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Smooth operators for the regular representation on homogeneous spaces

by

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Abstract. A necessary and sufficient condition for a bounded operator on $L^2(M)$, M a Riemannian compact homogeneous space, to be smooth under conjugation by the regular representation is given. It is shown that, if all formal "Fourier multipliers with variable coefficients" are bounded, then they are also smooth. In particular, they are smooth if M is a rank-one symmetric space.

1. Introduction. Consider the two unitary representations of $\mathbb{R}$ on $L^2(\mathbb{R})$: $(T_z u)(x) = u(x-z)$ and $(E_\zeta)u(x) = e^{ix\zeta}u(x)$. A bounded operator A on $L^2(\mathbb{R})$ is such that both mappings $z \mapsto T_z A T_z^{-1}$ and $\zeta \mapsto E_\zeta A E_\zeta^{-1}$ are smooth in the norm topology if and only if it is a pseudodifferential operator with symbol having bounded derivatives of all orders in $\mathbb{R}^2$. This remarkable result was proven by Cordes ([3], Theorem 1.2; see also [4], Theorem VIII.2.1) and was closely related to a previous abstract characterization (involving boundedness of commutators) of pseudodifferential operators due to Beals [2]. Other descriptions of pseudodifferential operators as bounded operators which give rise to smooth mappings when conjugated by Lie-group unitary representations have been called *Beals–Cordes-type characterizations* [15]. A class of operators characterized by such a smoothness condition naturally becomes ([4], Theorem VIII.6.6) a Ψ^*-algebra, in the sense of Gramsch [8]. As observed further by Payne [13], it also becomes a smooth tame Fréchet algebra.

A Beals–Cordes-type characterization for operators on the circle $\mathbb{S}^1$ was given in [12], Theorem 2: a bounded operator A on $L^2(\mathbb{S}^1)$ defines a smooth function when conjugated by the regular representation of $\mathbb{S}^1$ if and only if it is given by

$$Au(x) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} e^{ijx} a_j(x) \int_{-\pi}^{\pi} e^{-ijy} u(y)\, dy, \tag{1}$$

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for some *bounded* sequence $a_j \in C^\infty(\mathbb{S}^1)$ (bounded meaning that the sequence of derivatives of arbitrary order of a_j is bounded in sup-norm). Operators as in (1) not only are the formal discrete analogue of pseudodifferential operators, they are locally pseudodifferential operators as well [1, 11, 12].

On $L^2(\mathbb{S}^2)$, one can analogously define, given a bounded sequence $a_l \in C^\infty(\mathbb{S}^2)$, a formal "Fourier multiplier with variable coefficients" by

$$Au = \sum_{l=0}^{\infty} a_l P_l u, \tag{2}$$

with P_l denoting the orthogonal projection of $L^2(\mathbb{S}^2)$ onto the lth eigenspace of the Laplace operator on the sphere $\mathbb{S}^2$. Theorem 2 of [5] states that equation (2) indeed defines a bounded operator on $L^2(\mathbb{S}^2)$, which is "$SO(3)$-smooth", that is, such that $g \in SO(3) \mapsto T_g A T_g^{-1}$ is smooth, where T_g denotes the unitary operator $T_g u(x) = u(g^{-1}x)$, $u \in L^2(\mathbb{S}^2)$. In [5], we also defined the *signature* of a bounded operator with respect to a choice of a basis of spherical harmonics, and gave a necessary and sufficient condition for bounded operators on $L^2(\mathbb{S}^2)$ to be $SO(3)$-smooth, involving their signature.

