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Non-similarity of Walsh and trigonometric systems

by

P. WOJTASZCZYK (Warszawa)

Abstract. We show that in L_p for $p \neq 2$ the constants of equivalence between finite initial segments of the Walsh and trigonometric systems have power type growth. We also show that the Riemann ideal norms connected with those systems have power type growth.

1. Introduction. There are numerous similarities between the trigonometric system and the Walsh system; both are bounded orthonormal bases in L_2 and both are characters of a compact abelian group. Many results are parallel (cf. [4]). However it is also known (cf. [5]) that as bases in $L_p[0,1]$ they are not equivalent unless $p = 2$. In this note we are interested in the "quantitative" estimates for this non-equivalence. The argument in [5] gives only a "logarithmic" difference. Unfortunately we are unable to provide exact estimates, but our results give a much stronger "power type" non-equivalence. We also provide a power type non-equivalence in the language of certain operator ideals. This complements some observations and supports some conjectures made in [3].

We consider the classical Walsh system on the interval $[0,1]$ in the following classical order called the Paley order. If an integer $n = 0,1,2,\ldots$ can be written as $n = \sum_{k=0}^{\infty} p_k 2^k$ with $p_k = 0,1$ (obviously this sum is finite) then we take the nth Walsh function $w_n(t)$ to be $\prod_{k=0}^{\infty} r_k(t)^{p_k}$, where $r_0, r_1, \ldots$ are the Rademacher functions. The Walsh system in this order is called the *Walsh–Paley system* (cf. [4]).

We are interested in the comparison between norm convergence properties of Walsh series and trigonometric series. More precisely we are interested in best constants $K(p,N)$ and $C(p,N)$ in the following inequalities:

$$(1)\qquad \Big\| \sum_{n=0}^{2^N-1} a_n e^{in\alpha} \Big\|_p \le C(p,N) \Big\| \sum_{n=0}^{2^N-1} a_n w_n \Big\|_p$$

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and

$$\Big\|\sum_{n=0}^{2^N-1} a_n w_n\Big\|_p \le K(p,N)\Big\|\sum_{n=0}^{2^N-1} a_n e^{in\alpha}\Big\|_p. \tag{2}$$

Recall that for a trigonometric polynomial $f(\alpha)=\sum_{n=0}^{K} a_n e^{in\alpha}$ we mean by $\|f\|_p$ the quantity $\big(\frac{1}{2\pi}\int_0^{2\pi}|f(\alpha)|^p\,d\alpha\big)^{1/p}$, so the trigonometric system $(e^{in\alpha})_{n\in\mathbb{Z}}$ is a complete orthonormal system. The choice of dyadic sums is motivated by the the existence in this case of a simple representation for the kernels involved (cf. (8)). Clearly one can easily pass from estimates (1) and (2) to analogous estimates for sums of arbitrary length. Let us consider two sequences of operators

$$T_N(f)=\sum_{n=0}^{2^N-1}\langle f,w_n\rangle e^{in\alpha}, \tag{3}$$

$$S_N(f)=\sum_{n=0}^{2^N-1}\langle f,e^{in\alpha}\rangle w_n. \tag{4}$$

Then formally $T_N^*=S_N$. Let $\mathbf{W}_N=\operatorname{span}\{w_n : n=0,1,\ldots,2^N-1\}$ and $\mathbf{T}_N=\operatorname{span}\{e^{in\alpha} : n=0,1,\ldots,2^N-1\}$. The above spaces equipped with the L_p norm will be denoted by $\mathbf{W}_N^p$ and $\mathbf{T}_N^p$ respectively. Then clearly $C(p,N)=\|T_N:\mathbf{W}_N^p\to\mathbf{T}_N^p\|$ and $K(p,N)=\|S_N:\mathbf{T}_N^p\to\mathbf{W}_N^p\|$. Since $\mathbf{W}_N^p$ is norm one complemented in L_p, $1\le p\le\infty$, and $\mathbf{T}_N^p$ is complemented in L_p by the Riesz projection whose norms are for $1<p<\infty$ bounded uniformly in N and in L_1 and L_∞ are bounded by N (cf. [6], I, pp. 67 and 266), we infer that

$$C(p,N)=\|T_N:L_p\to L_p\| \tag{5}$$

and

$$\frac{\|S_N:L_p\to L_p\|}{\beta(N,p)}\le K(p,N)\le\|S_N:L_p\to L_p\| \tag{6}$$

where $\beta(N,p)=C_p$ if $1<p<\infty$ and $\beta(N,1)=\beta(N,\infty)=cN$. Also by duality we have

$$\|S_N:L_p\to L_p\|=\|T_N:L_{p'}\to L_{p'}\|=C(p',N)$$

where $1/p+1/p'=1$.

