A$;

(b) for some $\lambda \in \mathbb{R}, \lambda - A$ is injective and $D((\lambda - A)^{-1}) \supset R(C)$;

(c) for every $x \in (\lambda - A)^{-1}C(X), ACP_{\lambda_{\alpha-1}(\cdot)x, 0}$ has a unique solution $u(t; x)$ such that $\|u(t; x)\|$ and $\|u(t; x)\|_x$ are $O(e^{\omega t\cdot x})$ as $t \to \infty$.

Proof. The implication (i)$\Rightarrow$(ii) follows from Theorem 2.4 and Proposition 1.3, eq. (1.10). In order to show the converse, by Theorem 3.2, it suffices to show that $\{S(t) : t \geq 0\}$ defined by (3.2) is exponentially bounded. From condition (c) we deduce that

$$\|e^{-\omega t}Au(t; (\lambda - A)^{-1}Cz)\| = \|e^{-\omega t}[w(t; (\lambda - A)^{-1}Cz)] - \lambda_{\alpha-1}(\cdot)Cz\|$$

and $\|e^{-\omega t}u(t; (\lambda - A)^{-1}Cz)\|$ are bounded for $x \in X$ as $t \to \infty$. Thus, by the uniform boundedness principle, we have

$$\sup_{t \geq 0} \|S(t)\| = \sup_{t \geq 0} \sup_{\|x\| = 1} \|\lambda e^{-\omega t}u(t; (\lambda - A)^{-1}Cz) - e^{-\omega t}Au(t; (\lambda - A)^{-1}Cz)\| < \infty.$$