In this paper, the results of [5] are generalized to compact homogeneous or symmetric spaces. More precisely, the situation considered here is that of a Lie group G acting isometrically and transitively on a compact Riemannian manifold M. With $g \in G \mapsto T_g \in \mathcal{L}(L^2(M))$ denoting the regular representation, a necessary and sufficient condition for a bounded operator to be "G-smooth" is given, involving the signature of the operator, defined with respect to a choice of a basis of $\mathcal{L}(L^2(M))$ consisting of eigenfunctions of the Laplace operator on M. It is shown in Theorem 1 that a bounded operator $A \in \mathcal{L}(L^2(M))$ is G-smooth if and only if successive applications of certain differential operators (which are the images of a representation of the Lie algebra of G) to its signature yield the signatures of bounded operators, which turn out to be the directional derivatives of $T_g A T_g^{-1}$. In Section 3, as an application of Theorem 1, it is shown (Lemma 4) that the G-smoothness of the formal "Fourier multipliers with variable coefficients" follows if they are bounded. Their boundedness is proven for rank-one symmetric spaces, using a version of Schur's test for matrix blocks instead of scalars. It would be interesting to prove it with less strong assumptions on the growth of the eigenvalues of the Laplace operator. One available tool to estimate the norm of the corresponding matrix blocks is the integration-by-parts argument of Lemma 5, which is a slight improvement of Lemma 2 of [5].

As remarked in [5], the $SO(3)$-smooth operators do not necessarily have the pseudo-local property. That was already observed for the "Heisenberg-smooth" operators described in the first paragraph of this Introduction (boundedness of the derivatives of the symbol is not enough for a pseudodifferential operator to have Schwartz kernel smooth away from the diagonal). Already in [3], however, Cordes showed that, if smoothness is required for a larger group (containing rotations and dilations), pseudo-locality is achieved. Also for the $SO(3)$ case, it is true that, if the group is enlarged, one obtains pseudo-local operators. That was proven by Taylor [15] in a much broader context. His result implies that a bounded operator $A \in \mathcal{L}(L^2(\mathbb{S}^2))$ is smooth under conjugation by the representation of the conformal group if and only if it is a pseudodifferential operator with symbol in Hörmander's class of type $\varrho = 1$ and $\delta = 0$. More generally, Theorem 2.5 of [15] is a powerful tool to deal with the question: when is G big enough, or how can it be enlarged, so that the G-smooth operators are pseudo-local?

3. A smoothness criterion. Throughout this paper, M denotes an m-dimensional compact Riemannian manifold on which a Lie group G acts smoothly and transitively by isometries, that is, M is a compact homogeneous space which may be equipped with a Riemannian metric invariant for the action of G. That is possible, for example, if G is compact (see Theorem 2.43 of [7] for necessary and sufficient conditions), or if M is a globally symmetric space ([9], §IV.3). In such a case, the surface measure dS induced by the metric is also G-invariant and, consequently, the equation

$$T_g f(x) = f(g^{-1} \cdot x), \quad x \in M,\ g \in G,\ f \in \mathcal{H},$$

defines a unitary operator on the Hilbert space $\mathcal{H} = L^2(M, dS)$. It is easy to show that $g \mapsto T_g$ is a strongly continuous representation. Our goal in this section is to give necessary and sufficient conditions on a bounded operator $A \in \mathcal{L}(\mathcal{H})$ for the mapping

$$g \in G \mapsto_g A = T_g A T_g^{-1} \in \mathcal{L}(\mathcal{H}) \tag{3}$$

to be smooth with respect to the norm topology on $\mathcal{L}(\mathcal{H})$. The operators for which this condition is satisfied are called *G-smooth*.

Let $-\lambda_l$, $l = 0, 1, \ldots$, with $0 \le \lambda_l < \lambda_{l'}$ if $l < l'$, denote the eigenvalues of the Laplace operator Δ on M. For each l, let E_l denote the corresponding eigenspace and let $\beta_l = \{Y_{lk} : k = 1, \ldots, d_l\}$ denote an orthonormal basis of E_l. Given a bounded operator $A \in \mathcal{L}(\mathcal{H})$, define its *signature* (with respect to this choice of bases) by

$$\mathrm{sgn}(A) = (a_l)_{l \in \mathbb{N}} \in \prod_{l=0}^{\infty} L^2(M; \mathbb{C}^{d_l})$$

with $a_l = (AY_{l1}, \ldots, AY_{ld_l})$. It follows from the fact that the Laplace operator on a compact manifold is L^2-diagonalizable that the mapping $A \mapsto \mathrm{sgn}(A)$ is injective. Thus, we may describe a bounded operator by giving its signature.