Clearly the operator T_N is an integral operator given by the kernel

$$T_N(f)(\alpha)=\int_0^1\Big(\sum_{n=0}^{2^N-1} w_n(t)e^{in\alpha}\Big)f(t)\,dt.$$

We set

$$F_N(t,\alpha)=\sum_{n=0}^{2^N-1} w_n(t)e^{in\alpha}. \tag{7}$$

The following representation of $F_N(t,\alpha)$ will be of fundamental importance in our considerations:

$$\begin{aligned}F_N(t,\alpha)&=\sum_{n=0}^{2^{N-1}-1} w_n(t)e^{in\alpha}+r_{N-1}(t)e^{i2^{N-1}\alpha}\sum_{n=0}^{2^{N-1}-1} w_n(t)e^{in\alpha}\\&=(1+r_{N-1}(t)e^{i2^{N-1}\alpha})F_{N-1}(t,\alpha),\end{aligned}$$

so by induction we get

$$F_N(t,\alpha)=\prod_{k=0}^{N-1}(1+r_k(t)e^{i2^k\alpha}). \tag{8}$$

The following lemma will be used several times in this paper:

LEMMA 1. *For any A_j and B_j we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\prod_{j=1}^{N}(A_j+B_j\cos 2^j\alpha)\,d\alpha=\prod_{j=1}^{N}A_j.$$

This lemma is well known. It follows immediately if we write $\cos 2^j\alpha=\frac12(e^{i2^j\alpha}+e^{-i2^j\alpha})$ and expand the product. We see that there is no cancellation in the expansion so it is a trigonometric polynomial with constant term equal to $\prod_{j=1}^{N}A_j$. This gives the assertion.

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2. Non-equivalence. All our results will follow from estimates for various mixed norms of the function $F_N(t,\alpha)$. Let us start with some propositions which will be used in the proofs of Theorems 1 and 2.

PROPOSITION 2. *There exists a number $b<2$ such that*

$$(1+\sqrt{3}/2)^N\le\sup_\alpha\int_0^1|F_N(t,\alpha)|\,dt\le 2\cdot b^N \tag{9}$$

for $N=1,2,\ldots$

Proof. Using independence of Rademacher functions we get

$$\sup_\alpha \int_0^1 |F_N(t,\alpha)|\,dt = \sup_\alpha \prod_{k=0}^{N-1} \int_0^1 |1 + r_n(t)e^{i2^k\alpha}|\,dt = \sup_\alpha \prod_{k=0}^{N-1} \frac{|1+e^{i2^k\alpha}| + |1-e^{i2^k\alpha}|}{2}.$$

Since

$$(|1+e^{i\alpha}| + |1-e^{i\alpha}|)^2 = 4 + 2|1-e^{2i\alpha}| = 4(1+|\sin\alpha|)$$

we infer that

$$\sup_\alpha \int_0^1 |F_N(t,\alpha)|\,dt = \sqrt{\sup_\alpha \prod_{k=0}^{N-1} (1+|\sin 2^k\alpha|)}. \tag{10}$$

Let us start with the upper estimate. Fix ξ, $0<\xi<\pi/3$. If for some k we have

$$(n+1/2)\pi - \xi \le 2^k t \le (n+1/2)\pi + \xi \tag{11}$$

then for $l = k+1$ we have

$$(2n+1)\pi - 2\xi \le 2^l t \le (2n+1)\pi + 2\xi. \tag{12}$$

Now let s denote the number of k's in the product in (10) which satisfy (11). Clearly there are at least $s-1$ numbers l, $0 \le l \le N-1$, for which (12) holds. In particular $s \le 1+N/2$. The factors where (11) holds are estimated by 2 and factors where (12) holds are at most $1+\sin 2\xi$ each. All the other do not satisfy (11) so are at most $1+|\sin(\pi/2+\xi)|$. From this we infer that

$$\sup_\alpha \prod_{k=0}^{N-1} (1+|\sin 2^k\alpha|) \le 2^s[(1+\sin 2\xi)]^{s-1}(1+|\sin(\pi/2+\xi)|)^{N-2s+1}.$$

Set $A(\xi) = [2(1+\sin 2\xi)]^{1/2}$, $B(\xi) = 1+|\sin(\pi/2+\xi)|$ and $C(\xi) = \max[A(\xi), B(\xi)]$. With this notation we have

$$\sup_\alpha \prod_{k=0}^{N-1} (1+|\sin 2^k\alpha|) \le 2C(\xi)^N \tag{13}$$

for each $0<\xi<\pi/3$. Looking at the graphs of $A(\xi)$ and $B(\xi)$ we easily see that there exists ξ_0 such that $A(\xi_0) = B(\xi_0)$ and then $C(\xi_0) < 2$. This gives $b<2$.

Now let us work out the lower estimate. We take $\gamma = (\sum_{k=1}^\infty 4^{-k})/2 = 1/6$. For even $k=2s$ we have $2^k\gamma = 4^s\gamma = \text{integer} + 1/2 + \gamma$. For odd $k = 2s+1$ we have $2^k\gamma = \text{integer} + 2\gamma$. Since $1/2+\gamma = 2/3$ and $|\sin 2\pi/3| = |\sin\pi/3|$ we get

$$\prod_{k=0}^{N-1} (1+|\sin 2^k\pi\gamma|) = (1+\sin\pi/3)^N = (1+\sqrt{3}/2)^N. \ \blacksquare \tag{14}$$

REMARK. Estimating numerically we get $\xi_0 \sim 0.4609285$, which yields $b \sim 1.89564$. Since $1+(\sqrt{3}/2) \sim 1.866025$ we see that the arguments given above are quite precise.

PROPOSITION 3.

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} F_N(t,\alpha)^4\,d\alpha\,dt = 6^N.$$

Proof. First note that from (8) we get

$$\begin{aligned} F_N(t,\alpha)^4 &= \prod_{k=0}^{N-1} [(1+r_k(t)e^{i2^k\alpha})(1+r_k(t)e^{-i2^k\alpha})]^2 \\ &= \prod_{k=0}^{N-1} (2+2r_k(t)\cos 2^k\alpha)^2 \\ &= \prod_{k=0}^{N-1} (4+8r_k(t)\cos 2^k\alpha + 4\cos^2 2^k\alpha) \\ &= \prod_{k=0}^{N-1} (6+2\cos 2^{k+1}\alpha + 8r_k(t)\cos 2^k\alpha). \end{aligned}$$

Integrating this over t and using the independence of Rademacher functions we get

$$\int_0^1 F_N(t,\alpha)^4\,dt = \prod_{k=1}^{N} (6+2\cos 2^k\alpha). \tag{15}$$

Lemma 1 completes the proof. ■

PROPOSITION 4. *For every* $R > \frac{1}{4}(2+\sqrt{2}+\sqrt{6})$ *there exists* C_R *such that*

$$\sup_t \frac{1}{2\pi}\int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha \le C_R R^N \quad \textit{for } N=1,2,\ldots$$

Proof. Using (8) we have

$$|F_N(t,\alpha)| = \prod_{k=0}^{N-1} |1+r_k(t)e^{i2^k\alpha}| = \prod_{k=0}^{N-1} \sqrt{2+2r_k(t)\cos 2^k\alpha} = 2^{N/2}\prod_{k=0}^{N-1}\sqrt{1+r_k(t)\cos 2^k\alpha}. \tag{16}$$

LEMMA 5. *For* $|x| \le 1$ *we have*

$$\sqrt{1+x} \le 1 + \tfrac{1}{2}x - \gamma x^2 \quad \textit{for } \gamma = 1.5 - \sqrt{2}.$$

Proof. We consider the function $f(x) = \sqrt{1+x} - 1 - \frac{1}{2}x + \gamma x^2$. Differentiating we see that f has a local minimum at $x = 0$ and $f''(x) \ge 0$ on $[-1, \xi]$ and $f''(x) \le 0$ on $[\xi, 1]$ for a certain $\xi > 0$. This implies that in order to prove the inequality it suffices to check that $f(-1) \ge 0$, $f(1) \ge 0$ and $f(0) \ge 0$, which one verifies easily. ■

Using Lemma 5 and (16) we infer that

$$\sup_t \frac{1}{2\pi} \int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha \tag{17}$$

$$\le 2^{N/2} \sup_t \frac{1}{2\pi} \int_0^{2\pi} \prod_{k=0}^{N-1} \Big(1 + \frac{1}{2} r_k(t) \cos 2^k\alpha - \gamma \cos^2 2^k\alpha\Big)\,d\alpha$$

$$= 2^{N/2} \sup_t \frac{1}{2\pi} \int_0^{2\pi} \prod_{k=0}^{N-1} \Big[\Big(1 - \frac{\gamma}{2}\Big) + \frac{1}{2} r_k(t) \cos 2^k\alpha - \frac{\gamma}{2} \cos 2^{k+1}\alpha\Big]\,d\alpha$$

$$= 2^{N/2} \sup_t \frac{1}{2\pi} \int_0^{2\pi} \prod_{k=0}^{N-1} \Big[\Big(1 - \frac{\gamma}{2}\Big) + \frac{1}{4} r_k(t)(e^{i2^k\alpha} + e^{-i2^k\alpha}) - \frac{\gamma}{4}(e^{i2^{k+1}\alpha} + e^{-i2^{k+1}\alpha})\Big]\,d\alpha.$$

Now let us consider inductively the products

$$\varphi_N = \prod_{k=0}^{N-1} \Big[\Big(1 - \frac{\gamma}{2}\Big) + \frac{1}{4} r_k(t)(e^{i2^k\alpha} + e^{-i2^k\alpha}) - \frac{\gamma}{4}(e^{i2^{k+1}\alpha} + e^{-i2^{k+1}\alpha})\Big].$$