PROPOSITION 1. *If $A \in \mathcal{L}(\mathcal{H})$ is such that $g \mapsto {}_gA$ is smooth in the strong operator topology, then every component of its signature is smooth: $a_l \in C^\infty(M; \mathbb{C}^{d_l})$, $l \in \mathbb{N}$.*

Proof. Since G acts by isometries on M, every T_g commutes with Δ and hence leaves each E_l invariant. Let $R_l(g)$ denote the matrix of $T_g|_{E_l}$ with respect to the basis β_l. Since $g \mapsto T_g$ is strongly continuous, it follows that $g \mapsto R_l(g)$ is a continuous homomorphism between finite-dimensional Lie groups, and thus is smooth.

For each $g \in G$ let $\operatorname{sgn}({}_gA) = (a_l^g)_{l\in\mathbb{N}}$. It is straightforward to check that

$$a_l(g^{-1} \cdot x) = R_l(g)^t a_l^g(x), \quad x \in M. \tag{4}$$

It follows from the smoothness hypothesis that the mapping $g \in G \mapsto a_l^g \in L^2(M; \mathbb{C}^{d_l})$ is smooth. Thus, the proposition follows from (4), the fact that $R_l(\cdot)$ is smooth and the next lemma below. ■

LEMMA 2. *Let $a \in \mathcal{H}$ be such that $g \mapsto T_g a$ is smooth as a mapping from G into the Hilbert space $\mathcal{H}$. Then $a \in C^\infty(M)$.*

Proof. If k is an integer large enough, then $(1-\Delta)^{-k}$ is an integral operator with continuous kernel. Indeed, it follows from the pseudodifferential calculus that the inverse of the elliptic operator $(1-\Delta)^k$ is a pseudodifferential operator of order $-2k$ ([10], Theorem 18.1.24). It is well known that the kernel of a pseudodifferential operator is continuous if its order added to the dimension of the manifold is negative. So, let $k > m/2$ and let $K \in C(M \times M)$ be the kernel of $(1-\Delta)^{-k}$.

Given a satisfying the hypothesis, let $b = (1-\Delta)^{-k}a$. Since Δ commutes with every T_g, so does $(1-\Delta)^{-k}$. Using the invariance of the measure dS, we then have

$$b(g \cdot x) = \int_M K(x,y)a(g\cdot y)\,dS_y. \tag{5}$$

For each $x \in M$, let $\pi_x(g) = g \cdot x$. Equation (5) tells us that, for each $x \in M$, $b \circ \pi_x$ equals the inner product of a fixed element of $\mathcal{H}$ (namely, $K(x,\cdot)$) and $T_g a$. If a function f on M is such that $f \circ \pi_x$ is smooth for every x, it follows from the fact that π_x is a smooth submersion (since G acts transitively) that f is smooth. This proves that b is smooth. We are finished, since $a = (1-\Delta)^k b$. ■

Still assuming that the hypothesis of Proposition 1 is satisfied, let us compute the directional derivatives of the function ${}_gA = T_g A T_g^{-1}$. Let X be an element of the Lie algebra $\mathfrak{g}$ of the group G, and let Ξ_X denote the left invariant vector field on G induced by X. It follows immediately from the fact that $g \mapsto T_g$ is a group homomorphism that $\Xi_X({}_gA) = {}_gA^X$, where A^X denotes the value of $\Xi_X({}_gA)$ at the identity. The signature $(x_l)_{l\in\mathbb{N}}$ of A^X, in its turn, may be evaluated using