Clearly each φ_N is a trigonometric polynomial of the form

$$\varphi_N(\alpha) = \sum_{|s| < 2^{N+1}} a_s e^{is\alpha}. \tag{18}$$

Let us single out the coefficients corresponding to $s = 0$ and $s = \pm 2^N$ and write

$$\varphi_N(\alpha) = P_N + Q_N e^{i2^N\alpha} + Q_N e^{-i2^N\alpha} + \sum_{s\ne 0,\ s \ne \pm 2^N} a_s e^{is\alpha}. \tag{19}$$

Since

$$\varphi_{N+1}(\alpha) = \varphi_N(\alpha)\Big[\Big(1 - \frac{\gamma}{2}\Big) + \frac{1}{4} r_N(t)(e^{i2^N\alpha} + e^{-i2^N\alpha}) - \frac{\gamma}{4}(e^{i2^{N+1}\alpha} + e^{-i2^{N+1}\alpha})\Big]$$

from (18) and (19) we infer that

$$P_{N+1} = P_N\Big(1 - \frac{\gamma}{2}\Big) + 2 r_N(t) \frac{1}{4} Q_N \tag{20}$$

and

$$Q_{N+1} = \frac{1}{4} r_N(t) Q_N + \frac{\gamma}{4} P_N. \tag{21}$$

To estimate P_N from above we define inductively two sequences p_n and q_n by the conditions:

$$p_0 = 1 - \frac{\gamma}{2}, \quad q_0 = \frac{1}{4}, \tag{22}$$

$$p_{n+1} = p_n\Big(1 - \frac{\gamma}{2}\Big) + \frac{1}{2} q_n, \tag{23}$$

$$q_{n+1} = \frac{1}{4} q_n + \frac{\gamma}{4} p_n. \tag{24}$$

We easily see that for each $t \in [0,1]$ we have $|P_N| \le p_N$ and also $|Q_N| \le q_N$.

LEMMA 6. *Let* p_n *and* q_n *be defined as in* (22)–(24). *Then for* $n = 0, 1, 2, \ldots$,

$$p_n \le C_a a^n \quad \textit{for } a > 1 - \frac{\gamma}{2} + \frac{1}{4}\big(\sqrt{5 - 2\sqrt{2}} - \sqrt{2}\big). \tag{25}$$

Since $\frac{1}{2\pi}\int_0^{2\pi} \varphi_N(\alpha)\,d\alpha = P_N$ (see Lemma 1), from (17) and Lemma 6 we infer that

$$\sup_t \frac{1}{2\pi} \int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha \le C 2^{N/2} a^N. \ ■$$

Proof of Lemma 6. Define $\Gamma_n = q_n/p_n$. Then from (22)–(24) we have $\Gamma_0 = [4(1 - \gamma/2)]^{-1}$ and $\Gamma_{n+1} = f(\Gamma_n)$ where

$$f(x) = \frac{x + \gamma}{4 - 2\gamma + 2x}.$$

One easily checks that for $x \ge 0$ we have $0 < f'(x) < 1$. This implies that Γ_n converges to the fixed point of f, i.e. the solution of the equation $f(x) = x$; call it g. A standard computation yields

$$g = \frac{\sqrt{5 - 2\sqrt{2}} - \sqrt{2}}{2}.$$

So for any $a > g$ we have $q_n \le a p_n$ for large n's. So from (23) we infer that for large n's we have $p_{n+1} \le [(1 - \gamma/2) + a/2] p_n$, which gives the assertion. ■

PROPOSITION 7. *For* $1 \le p \le 2$ *we have* $C(p, N) \ge (2^{-p/2} + 1/2)^{N/p}$.

Proof. For each $t \in [0,1]$ we have

$$\Big\| \sum_{n=0}^{2^N-1} w_n(t) e^{in\cdot} \Big\|_p^p \le C(p,N)^p \Big\| \sum_{n=0}^{2^N-1} w_n(t) w_n \Big\|_p^p. \tag{26}$$

It is easy and well known (see e.g. [4], p. 7) that $\|\sum_{n=0}^{2^N-1} w_n(t)w_n\|_p^p$ does not depend on t and equals $2^{-N}\cdot 2^{Np}$. Note that

$$\begin{aligned}\int_0^1 \Big\| \sum_{n=0}^{2^N-1} w_n(t)e^{in\cdot}\Big\|_p^p dt &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |F_N(t,\alpha)|^p\,dt\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\frac{|1+e^{i2^k\alpha}|^p+|1-e^{i2^k\alpha}|^p}{2}\,d\alpha \\ &= 2^{-N}2^{Np/2}\frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}(|1+\cos 2^k\alpha|^{p/2}+|1-\cos 2^k\alpha|^{p/2})\,d\alpha.\end{aligned}\tag{27}$$

To estimate it further we will need the following lemma:

LEMMA 8. *For $0 \le s \le 1$ and $|x| \le 1$ the following inequality holds*:

$$|1+x|^s + |1-x|^s \ge 2-(2-2^s)x^2. \tag{28}$$

We postpone the proof of this lemma for a while.