$$x_l = \frac{d}{dt}\bigg|_{t=0} a_l^{\exp tX}, \tag{6}$$

which follows from the fact that ${}_gA$ is smooth in the strong operator topology, and the fact that right multiplication by $\exp tX$ gives the flow of Ξ_X. The derivative in (6) has to be understood as a limit in the Hilbert space $L^2(M; \mathbb{C}^{d_l})$. Since a_l is smooth, however, one can compute that limit at each point $x \in M$, using

$$a_l^g(x) = \overline{R_l(g)}a_l(g^{-1}\cdot x), \tag{7}$$

which follows from $(R_l(g)^t)^{-1} = \overline{R_l(g)}$ and (4). Since G acts smoothly on M and M is compact, that pointwise limit is in fact uniform in x, and thus we also get the limit in L^2-sense. We obtain

$$x_l = L^X a_l + P_l^X a_l, \tag{8}$$

where $P_l^X = \frac{d}{dt}\big|_{t=0}\overline{R_l(\exp tX)}$ and L^X denotes the vector field on M given by

$$L^X f(x) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX)\cdot x)$$

(strictly speaking, $L^X a_l$ stands for $(L^X \otimes I_{d_l})a_l$). Both mappings $X \mapsto L^X$ and $X \mapsto P_l^X$ are representations of the Lie algebra $\mathfrak{g}$; the first being the derivative of the *regular representation* $g \mapsto T_g$ and the second being the derivative of the *contragradient* [6] of its restriction to E_l.

For each $X \in \mathfrak{g}$ and each $l \in \mathbb{N}$, let ∇_l^X denote the differential operator $\nabla_l^X a = L^X a + P_l^X a$, $a \in C^\infty(M; \mathbb{C}^{d_l})$. We are ready to state the main result of this section.

THEOREM 1. *The following conditions are equivalent, given an operator $A \in \mathcal{L}(\mathcal{H})$.*

(i) *The mapping in (3) is smooth in the norm topology.*
(ii) *The mapping in (3) is smooth in the strong operator topology.*
(iii) *Every component a_l of the signature of A is smooth and, for every finite sequence $X_1, \ldots, X_p$ in $\mathfrak{g}$, there is an $A^{X_1,\ldots,X_p} \in \mathcal{L}(\mathcal{H})$ whose signature equals $(\nabla_l^{X_p} \ldots \nabla_l^{X_1} a_l)_{l\in\mathbb{N}}$.*

Proof. Our previous computations show that, if (ii) holds, then (iii) holds if we take $A^{X_1,\ldots,X_p}$ equal to the value of $\Xi_{X_p}(\ldots(\Xi_{X_1}({}_gA))\ldots)$ at the identity.

Let us assume (iii) and prove (i). It is enough to show that, if an arbitrary $X \in \mathfrak{g}$ is given, then $\Xi_X({}_gA)$ exists in the norm topology and its value at the identity also satisfies condition (iii). Let D denote the linear span of $\bigcup_{l=0}^{\infty}\beta_l$. For $u\in D$ and $g\in G$, we have

$$(9)\qquad \lim_{h\to 0}\frac{T_{g\exp hX}AT_{g\exp hX}^{-1}-T_gAT_g^{-1}}{h}(u)={}_gA^X(u).$$

Indeed, at the identity, (9) follows from (4), from the fact that $A\mapsto \operatorname{sgn}(A)$ is injective and from the definition of the differential operators ∇_l^X. For arbitrary g, we also need to use the fact that D is invariant under T_g^{-1} and the continuity of T_g.

Given $u\in D$, define $f(t)=T_{\exp tX}AT_{\exp tX}^{-1}u$, $t\in\mathbb{R}$. It follows from (9) that $f'(t)=T_{\exp tX}A^XT_{\exp tX}^{-1}u$. Since A^X also satisfies (iii), we get $f''(t)=T_{\exp tX}A^{X,X}T_{\exp tX}^{-1}u$, and, hence, $\sup_t\|f''(t)\|\le\|A^{X,X}\|\cdot\|u\|$. It then follows that

$$(10)\qquad \left\|T_g\left(\frac{{}_{\exp tX}A-A}{t}-A^X\right)T_g^{-1}u\right\|\le\|A^{X,X}\|\cdot\|u\|\cdot|h|$$

for $g\in G$, $h\in\mathbb{R}$ and $u\in D$. Since D is dense, (10) implies that $\Xi_X({}_gA)$ exists in the norm topology and equals ${}_gA^X$. ∎

REMARK 3. If a Banach-space-valued function has bounded directional derivatives of all orders with respect to a family of vector fields that span the tangent space at every point, then it is smooth. In Theorem 1, we have used this elementary result for the family $\{\Xi_X : X\in\mathfrak{g}\}$. It is clear, however, that it is enough to take X belonging to a basis of $\mathfrak{g}$, and that, in (iii), it suffices to consider sequences $X_1,\ldots,X_p$ taking values in such a basis.

3. A class of examples. For each bounded sequence $\alpha_l\in C^\infty(M)$, $l\in\mathbb{N}$ (bounded meaning that $p(\alpha_l)$ is bounded for every continuous seminorm p on $C^\infty(M)$), define a *Fourier multiplier* A by

$$(12)\qquad Au=\sum_{l=0}^{\infty}\alpha_lP_lu\,,$$

where P_l denotes the orthogonal projection of $\mathcal{H}$ onto E_l. Expression (11) has a meaning for $u\in D$; so that, *a priori*, A is a densely defined unbounded operator on $\mathcal{H}$.

LEMMA 4. *If all Fourier multipliers are bounded, then they are also G-smooth.*

Proof. Let $\mathbb{Y}_l$ denote the vector-valued function $(Y_{l1},\ldots,Y_{ld_l})\in C^\infty(M;\mathbb{C}^{d_l})$. It follows from the definition of the matrix $R_l(g)$ that

$$(12)\qquad \overline{R_l(\exp tX)}\cdot\mathbb{Y}_l(\exp(-tX)\cdot x)=\mathbb{Y}_l(x)$$

for all $t\in\mathbb{R}$ and $x\in M$. It follows from the definition of ∇_l^X that $\nabla_l^X\mathbb{Y}_l(x)$ equals the derivative with respect to t at $t=0$ of the left-hand side of (12) and, hence, $\nabla_l^X\mathbb{Y}_l=0$.

Given a bounded sequence $\alpha_l\in C^\infty(M)$, let A denote the corresponding Fourier multiplier. The signature (a_l) of A equals $(\alpha_l\mathbb{Y}_l)$ and we have

$$\nabla_l^Xa_l=L^X(\alpha_l\mathbb{Y}_l)+\alpha_lP_l^X\mathbb{Y}_l=L^X(\alpha_l)\mathbb{Y}_l+\alpha_l\nabla_l^X\mathbb{Y}_l=L^X(\alpha_l)\mathbb{Y}_l.$$

This shows that $(\nabla_l^Xa_l)$ is the signature of the Fourier multiplier associated with the bounded sequence $L^X\alpha_l\in C^\infty(M)$; and similarly for higher orders. By Theorem 1, A is G-smooth. ∎

Given a Fourier multiplier A, define, for each $(l,l')\in\mathbb{N}^2$, $A_{l,l'}$ as the bounded operator $P_lAP_{l'}$. In Theorem 2 below it is proven that A is bounded (under an additional hypothesis on M) using the following version of Schur's test: If there exists an $M>0$ such that

$$(13)\qquad \sup_l\sum_{l'}\|A_{l,l'}\|<M\quad\text{and}\quad\sup_{l'}\sum_l\|A_{l,l'}\|<M,$$

then A is bounded and $\|A\|<M$. The norms $\|A_{l,l'}\|$ may be estimated using the following lemma.