Using the lemma we can continue (27):

$$\begin{aligned}&\ge 2^{N(p/2-1)}\frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}[2-(2-2^{p/2})\cos^2 2^k\alpha]\,d\alpha \\ &= 2^{N(p/2-1)}\frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\Big[2-\frac12(2-2^{p/2})-\frac12(2-2^{p/2})\cos 2^{k+1}\alpha\Big]\,d\alpha.\end{aligned}$$

From Lemma 1 we see that this integral equals $[2-\frac12(2-2^{p/2})]^N$. This gives

$$\begin{aligned}\int_0^1 \Big\| \sum_{n=0}^{2^N-1} w_n(t)e^{in\cdot}\Big\|_p^p dt &\ge 2^{N(p/2-1)}\Big[2-\frac12(2-2^{p/2})\Big]^N = \Big[\frac12\cdot 2^{p/2}+\frac14\cdot 2^p\Big]^N.\end{aligned}\tag{29}$$

Thus we get

$$C(p,N)^p \ge 2^N 2^{-Np}\Big[\frac12\cdot 2^{p/2}+\frac14\cdot 2^p\Big]^N = \Big(\frac12+2^{-p/2}\Big)^N$$

which gives the assertion of Proposition 7. ∎

Proof of Lemma 8. Consider the function

$$f(x) = |1+x|^s+|1-x|^s-2+(2-2^s)x^2.$$

Clearly f is even. Differentiating twice we see that $f(x)$ is convex on a certain interval $(-\mu,\mu)$ with $\mu > 0$ and concave for $|x| \ge \mu$. From this we infer that for (28) to hold it suffices to have $f(0) \ge 0$ and $f(1) \ge 0$, which is easily seen to be true. ∎

Before we start with our main theorem let us recall the following lemma.

LEMMA 9. *Let $\psi(\alpha) = \sum_{k=0}^{2^N-1} a_k e^{ik\alpha}$. Then for $1 \le p \le \infty$ we have*

$$\|\psi\|_\infty \le 2\cdot 2^{N/p}\|\psi\|_p.$$

This lemma was proved by Jackson in [2] and is a special case of classical Nikol'skiĭ estimates (see [1], Theorem 2.6).

THEOREM 1. *The quantities $C(p,N)$ satisfy the following estimates*:

(i) *For $1 \le p \le 2$ we have*

$$(2^{-p/2}+1/2)^{N/p} \le C(p,N) \le 2^{2/p-1}b^{(2/p-1)N} \tag{30}$$

where $b < 2$ is the constant from Proposition 2.

(ii) *For $2 \le p \le \infty$ we have*

$$c2^{\xi(p)N} \le C(p,N) \le C_R R^{(1-2/p)N} \tag{31}$$

where R is any number $> \frac14(2+\sqrt2+\sqrt6)$ and $\xi(p) > 1$ for all $p > 2$. The function $\xi(p)$ satisfies

$$\xi(p) \ge \begin{cases} \log_2\Big(\dfrac{2}{\sqrt3}\Big(\dfrac34\Big)^{1/p}\Big) & \text{for } 2 < p \le 4,\\ 0.44-1/p & \text{for } 4 \le p < \infty.\end{cases} \tag{32}$$

Proof. The left hand inequality in (30) is Proposition 7. Clearly $C(2,N) = \|T_N : L_2 \to L_2\| = 1$. From (5) we infer that

$$C(1,N) = \|S_N : L_\infty \to L_\infty\| = \sup_\alpha \int_0^1 |F_N(t,\alpha)|\,dt$$

so from Proposition 2 we infer that $C(1,N) \le b^N\cdot 2^N$. Using the Riesz–Thorin interpolation theorem (cf. [6], Vol. II, p. 93) we get

$$C(p,N) \le 2^{2/p-1}\cdot b^{(2/p-1)N}.$$

The upper estimate in (31) follows directly from the Riesz–Thorin theorem and Proposition 4. To prove the lower estimate in (30) we use Lemma 9 to find that for any bounded f we have

$$\begin{aligned}\|T_N(f)\|_\infty &\le 2\cdot 2^{N/p}\|T_N(f)\|_p \le 2\cdot 2^{N/p}\|T_N : L_p\to L_p\|\cdot\|f\|_p\\ &\le 2\cdot 2^{N/p}\|T_N : L_p\to L_p\|\cdot\|f\|_\infty\end{aligned}$$

so

$$\|T_N : L_p\to L_p\| \ge \frac{1}{2}\cdot 2^{-N/p}\|T_N : L_\infty\to L_\infty\|.$$

So from Proposition 2 we get

$$C(p,N)\ge \frac12\cdot 2^{-N/p}2^{aN} = \frac12\cdot 2^{N(a-1/p)}. \tag{33}$$

This is a sensible estimate for $p > 1/\alpha$ but not for all $p > 2$. From Remark after the proof of Proposition 2 we see that $1/a < 4$ so we will use (33) for $p \ge 4$. For $2 < p \le 4$ we take $a_n = e^{ins}$ in (1) to get