LEMMA 5. *Given a non-negative integer k, there exists a continuous seminorm p_k on $C^\infty(M)$ such that*

$$(14)\qquad \left|(\lambda_l-\lambda_{l'})^k\int_M auv\,dS\right|\le p_k(a)(1+\lambda_{l'})^{k/2}\|u\|\cdot\|v\|$$

for all $a\in C^\infty(M)$, $u\in E_l$ and $v\in E_{l'}$, with $\|\cdot\|$ denoting the $\mathcal{H}$-norm.

Proof. Let L_0 denote the 0th order partial differential operator on M given by $L_0u=au$, and define by induction $L_k=\Delta L_{k-1}-L_{k-1}\Delta$. Then L_k is a kth order differential operator whose coefficients in local coordinates involve derivatives of order up to $2k$ of a. Taking a finite number of charts and a partition of unity, one can show that there is a continuous seminorm on $C^\infty(M)$ such that $\|L_kv\|\le p_k(a)\|v\|_k$, with $\|v\|_k$ denoting a kth order L^2-Sobolev norm of v, which we may suppose to be equal to $\|(1-\Delta)^{-k/2}v\|$.

Using the fact that u and v are eigenfunctions for Δ and successive integrations by parts, one sees that the left-hand side of (14) equals $\int_M uL_kv$. The proposition then follows from the Cauchy–Schwarz inequality and $\|(1-\Delta)^{-k/2}v\|=(1+\lambda_{l'})^{k/2}\|v\|$. ∎

THEOREM 2. *If M as in Section 2 is in addition a rank-one symmetric space, and if $(\alpha_l)_{l\in\mathbb{N}}$ is a bounded sequence in $C^\infty(M)$, then (11) defines a bounded operator $A\in\mathcal{L}(\mathcal{H})$ which is G-smooth.*

Proof. Let $\|\cdot\|_\infty$ denote the supremum norm for functions on M. A rough and obvious estimate for $\|A_{l,l'}\|$ is that they are all bounded by

$\sup_l \|\alpha_l\|_\infty$, which is finite, by our boundedness hypothesis on (α_l). Therefore, when we estimate the sums in (13), we will be allowed to omit a (fixed) finite number of terms.

Let $\langle\cdot,\cdot\rangle$ denote the inner product of $\mathcal{H}$ (linear for the second argument). Using the fact that, if $v \in E_{l'}$, then $\|A_{l,l'}v\|^2 = \langle P_l(\alpha_{l'}v), \alpha_{l'}v\rangle$, and applying Lemma 5 with $u = \overline{P_l(\alpha_{l'}v)}$ and $a = \alpha_{l'}$, we get

$$|\lambda_l - \lambda_{l'}|^{3/2}\|A_{l,l'}\| \le K(1+\lambda_l)^{3/4}, \tag{15}$$

where $K = \sup_{l\in\mathbb{N}} \sqrt{p_3(\alpha_l)\|\alpha_l\|_\infty}$, which is finite since (α_l) is bounded.

Since M is a rank-one symmetric space, there are positive numbers a and b such that $\lambda_l = al^2 + bl$ ([14], Theorem III.3.5). From this explicit formula for λ_l, one proves that $\sup_{l\neq l'} \sum_{l'} |\lambda_l^{1/2} - \lambda_{l'}^{1/2}|^{-3/2}$ is finite. Writing $|\lambda_l - \lambda_{l'}| = |\lambda_l^{1/2} - \lambda_{l'}^{1/2}|\cdot|\lambda_l^{1/2} + \lambda_{l'}^{1/2}|$, and noticing that $\sup_{l\neq l'}(1+\lambda_{l'})^{1/2}(\lambda_l^{1/2}+\lambda_{l'}^{1/2})^{-1}$ is finite, one sees from (15) that $\sup_l \sum_{l'} \|A_{l,l'}\|$ is finite, and similarly for the other supremum in (13). ■

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