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{2^N-1} e^{ins}e^{in\alpha}\Big|^p\,d\alpha \le C(p,N)^p\int_0^1\Big|\sum_{n=0}^{2^N-1} e^{ins}w_n(t)\Big|^p\,dt. \tag{34}$$

Classical estimates for the Dirichlet kernel (cf. [6], Vol. I, p. 67) give

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{2^N-1} e^{in(s+\alpha)}\Big|^p\,d\alpha \ge c2^{N(p-1)}.$$

Using this, integrating (34) over s and applying Hölder's inequality we get

$$\begin{aligned}c2^{N(p-1)} &\le C(p,N)^p\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |F_N(t,\alpha)|^p\,dt\,d\alpha\\ &\le C(p,N)^p\Big(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1|F_N(t,\alpha)|^4\,dt\,d\alpha\Big)^{p/2-1}\\ &\quad\times\Big(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1|F_N(t,\alpha)|^2\,dt\,d\alpha\Big)^{2-p/2}.\end{aligned}$$

The second integral above clearly equals 2^N so from Proposition 3 we get

$$C(p,N)^p \ge c2^{N(p-1)}6^{N(1-p/2)}2^{N(p/2-2)},$$

which yields

$$C(p,N)\ge c\bigg(\frac{2}{\sqrt3}\Big(\frac34\Big)^{1/p}\bigg)^N. \ \blacksquare$$

REMARK. Note that one can extract from our arguments a very simple proof of the main result of [5] that the trigonometric system and the Walsh system are equivalent in L_p only when $p = 2$. By duality it suffices to consider only $p > 2$. Then Proposition 3 and the argument starting from (34) give the non-equivalence for $2 < p \le 4$. The Riesz–Thorin interpolation theorem shows that this implies the non-equivalence also for $p > 4$.

3. Riemann norms. In this section we want to express the "power type" difference between Walsh and trigonometric systems in the language of Riemann ideal norms (cf. [3]).

For two orthonormal systems $\mathcal{A}_n = \{\varphi_i\}_{i=1}^n$ and $\mathcal{B}_n = \{\psi_i\}_{i=1}^n$ and a Banach space X we define $\varrho(X\,|\,\mathcal{A}_n,\mathcal{B}_n)$ to be the least constant c such that the inequality

$$\Big(\int\Big\|\sum_{i=1}^n x_i\varphi_i(t)\Big\|_X^2\,dt\Big)^{1/2} \le c\Big(\int\Big\|\sum_{i=1}^n x_i\psi_i(t)\Big\|_X^2\,dt\Big)^{1/2} \tag{35}$$

holds for all sequences $(x_i)_{i=1}^n\subset X$. It is clear (take $n = 1$ and $x_1\ne 0$) that $c\ge 1$. One can also easily check that such a smallest constant actually exists (if all c's greater than a work in (35) then a also works).

This concept, its variants and ramifications are discussed in [3], 3.3. Let us recall that for an operator $T : X\to Y$ we define its *Riemann ideal norm* (with respect to the orthogonal systems $\mathcal{A}_n$ and $\mathcal{B}_n$) as the least constant c such that the inequality

$$\Big(\int\Big\|\sum_{i=1}^n T(x_i)\varphi_i(t)\Big\|_Y^2\,dt\Big)^{1/2} \le c\Big(\int\Big\|\sum_{i=1}^n x_i\psi_i(t)\Big\|_X^2\,dt\Big)^{1/2}$$

holds for all sequences $(x_i)_{i=1}^n\subset X$. Thus $\varrho(X\,|\,\mathcal{A}_n,\mathcal{B}_n)$ is the Riemann ideal norm of the identity operator on the space X. We will be interested only in $n = 2^N$ and $\mathcal{A}_n$ and $\mathcal{B}_n$ being either the initial segment of the Walsh system or the initial segment of the trigonometric system. We want to contribute to the problem of estimating such quantities from below for $X = L_1$ and $X = L_\infty$. This question is discussed in Section 6.5.4 of [3].

Let us introduce the following notation: $\mathcal{W}_N = \{w_n\}_{n=0}^{2^N-1}$ and $\mathcal{T}_N = \{e^{in\alpha}\}_{n=0}^{2^N-1}$.

THEOREM 2. *The following inequalities hold*:

$$\varrho(L_\infty\,|\,\mathcal{W}_N,\mathcal{T}_N)\ge C\bigg(\sqrt{\frac{6\sqrt3}{\sqrt{86}}}\bigg)^N \sim C(1.058599)^N, \tag{36}$$

$$\varrho(L_\infty\,|\,\mathcal{T}_N,\mathcal{W}_N)\ge C\bigg(\frac{2}{\sqrt{2+\sqrt2}}\bigg)^N \sim C(1.082392)^N, \tag{37}$$

$$\varrho(L_1\,|\,\mathcal{W}_N,\mathcal{T}_N)\ge \frac{c}{N}\big(\sqrt{1+1/\sqrt2}\big)^N \sim \frac{c}{N}(1.306563)^N, \tag{38}$$

$$\varrho(L_1\,|\,\mathcal{T}_N,\mathcal{W}_N)\ge \big(\sqrt{1+1/\sqrt2}\big)^N \sim C(1.306563)^N. \tag{39}$$

Proof. First we reduce the estimates (36)–(39) to estimates of some mixed norms of $F_N(t,\alpha)$. Consider (36) and take $x_j = e^{ij\alpha}$. Then from (35) we infer that

$$\varrho(L_\infty \mid \mathcal{W}_N, \mathcal{T}_N) \ge \left(\frac{1}{2\pi}\int_0^{2\pi}\Big(\sup_\alpha\Big|\sum_{j=0}^{2^N-1} e^{ij\alpha}e^{ij\beta}\Big|\Big)^2 d\beta\right)^{1/2} \times \left(\int_0^1 \sup_\alpha\Big|\sum_{j=0}^{2^N-1} e^{ij\alpha}w_n(t)\Big|^2 dt\right)^{-1/2}.$$

Since the sup in the first integral does not depend on β and equals 2^N we get

$$\varrho(L_\infty \mid \mathcal{W}_N, \mathcal{T}_N) \ge 2^N\Big(\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt\Big)^{-1/2}. \tag{40}$$

Analogously, taking $x_j = w_j$ we get

$$\varrho(L_\infty \mid \mathcal{T}_N, \mathcal{W}_N) \ge 2^N\Big(\frac{1}{2\pi}\int_0^1 \sup_t |F_N(t,\alpha)|^2\,d\alpha\Big)^{-1/2}. \tag{41}$$

To prove (38) we take $x_j = e^{ij\cdot}$ and obtain from (35)

$$\varrho(L_1 \mid \mathcal{W}_N, \mathcal{T}_N) \ge \Big(\int_0^1 \Big\|\sum_{j=0}^{2^N-1} e^{ij\cdot}w_j(t)\Big\|_1^2 dt\Big)^{1/2} \times \Big(\frac{1}{2\pi}\int_0^{2\pi}\Big\|\sum_{j=0}^{2^N-1} e^{ij\cdot}e^{ij\alpha}\Big\|_1^2 d\alpha\Big)^{-1/2}.$$

Since $\sum_{j=0}^{2^N-1} e^{ij\cdot}e^{ij\alpha}$ is a translate of the Dirichlet kernel we see that its L_1 norm does not depend on α and is $\ge CN$ by the classical estimates of the Dirichlet kernel (see [6], Vol. I, p. 67), so we get

$$\varrho(L_1 \mid \mathcal{W}_N, \mathcal{T}_N) \ge \frac{c}{N}\Big(\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|F_N(t,\alpha)|\,d\alpha\Big)^2 dt\Big)^{1/2}. \tag{42}$$

Analogously, to prove (39) we take $x_j = w_j$ and obtain from (35)

$$\varrho(L_1 \mid \mathcal{T}_N, \mathcal{W}_N) \ge \Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^1 |F_N(t,\alpha)|\,dt\Big)^2 d\alpha\Big)^{1/2} \times \Big(\int_0^1\Big(\int_0^1\Big|\sum_{n=0}^{2^N-1} w_n(s)w_n(t)\Big|\,ds\Big)^2 dt\Big)^{-1/2}.$$

Since $\sum_{n=0}^{2^N-1} w_n(s)w_n(t)$ is a (dyadic) translation of the Dirichlet kernel of the Walsh system it is well known and easy (see [4], Paley's Lemma, p. 7) that its L_1 norm equals 1 so we get

$$\varrho(L_1 \mid \mathcal{T}_N, \mathcal{W}_N) \ge \Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^1 |F_N(t,\alpha)|\,dt\Big)^2 d\alpha\Big)^{1/2}. \tag{43}$$

From estimates (40)–(43) we see that Theorem 2 immediately follows from Proposition 10 below.

PROPOSITION 10. *The following inequalities hold:*

$$\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt \le 2^N\cdot\Big(\frac{1}{3}\sqrt{\frac{86}{3}}\Big)^N, \tag{44}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sup_t |F_N(t,\alpha)|^2\,d\alpha \le 2^N\Big(1+\frac{1}{\sqrt{2}}\Big)^N, \tag{45}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^1 |F_N(t,\alpha)|\,dt\Big)^2 d\alpha \ge \Big(\frac{1}{2}+\frac{1}{\sqrt{2}}\Big)^N, \tag{46}$$

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|F_N(t,\alpha)|\,d\alpha\Big)^2 dt \ge \Big(\frac{1}{2}+\frac{1}{\sqrt{2}}\Big)^N. \tag{47}$$

Proof. Let us start with the proof of (44). First observe that

$$\begin{aligned}|F_N(t,\alpha)|^2 &= \prod_{n=0}^{N-1}|1+r_n(t)e^{i2^n\alpha}|^2\\ &= \prod_{n=0}^{N-1}(2+2r_n(t)\cos 2^{n+1}\alpha)\\ &= 2^N\prod_{n=1}^{N}(1+r_{n+1}(t)\cos 2^n\alpha).\end{aligned} \tag{48}$$

Let $\varphi_s(t,\alpha) = (1+r_{s+1}(t)\cos 2^s\alpha)(1+r_{s+2}(t)\cos 2^{s+1}\alpha)$. With this notation we have

$$\sup_\alpha |F_N(t,\alpha)|^2 \le 2^N\prod_{s=0}^{[N/2]+1}\sup_\alpha \varphi_{2s+1}(t,\alpha). \tag{49}$$

Clearly the functions $\sup_\alpha \varphi_{2s+1}(t,\alpha)$ are stochastically independent as functions of t for $s=0,1,\ldots,[N/2]$ and have the same distribution in t as the function $\psi(t) = \sup_\alpha(1+r_1(t)\cos\alpha)(1+r_2(t)\cos 2\alpha)$. Thus we have to find maxima in α of the four functions corresponding to all possible choices of $r_i(t)=\pm1$. After some tedious calculations we conclude that two of them have maximum 4 and two have maximum $B = 2\frac{10}{27}$. Together with (49) this gives

$$(50)\qquad \int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt \le 2^N \prod_{s=0}^{[N/2]+1} \int_0^1 \sup_\alpha \varphi_{2s+1}(t,\alpha)\,dt = 2^N\Big(2+\frac{1}{2}B\Big)^{N/2} = 2^N\Big(\frac{1}{3}\sqrt{\frac{86}{3}}\Big)^N.$$

Now let us prove (45). From (48) we get

$$(51)\qquad \sup_t |F_N(t,\alpha)|^2 = 2^N \prod_{n=1}^N (1+|\cos 2^n\alpha|).$$

Expanding the product we get

$$(52)\qquad \frac{1}{2\pi}\int_0^{2\pi} \prod_{n=1}^N (1+|\cos 2^n\alpha|)\,d\alpha = 1 + \sum_{1\le k_1<\ldots<k_s\le N} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^s |\cos 2^{k_j}\alpha|\,d\alpha.$$

Each integral in (52) is estimated using the Cauchy inequality and Lemma 1 as follows:

$$(53)\qquad \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^s |\cos 2^{k_j}\alpha|\,d\alpha \le \Big(\frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^s \cos^2 2^{k_j}\alpha\,d\alpha\Big)^{1/2} = \Big(\frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^s \frac{1}{2}(1+\cos 2\cdot 2^{k_j}\alpha)\,d\alpha\Big)^{1/2} = 2^{-s/2}.$$

Thus substituting (53) into (52) we get

$$\frac{1}{2\pi}\int_0^{2\pi} \prod_{n=1}^N (1+|\cos 2^k\alpha|)\,d\alpha \le 1 + \sum_{1\le k_1<\ldots<k_s\le N} 2^{-s/2} = \Big(1+\frac{1}{\sqrt{2}}\Big)^N$$

so from (51) we get (45).

Both (46) and (47) are estimated from below by

$$\Big(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |F_N(t,\alpha)|\,dt\,d\alpha\Big)^2$$

so (29) gives the claim. ■

REMARK. If we split the product in (49) into triples, and not in pairs like we did, we will have to analyze eight functions

$$(1\pm\cos\alpha)(1\pm\cos 2\alpha)(1\pm\cos 4\alpha).$$

It can be checked (I used a computer) that two of them have maximum 8, two have maximum ≤ 4.33, two ≤ 4.186 and two ≤ 3.78. This leads to the estimate

$$\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt \le 2^N(5.074)^{N/3} \sim 2^N(1.7718371)^N,$$

which is slightly better than (50) because numerically (50) gives $\le 2^N(1.784709)^N$. Obviously also this estimate is not optimal.

3.1. *Concluding remarks.* 1. Our estimates are quite sloppy and do not give the asymptotically correct values. It would be interesting to have precise estimates.

2. The Paley order of the Walsh functions was crucial in our considerations. This order may be considered natural, but there are other ones, just as natural. We conjecture, however, that in reality our results are essentially independent of order. The following question seems to be natural and very interesting: *does there exist a permutation σ of the natural numbers such that $\{e^{in\alpha}\}_{n=0}^\infty$ is equivalent in L_p for some $p\neq 2$ to $\{w_{\sigma(n)}\}_{n=0}^\infty$?*

3. Both systems we consider in this paper are sets of characters of an abelian compact group. It is an interesting question whether character sets of non-isomorphic topological groups can be equivalent systems in L_p. In particular one may investigate the equivalence of Vilenkin systems (cf. [4]) with the trigonometric system or with the Walsh system or between themselves.

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Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: wojtaszczyk@mimuw.edu.pl

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland

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